Research Article

A Mathematical Overview of the Monogamous Marriage in a Multiregion Framework: Modelling and Control

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1. Introduction

Mathematical models are useful tools for understanding the functioning of natural systems and for predicting their evolution. Among these models are those that study the dynamics of populations and ecosystems. Many researchers have studied models of population dynamics: prey and predator dynamics [1–4], epidemic dynamics in a population [5–8], molecular systems [9–11], and so forth.

Civil status is the situation of the person in the family and/or society. A person’s marital status is positioned in one of four categories: virgin, married, divorced, or widowed (VMDW). Several authors have discussed the social functioning, marital status, family stability, and social control of health behavior [12–18]. In [19], the authors have given a discrete mathematical model that describes the marital status of the monogamous marriage case and they also studied the optimal control that reduces the number of unmarried and divorced people and increases the number of married people. In this paper, we generalize the results established in [19] to the case of a multiregion system. We studied the effect of the population travel to different regions for the marriage, divorce, and widow.

Many studies on civil status have given statistics on the family situation of different regions (cities, rural regions, industrial regions, tourist regions, etc.), according to the age group of the population. For example in Morocco, the High Commissioner for Planning (HCP, public institution) [14] gives different statistics according to the age groups of the population; there are 76.2% of single people for the age group of 20 to 24 and 54% for the group of 25 to 29 years old. The percentage of divorce is 30% for couples with less than 5 years of marriage and 3% for married couples who have exceeded 20 years of marriages. 41.4% of married men
41.8% of married women are identified by the HCP in the urban areas. The percentage of divorce is 0.8% for men and 3.1% for women. The widowed men are 0.6% of the population and women are 7.3% in urban areas. For the rural area, there are 40% of married men and 42.5% of married women; the percentage of divorce is 0.4% for married men and 1.4% for married women. Widowed men are represented by 0.6% of the population and women by 6.8% in rural areas.

In recent years, many attempts have been made to develop control strategies for different systems [5, 20–27]. There are a number of different methods for calculating the optimal control for a specific mathematical model. For example, Pontryagin et al.’s maximum principle [28] allows the characterization of the optimal control for an ordinary equation model system with a given constraint.

In [29], the authors have described a new modelling approach based on multiregion discrete-time SIR models aiming to describe the spatial-temporal evolution of epidemics that emerge in different geographical regions and to show the influence of one region on another region via infection travel.

In this paper and inspired by [29], we investigate an approach that determines an optimal control relative to a discrete VMDW model in a multiregion framework, which defines the evolution of the marital status of the marriage in a population, enabling decision-makers to develop very useful control strategies to reduce the virgin individuals and to increase the number of married individuals in a specific region.

The first control can be considered as public awareness campaigns showing to individuals the benefits of marriage on the psychic and social stability on persons and the society, or cultural entertained events to give people the chance to meet and to know and to allow themselves to get married. The second control is determined for the persons who have initiated the divorce proceedings; this control is considered as a long and costly legal procedure.

The optimal control problem was the subject of an optimization criterion represented by the minimization of an objective function. The optimality system is solved based on an iterative discrete schema that converges following an appropriate test similar to the one related to the Forward-Backward Sweep Method (FBSM).

The paper is organized as follows: in Section 2, the model VMDW is described for a multiregion discrete model. In Section 3, we give some results concerning the existence of optimal control and we use Pontryagin’s maximum principle to investigate the analysis of control strategies and to determine the necessary condition for optimal control. Numerical simulations are given in Section 4. Finally, we conclude the paper in Section 5.

2. Mathematical Modelling

We consider a discrete-time model VMDW of the marital status of the family dynamic within a domain of interest Ω which represents a country, a city, a town, or a small domain. We assume that there are p geographical regions (domains) of the domain studied Ω. Let Ω = Ωp i=1 Ωi, and let Ni(Ωi) be the population of domain Ωi at time i; that is, the number of individuals who are residents in Ωi.

This model classifies the marital status of the family dynamics of a population into eight compartments in each region Ωi: virgin men VM(Ωi), virgin women VW(Ωi), married men MM(Ωi), married women MW(Ωi), divorced men DM(Ωi), divorced women DW(Ωi), widowed men WM(Ωi), and widowed women WW(Ωi).

The unit of time i can correspond to days, months, or years; it depends on the frequency of the survey and demographic studies as needed. However demographic statistics are generally done annually so the units i, i + 1, . . . can be considered as years. The following system describes a discrete model of the marital status of the monogamous marriage case of a region Ωi:

\[
V_{i+1}^M(Ω_k) = \Lambda_{k1} + V_i^M(Ω_k) - \sum_{j=1}^{p} \alpha_{kj}V_i^{W}(Ω_j) + \gamma_{kj}D_i^{W}(Ω_j) + \delta_{kj}W_i^{W}(Ω_j) \frac{N_i(Ω_k)}{N_i(Ω_j) + N_i(Ω_j)} V_i^M(Ω_k) - d_k V_i^M(Ω_k),
\]

\[
V_{i+1}^W(Ω_k) = \Lambda_{k2} + V_i^W(Ω_k) - \sum_{j=1}^{p} \alpha_{kj}V_i^{W}(Ω_j) + \beta_{kj}D_i^{M}(Ω_j) + \eta_{kj}W_i^{M}(Ω_j) \frac{N_i(Ω_k)}{N_i(Ω_j) + N_i(Ω_j)} V_i^W(Ω_k) - d_k V_i^W(Ω_k),
\]

\[
D_{i+1}^M(Ω_k) = D_i^M(Ω_k) - \sum_{j=1}^{p} \beta_{kj}V_i^{W}(Ω_j) + \gamma_{kj}D_i^{W}(Ω_j) + \theta_{kj}W_i^{W}(Ω_j) \frac{N_i(Ω_k)}{N_i(Ω_j) + N_i(Ω_j)} D_i^M(Ω_k) + \lambda_k M_i^M(Ω_k) - d_k D_i^M(Ω_k),
\]
\begin{align*}
D_{i+1}^W(\Omega_k) &= D_i^W(\Omega_k) - \sum_{j=1}^{p} \frac{\gamma_{jk} V_i^W(\Omega_j) + \mu_{jk} D_i^M(\Omega_j) + \nu_{jk} W_i^M(\Omega_j)}{N_i(\Omega_j) + N_i(\Omega_k)} D_i^W(\Omega_k) \\
&\quad + \sum_{j=1}^{p} \chi_{ij} M_i^W(\Omega_j) - d_k D_i^W(\Omega_k),
\end{align*}

\begin{align*}
W_{i+1}^M(\Omega_k) &= W_i^M(\Omega_k) - \sum_{j=1}^{p} \frac{\eta_{jk} V_i^W(\Omega_j) + \nu_{jk} D_i^W(\Omega_j) + \sigma_{jk} W_i^W(\Omega_j)}{N_i(\Omega_j) + N_i(\Omega_k)} W_i^M(\Omega_k) \\
&\quad + \omega_k M_i^W(\Omega_k) - d_k W_i^M(\Omega_k),
\end{align*}

\begin{align*}
W_{i+1}^W(\Omega_k) &= W_i^W(\Omega_k) - \sum_{j=1}^{p} \frac{\delta_{jk} V_i^M(\Omega_j) + \theta_{jk} D_i^M(\Omega_j) + \sigma_{jk} W_i^M(\Omega_j)}{N_i(\Omega_j) + N_i(\Omega_k)} W_i^W(\Omega_k) \\
&\quad + \sum_{j=1}^{p} \rho_{ij} M_i^W(\Omega_j) - d_k W_i^W(\Omega_k),
\end{align*}

\begin{equation}
M_{i+1}^M(\Omega_k) = M_i^M(\Omega_k) + \sum_{j=1}^{p} \left[ \frac{\alpha_{jk} V_i^W(\Omega_j) + \beta_{jk} D_i^W(\Omega_j) + \delta_{jk} W_i^W(\Omega_j)}{N_i(\Omega_j) + N_i(\Omega_k)} V_i^M(\Omega_k) \right] - (\lambda_k + \rho_k) M_i^M(\Omega_k),
\end{equation}

\begin{equation}
M_{i+1}^W(\Omega_k) = M_i^W(\Omega_k) + \sum_{j=1}^{p} \left[ \frac{\alpha_{jk} V_i^W(\Omega_j) + \beta_{jk} D_i^W(\Omega_j) + \delta_{jk} W_i^W(\Omega_j)}{N_i(\Omega_j) + N_i(\Omega_k)} W_i^M(\Omega_k) \right] - (\lambda_k + \omega_k) M_i^W(\Omega_k),
\end{equation}
have Ni after contact with a virgin, divorced, or widowed men in
we have DW number of virgin women at the instant
the number of virgin men decreases and the number
\( \beta_{ki} \) Marriage rate of divorced men of region \( \Omega_i \) to virgin women of region \( \Omega_j \) 
\( \gamma_{ki} \) Marriage rate of virgin men of region \( \Omega_i \) to divorced women of region \( \Omega_j \) 
\( \delta_{ki} \) Marriage rate of virgin men of region \( \Omega_i \) to widowed women of region \( \Omega_j \) 
\( \eta_{ki} \) Marriage rate of widowed men of region \( \Omega_i \) to virgin women of region \( \Omega_j \) 
\( \mu_{ki} \) Marriage rate of divorced men of region \( \Omega_i \) to divorced women of region \( \Omega_j \) 
\( \theta_{ki} \) Marriage rate of divorced men of region \( \Omega_i \) to widowed women of region \( \Omega_j \) 
\( \nu_{ki} \) Marriage rate of widowed men of region \( \Omega_i \) to divorced women of region \( \Omega_j \) 
\( \sigma_{ki} \) Marriage rate of widowed men of region \( \Omega_i \) to widowed women of region \( \Omega_j \) 
\( \lambda_i \) Divorce rate of married men of region \( \Omega_i \) 
\( \lambda_j \) Divorce rate of women of region \( \Omega_j \) who return to the region \( \Omega_j \) 
\( \rho_i \) Widow rate of married women of region \( \Omega_i \) 
\( \omega_i \) Widow rate of married men of region \( \Omega_i \) 

In the model we propose here, it was considered that a
divorced man remains in his region and the divorced woman
returns to the region of her parents; this is the case for the
majority of regions whether they are conservative or not.
And so the number \( \lambda_i M_i^w(\Omega_i) \) is added to the number of
divorced men and the number \( \lambda_i M_i^w(\Omega_j) \) of divorced
women is added to the number of divorced women of the
region \( \Omega_i \) with \( \lambda_i \) being the rate that a woman divorces
a man from the region \( \Omega_k \) and returns to the region \( \Omega_j \).
Consequently, the total number of \( \sum_{i=1}^{p} \lambda_i M_i^w(\Omega_i) \) is added
to the number of divorced women in the region \( \Omega_j \) with
the relation \( \lambda_j = \sum_{i=1}^{p} \lambda_i \). The same principle applies to
equations (5) and (6) with \( \rho_i = \sum_{i=1}^{p} \rho_i \).

For equation (7), the number of married men in the region
\( \Omega_k \) increases at the instant \( t + 1 \) by the number of virgin,
divorced, and widowed men who are married by contacting
virgin, divorced, or widowed women of the region \( \Omega_j \), and
decreases the natural mortality with a \( \rho_i \) and divorce rate with
a \( \lambda_i \) rate. The same principle can be applied to equation (8).

In equations (7) and (8) from the model, the natural
mortality of married men and women was considered.
The mortality rate for married men is \( \rho_i \) and the mortality rate
for married women is \( \omega_i \). These rates appear in equations (5) and
(6) which correspond to widowed men and women, respectively,
so the number of mortalities which is \( \rho_i M_i^w(\Omega_i) \) married
men and the number \( \sum_{i=1}^{p} \rho_i M_i^w(\Omega_i) \) are added
to the number of widowed women and similarly the number
\( \omega_i M_i^w(\Omega_i) \) is added to the number of widowed men.

3. Methods and Results

3.1. An Optimal Control Approach. An optimal control
approach has been applied to models (1)–(8) to reduce the
virgin and divorced individuals and increase the number of
married individuals along the control strategy period. For this,
we introduce a control variable \( (u_i, v_i) \) that characterizes the
benefits of an awareness campaign to educate virgin men and
women about the benefits of marriage for the individual and
the society, especially the legal procedures, administrative
complications, and the heavy financial and social consequences
of divorces, respectively, in the abovementioned models
(1)–(8). Then, the model is given by the following equations:
\begin{align*}
V_{i+1}^M(\Omega_k) &= \Lambda_{k1} + V_i^M(\Omega_k) - \sum_{j=1}^{P} \frac{\alpha_k V_i^W(\Omega_j) + \gamma_k D_i^W(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} V_i^M(\Omega_k) \\
&\quad - d_k V_i^M(\Omega_k) - u_i V_i^M(\Omega_k),
\end{align*}

\begin{align*}
V_{i+1}^W(\Omega_k) &= \Lambda_{k2} + V_i^W(\Omega_k) - \sum_{j=1}^{P} \frac{\alpha_k V_i^M(\Omega_j) + \beta_k D_i^M(\Omega_j) + \delta_k W_i^M(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} V_i^W(\Omega_k) \\
&\quad - d_k V_i^W(\Omega_k) - u_i V_i^W(\Omega_k),
\end{align*}

\begin{align*}
D_{i+1}^M(\Omega_k) &= D_i^M(\Omega_k) - \sum_{j=1}^{P} \frac{\beta_k V_i^W(\Omega_j) + \mu_k D_i^W(\Omega_j) + \theta_k W_i^W(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} D_i^M(\Omega_k) \\
&\quad + \lambda_k M_i^M(\Omega_k) - d_k D_i^M(\Omega_k) - v_i D_i^M(\Omega_k),
\end{align*}

\begin{align*}
D_{i+1}^W(\Omega_k) &= D_i^W(\Omega_k) - \sum_{j=1}^{P} \frac{\gamma_k V_i^M(\Omega_j) + \mu_k D_i^M(\Omega_j) + \nu_k W_i^M(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} D_i^W(\Omega_k) \\
&\quad + \sum_{j=1}^{P} \lambda_k M_i^W(\Omega_j) - d_k D_i^W(\Omega_k) - v_i D_i^W(\Omega_k),
\end{align*}

\begin{align*}
W_{i+1}^M(\Omega_k) &= W_i^M(\Omega_k) - \sum_{j=1}^{P} \frac{\eta_k V_i^W(\Omega_j) + \phi_k D_i^W(\Omega_j) + \sigma_k W_i^W(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} W_i^M(\Omega_k) \\
&\quad + \omega_k M_i^W(\Omega_k) - d_k W_i^M(\Omega_k),
\end{align*}

\begin{align*}
W_{i+1}^W(\Omega_k) &= W_i^W(\Omega_k) - \sum_{j=1}^{P} \frac{\delta_k V_i^M(\Omega_j) + \phi_k D_i^M(\Omega_j) + \sigma_k W_i^W(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} W_i^W(\Omega_k) \\
&\quad + \sum_{j=1}^{P} \rho_k M_i^W(\Omega_j) - d_k W_i^W(\Omega_k),
\end{align*}

\begin{align*}
M_{i+1}^M(\Omega_k) &= M_i^M(\Omega_k) + \sum_{j=1}^{P} \frac{\alpha_k V_i^W(\Omega_j) + \gamma_k D_i^W(\Omega_j) + \delta_k W_i^W(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} V_i^M(\Omega_k) \\
&\quad + \sum_{j=1}^{P} \frac{\beta_k V_i^M(\Omega_j) + \mu_k D_i^M(\Omega_j) + \nu_k W_i^M(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} W_i^M(\Omega_k) \\
&\quad + \sum_{j=1}^{P} \frac{\gamma_k V_i^M(\Omega_j) + \phi_k D_i^M(\Omega_j) + \sigma_k W_i^M(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} W_i^W(\Omega_k) \\
&\quad + (\lambda_k + \rho_k) M_i^M(\Omega_k) + u_i V_i^M(\Omega_k) + v_i D_i^M(\Omega_k),
\end{align*}

\begin{align*}
M_{i+1}^W(\Omega_k) &= M_i^W(\Omega_k) + \sum_{j=1}^{P} \frac{\alpha_k V_i^W(\Omega_j) + \gamma_k D_i^W(\Omega_j) + \delta_k W_i^W(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} V_i^W(\Omega_k) \\
&\quad + \sum_{j=1}^{P} \frac{\beta_k V_i^M(\Omega_j) + \mu_k D_i^M(\Omega_j) + \nu_k W_i^M(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} W_i^W(\Omega_k) \\
&\quad + \sum_{j=1}^{P} \frac{\gamma_k V_i^M(\Omega_j) + \phi_k D_i^M(\Omega_j) + \sigma_k W_i^W(\Omega_j)}{N_i(\Omega_k) + N_i(\Omega_j)} W_i^W(\Omega_k) \\
&\quad + (\lambda_k + \rho_k) M_i^W(\Omega_k) + u_i V_i^W(\Omega_k) + v_i D_i^W(\Omega_k),
\end{align*}
\[ M^W_{in}(\Omega_k) = M^W_i(\Omega_k) + \sum_{j=1}^{D} \left[ \frac{\alpha^k_j V^W_i(\Omega_j) + \gamma^k_j D^W_i(\Omega_j) + \delta^k_j W^W_i(\Omega_j)}{N_i(\Omega_j) + N_\omega(\Omega_j)} \right] \]

subject to systems (9)–(16). Here, \( A_1, A_2, \) and \( A_3 \) are positive constants to keep a balance in the size of \( V^W_i(\Omega_k), D^W_i(\Omega_k), \) and \( M^W_i(\Omega_k), \) respectively. In the objective functional, \( \tau_1 \) and \( \tau_2 \) are the positive weight parameters which are associated with the controls \( u_i \) and \( v_i. \)

3.2. Characterization of the Optimal Control. For an initial state \( (V^M_0(\Omega_k), V^W_0(\Omega_k), M^M_0(\Omega_k), M^W_0(\Omega_k), D^M_0(\Omega_k), D^W_0(\Omega_k), W^M_0(\Omega_k), W^W_0(\Omega_k)) \), we consider an optimization criterion defined by the following objective function:

\[
J(u, v) = \sum_{i=0}^{N} \left( A_1 V^W_i(\Omega_k) + A_2 D^W_i(\Omega_k) - A_3 M^W_i(\Omega_k) \right) + \sum_{i=0}^{N-1} \left( \frac{\tau_1}{2} (u_i)^2 + \frac{\tau_2}{2} (v_i)^2 \right),
\]

subject to systems (9)–(16). Here, \( A_1, A_2, \) and \( A_3 \) are positive constants to keep a balance in the size of \( V^W_i(\Omega_k), D^W_i(\Omega_k), \) and \( M^W_i(\Omega_k), \) respectively. In the objective functional, \( \tau_1 \) and \( \tau_2 \) are the positive weight parameters which are associated with the controls \( u_i \) and \( v_i. \)

In other words, we seek the optimal controls \((u^*, v^*)\) such that

\[
J(u^*, v^*) = \min \{ J(u, v) | (u, v) \in \mathcal{U}_{ad} \},
\]

where \( \mathcal{U}_{ad} \) is the set of admissible controls defined by

\[
\mathcal{U}_{ad} = \left\{ (u, v) | u_{\min} \leq u_i \leq u_{\max}, v_{\min} \leq v_i \leq v_{\max}, i \in \{0, \ldots, N-1\} \right\},
\]

where \( (u_{\min}, u_{\max}, v_{\min}, v_{\max}) \in [0, 1]^4 \).

The sufficient condition for existence of an optimal control \((u^*, v^*)\) for problem (18) follows from standard results of [21]. In order to find an optimal solution, first we find the Hamiltonian for the optimal control problem (18). In fact, the Hamiltonian \( H \) of the optimal problem is given by

\[
- (\lambda_k + \omega_k) M^W_i(\Omega_k) + u_i V^W_i(\Omega_k) + v_i D^W_i(\Omega_k).
\]
\[ H = A_i V_i^M (\Omega_i) + A_i D_i^M (\Omega_i) - A_i M_i^M (\Omega_i) + \frac{T_p}{2} q_i^p + \frac{T_p}{2} p_i^p \]

\[ \begin{aligned}
+ \frac{\gamma_{\Omega_i}}{N_i(\Omega_i) + N_i(\Omega_{\bar{i}})} \left[ \begin{array}{c}
\delta_{\Omega_i} V_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} D_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} M_{i,\Omega_i}^M (\Omega_i) \\
N_i(\Omega_i) + N_i(\Omega_{\bar{i}})
\end{array} \right] \\
+ \frac{\gamma_{\Omega_i}}{N_i(\Omega_i) + N_i(\Omega_{\bar{i}})} \left[ \begin{array}{c}
\delta_{\Omega_i} V_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} D_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} M_{i,\Omega_i}^M (\Omega_i) \\
N_i(\Omega_i) + N_i(\Omega_{\bar{i}})
\end{array} \right] \\
+ \frac{\gamma_{\Omega_i}}{N_i(\Omega_i) + N_i(\Omega_{\bar{i}})} \left[ \begin{array}{c}
\delta_{\Omega_i} V_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} D_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} M_{i,\Omega_i}^M (\Omega_i) \\
N_i(\Omega_i) + N_i(\Omega_{\bar{i}})
\end{array} \right] \\
+ \frac{\gamma_{\Omega_i}}{N_i(\Omega_i) + N_i(\Omega_{\bar{i}})} \left[ \begin{array}{c}
\delta_{\Omega_i} V_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} D_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} M_{i,\Omega_i}^M (\Omega_i) \\
N_i(\Omega_i) + N_i(\Omega_{\bar{i}})
\end{array} \right] \\
+ \frac{\gamma_{\Omega_i}}{N_i(\Omega_i) + N_i(\Omega_{\bar{i}})} \left[ \begin{array}{c}
\delta_{\Omega_i} V_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} D_{i,\Omega_i}^M (\Omega_i) + \delta_{\Omega_i} M_{i,\Omega_i}^M (\Omega_i) \\
N_i(\Omega_i) + N_i(\Omega_{\bar{i}})
\end{array} \right]
\end{aligned} \]
where $\zeta_1^k, \zeta_2^k, \ldots, \zeta_8^k$ are the adjoint functions to be determined suitably.

At the same time by using Pontryagin et al.’s maximum principle [28], we derive necessary conditions for our optimal control. We have the following theorem.

**Theorem 1.** Let $V^M(\Omega_k), V^W(\Omega_k)$, $M^M(\Omega_k), M^W(\Omega_k), D^M(\Omega_k), D^W(\Omega_k)$, and $W^M(\Omega_k)$, be optimal state solutions with associated optimal control $(u^*, v^*)$ for the optimal control problem (18). Then, there exist adjoint variables $\zeta^k, \zeta^k_2, \ldots, \zeta^k_8$ that satisfy

\[
\Delta \zeta^k_{ij} = - \zeta^k_{ij} \left( 1 - \sum_{j=1}^N \frac{\alpha_{ji} V^M_j(\Omega_i) + \beta_{ji} D^M_j(\Omega_i) + \eta_{ji} W^M_j(\Omega_i)}{N_i(\Omega_i) + N_j(\Omega_j)} - d_k - u_i \right) \\
+ \zeta^k_{1,i+1} \sum_{j=1}^N \frac{\alpha_{ji} V^W_j(\Omega_i) + \beta_{ji} D^W_j(\Omega_i) + \delta_{ji} W^W_j(\Omega_i)}{N_j(\Omega_j) + N_i(\Omega_i)} - d_k - u_i \\
- \zeta^k_{i,j+1} \Omega_i,
\]

\[
\Delta \zeta^k_{2j} = - A_1 - \zeta^k_{2j+1} \left( 1 - \sum_{j=1}^N \frac{\alpha_{ij} V^M_j(\Omega_j) + \beta_{ij} D^M_j(\Omega_j) + \eta_{ij} W^M_j(\Omega_j)}{N_i(\Omega_i) + N_j(\Omega_j)} - d_k - u_i \right) \\
+ \zeta^k_{1,j+1} \sum_{i=1}^N \frac{\alpha_{ij} V^W_i(\Omega_j) + \beta_{ij} D^W_i(\Omega_j) + \delta_{ij} W^W_i(\Omega_j)}{N_i(\Omega_i) + N_j(\Omega_j)} - d_k - u_i \\
- \zeta^k_{j+1,i} \Omega_i,
\]

\[
\Delta \zeta^k_{3j} = - \zeta^k_{3j+1} \left( 1 - \sum_{j=1}^N \frac{\alpha_{ij} V^M_j(\Omega_j) + \beta_{ij} D^M_j(\Omega_j) + \theta_{ij} W^M_j(\Omega_j)}{N_i(\Omega_i) + N_j(\Omega_j)} - d_k - v_i \right) \\
+ \zeta^k_{2,j+1} \sum_{i=1}^N \frac{\alpha_{ij} V^W_i(\Omega_k) + \beta_{ij} D^W_i(\Omega_k) + \delta_{ij} W^W_i(\Omega_k)}{N_i(\Omega_i) + N_j(\Omega_j)} - d_k - v_i \\
- \zeta^k_{j+1,i} \Omega_i,
\]

\[
\Delta \zeta^k_{4j} = - A_2 - \zeta^k_{4j+1} \left( 1 - \sum_{j=1}^N \frac{\alpha_{ij} V^M_j(\Omega_j) + \beta_{ij} D^M_j(\Omega_j) + \eta_{ij} W^M_j(\Omega_j)}{N_i(\Omega_i) + N_j(\Omega_j)} - d_k - v_i \right) \\
+ \zeta^k_{1,j+1} \sum_{i=1}^N \frac{\alpha_{ij} V^W_i(\Omega_j) + \beta_{ij} D^W_i(\Omega_j) + \delta_{ij} W^W_i(\Omega_j)}{N_i(\Omega_i) + N_j(\Omega_j)} - d_k - v_i \\
- \zeta^k_{j+1,i} \Omega_i,
\]
\[ \Delta \zeta_{5,i} = -\zeta_{5,i+1}^k \left( \sum_{j=1}^{p} \eta_{ij} V_i^W(\Omega_j) + \nu_{ij} D_i^W(\Omega_j) + \sigma_{ij} W_i^W(\Omega_j) - d_k \right) 
+ \frac{\zeta_{2,i+1}^k}{2N_i(\Omega_k)} \frac{\eta_{ik} V_i^W(\Omega_k)}{N_i(\Omega_k)} + \frac{\zeta_{4,i+1}^k}{2N_i(\Omega_k)} \frac{\nu_{ik} D_i^W(\Omega_k)}{N_i(\Omega_k)} + \frac{\zeta_{6,i+1}^k}{2N_i(\Omega_k)} \frac{\sigma_{ik} W_i^W(\Omega_k)}{N_i(\Omega_k)} 
- \left( \zeta_{7,i+1}^k + \zeta_{8,i+1}^k \right) \left( \sum_{j=1}^{p} \eta_{ij} V_i^W(\Omega_j) + \nu_{ij} D_i^W(\Omega_j) + \sigma_{ij} W_i^W(\Omega_j) + \eta_{ik} V_i^W(\Omega_k) + \nu_{ik} D_i^W(\Omega_k) + \sigma_{ik} W_i^W(\Omega_k) \right) \right), \]  
(25) 

\[ \Delta \zeta_{6,i+1} = -\zeta_{6,i+1}^k \left( \sum_{j=1}^{p} \delta_{jk} V_i^M(\Omega_j) + \theta_{jk} D_i^M(\Omega_j) + \sigma_{jk} W_i^M(\Omega_j) - d_k \right) 
+ \frac{\zeta_{1,i+1}^k}{2N_i(\Omega_k)} \frac{\delta_{ik} V_i^M(\Omega_k)}{N_i(\Omega_k)} + \frac{\zeta_{3,i+1}^k}{2N_i(\Omega_k)} \frac{\theta_{ik} D_i^M(\Omega_k)}{N_i(\Omega_k)} + \frac{\zeta_{5,i+1}^k}{2N_i(\Omega_k)} \frac{\sigma_{ik} W_i^M(\Omega_k)}{N_i(\Omega_k)} 
- \left( \zeta_{7,i+1}^k + \zeta_{8,i+1}^k \right) \left( \sum_{j=1}^{p} \delta_{jk} V_i^M(\Omega_j) + \theta_{jk} D_i^M(\Omega_j) + \sigma_{jk} W_i^M(\Omega_j) + \delta_{ik} V_i^M(\Omega_k) + \theta_{ik} D_i^M(\Omega_k) + \sigma_{ik} W_i^M(\Omega_k) \right) \right), \]  
(26) 

\[ \Delta \zeta_{7,i+1} = -\zeta_{7,i+1}^k \lambda_k - \zeta_{6,i+1}^k \rho_k - \zeta_{7,i+1}^k (1 - \lambda_k - \rho_k), \]  
(27) 

\[ \Delta \zeta_{8,i+1} = A_3 - \zeta_{4,i+1}^k \lambda_k - \zeta_{5,i+1}^k \omega_k - \zeta_{8,i+1}^k (1 - \lambda_k - \omega_k), \]  
(28) 

**with transversality conditions**

\[ \begin{align*} 
\zeta_{1,N} &= 0, \\
\zeta_{2,N} &= A_1, \\
\zeta_{3,N} &= 0, \\
\zeta_{4,N} &= A_2, \\
\zeta_{5,N} &= 0, \\
\zeta_{6,N} &= 0, \\
\zeta_{7,N} &= 0, \\
\zeta_{8,N} &= -A_3. 
\end{align*} \]  
(29) 

Furthermore, the optimal control \((u_1^*, v_1^*)\) is given by

\[ u_1^* = \min \left\{ \max \left\{ \frac{V_i^M(\Omega_k)(\zeta_{1,i+1} - \zeta_{7,i+1}) + V_i^W(\Omega_k)(\zeta_{2,i+1} - \zeta_{8,i+1})}{C_1}, u_{min}, u_{max} \right\}, \right\} \]  
(30) 

\[ v_1^* = \min \left\{ \max \left\{ \frac{D_i^M(\Omega_k)(\zeta_{1,i+1} - \zeta_{7,i+1}) + D_i^W(\Omega_k)(\zeta_{2,i+1} - \zeta_{8,i+1})}{C_2}, v_{min}, v_{max} \right\}, \right\} \]  
(31) 

for \(i = 0, \ldots, N-1\).

**Proof.** Using Pontryagin et al.’s maximum principle [28] and setting \(V_i^{M*}(\Omega_k), V_i^{W*}(\Omega_k), M_i^{M*}(\Omega_k), M_i^{W*}(\Omega_k)\), we obtain the following adjoint equations:

\[ \begin{align*} 
D_i^{M*}(\Omega_k), D_i^{W*}(\Omega_k), W_i^{M*}(\Omega_k), W_i^{W*}(\Omega_k), \text{ and } (u^*, v^*), 
\end{align*} \]
\[ \Delta \xi_{1,j}^k = \frac{\partial H}{\partial V_i^M(\Omega_k)} \left[ \xi_{1,j+1}^k \left( 1 - \sum_{j=1}^{p} \frac{\alpha_{jk} V_j^M(\Omega) + \gamma_{jk} D_j^M(\Omega) + \delta_{jk} W_j^M(\Omega)}{N_i(\Omega_k) + N_i(\Omega_j)} - d_k - u_i \right) \right] \]

\[ = - \xi_{1,j+1}^k \left( \sum_{j=1}^{p} \frac{\alpha_{jk} V_j^M(\Omega) + \gamma_{jk} D_j^M(\Omega) + \delta_{jk} W_j^M(\Omega)}{N_i(\Omega_k) + N_i(\Omega_j)} - d_k - u_i \right) \]

\[ + \xi_{2,j+1}^k \frac{\alpha_{kk} V_k^M(\Omega_k)}{2N_i(\Omega_k)} + \xi_{4,j+1}^k \frac{\gamma_{kk} D_k^M(\Omega_k)}{2N_i(\Omega_k)} + \xi_{5,j+1}^k \frac{\delta_{kk} W_k^M(\Omega_k)}{2N_i(\Omega_k)} \]

\[ - (\xi_{7,j+1}^k + \xi_{8,j+1}^k) \left( \sum_{j=1}^{p} \frac{\alpha_{jk} V_j^M(\Omega) + \gamma_{jk} D_j^M(\Omega) + \delta_{jk} W_j^M(\Omega)}{N_i(\Omega_k) + N_i(\Omega_j)} + \frac{\alpha_{kk} V_k^M(\Omega_k) + \gamma_{kk} D_k^M(\Omega_k) + \delta_{kk} W_k^M(\Omega_k)}{2N_i(\Omega_k)} \right) - \xi_{7,j+1}^k u_i, \]

\[ \Delta \xi_{2,j}^k = \frac{\partial H}{\partial V_i^M(\Omega_k)} \left[ A_1 - \xi_{1,j+1}^k \frac{\alpha_{kk} V_k^M(\Omega_k)}{2N_i(\Omega_k)} V_i^M(\Omega_k) \xi_{1,j+1}^k \left( 1 - \sum_{j=1}^{p} \frac{\alpha_{jk} V_j^M(\Omega) + \beta_{jk} D_j^M(\Omega) + \eta_{jk} W_j^M(\Omega)}{N_i(\Omega_k) + N_i(\Omega_j)} - d_k - u_i \right) \right] \]

\[ = - A_1 - \xi_{1,j+1}^k \left( \sum_{j=1}^{p} \frac{\alpha_{jk} V_j^M(\Omega) + \beta_{jk} D_j^M(\Omega) + \eta_{jk} W_j^M(\Omega)}{N_i(\Omega_k) + N_i(\Omega_j)} - d_k - u_i \right) \]

\[ + \xi_{2,j+1}^k \frac{\alpha_{kk} V_k^M(\Omega_k)}{2N_i(\Omega_k)} \left( \sum_{j=1}^{p} \frac{\alpha_{jk} V_j^M(\Omega) + \beta_{jk} D_j^M(\Omega) + \eta_{jk} W_j^M(\Omega)}{N_i(\Omega_k) + N_i(\Omega_j)} + \frac{\alpha_{kk} V_k^M(\Omega_k) + \beta_{kk} D_k^M(\Omega_k) + \eta_{kk} W_k^M(\Omega_k)}{2N_i(\Omega_k)} \right) \]

\[ + \xi_{4,j+1}^k \frac{\beta_{kk} D_k^M(\Omega_k)}{2N_i(\Omega_k)} + \xi_{5,j+1}^k \frac{\eta_{kk} W_k^M(\Omega_k)}{2N_i(\Omega_k)} \]

\[ - (\xi_{7,j+1}^k + \xi_{8,j+1}^k) \left( \sum_{j=1}^{p} \frac{\alpha_{jk} V_j^M(\Omega) + \beta_{jk} D_j^M(\Omega) + \eta_{jk} W_j^M(\Omega)}{N_i(\Omega_k) + N_i(\Omega_j)} + \frac{\alpha_{kk} V_k^M(\Omega_k) + \beta_{kk} D_k^M(\Omega_k) + \eta_{kk} W_k^M(\Omega_k)}{2N_i(\Omega_k)} \right) - \xi_{7,j+1}^k u_i, \]
\[ \Delta \xi_{k,i} = \frac{\partial H}{\partial D_{ji}^M}(\Omega_k) \]

\[
\Delta \xi_{k,i} = \frac{\partial H}{\partial D_{ji}^M}(\Omega_k)
= \begin{pmatrix}
\rho_{ik}^k - \frac{\rho_{ik}^k}{2N_i(\Omega_k)} V_i^M(\Omega_k) + \xi_{j,i+1} \left( 1 - \sum_{j=1}^p \frac{\rho_{ik}^j V_i^M(\Omega_j) + \mu_{ik} D_{ji}^M(\Omega_j) + \theta_{ik} W_i^M(\Omega_j)}{N_i(\Omega_j) + N_j(\Omega_j)} - d_k - v_i \right)
\end{pmatrix}
\]

\[\Delta \xi_{k,i} = \frac{\partial H}{\partial D_{ji}^M}(\Omega_k) \]

\[
\Delta \xi_{k,i} = \frac{\partial H}{\partial D_{ji}^M}(\Omega_k)
= \begin{pmatrix}
A_2 - \xi_{j,i+1} \frac{\rho_{ik}^k}{2N_i(\Omega_k)} V_i^M(\Omega_k) - \xi_{j,i+1} \frac{\rho_{ik}^k}{2N_i(\Omega_k)} D_{ji}^M(\Omega_k) + \xi_{j,i+1} \left( 1 - \sum_{j=1}^p \frac{\rho_{ik}^j V_i^M(\Omega_j) + \mu_{ik} D_{ji}^M(\Omega_j) + \theta_{ik} W_i^M(\Omega_j)}{N_i(\Omega_j) + N_j(\Omega_j)} - d_k - v_i \right)
\end{pmatrix}
\]

(34)

(35)
\[ \Delta \xi_{5,i}^k = - \frac{\partial H}{\partial W_i^M(\Omega_k)} \]

\[ = \left[ -\zeta_{i,1}^{\delta_{i,1}} \left( \frac{\eta_{i,1}}{2N_i(\Omega_k)} V_i^M(\Omega_k) - \zeta_{i,1}^{\delta_{i,1}} \frac{\eta_{i,1}}{2N_i(\Omega_k)} D_i^M(\Omega_k) \right) + \zeta_{i,1}\left( 1 - \sum_{j=1}^p \frac{\eta_{j,1}}{N_j(\Omega_k)} + \frac{\eta_{i,1}}{N_i(\Omega_k)} \right) - d_k \right] \]

\[ = \left[ -\zeta_{i,1}^{\delta_{i,1}} \left( \frac{\eta_{i,1}}{2N_i(\Omega_k)} V_i^M(\Omega_k) + \frac{\eta_{i,1}}{2N_i(\Omega_k)} D_i^M(\Omega_k) \right) + \zeta_{i,1}\left( 1 - \sum_{j=1}^p \frac{\eta_{j,1}}{N_j(\Omega_k)} + \frac{\eta_{i,1}}{N_i(\Omega_k)} \right) - d_k \right] \]

\[ = -\zeta_{5,i}^k \left( 1 - \sum_{j=1}^p \frac{\eta_{j,1}}{N_j(\Omega_k)} + \frac{\eta_{i,1}}{N_i(\Omega_k)} \right) - d_k \]

\[ \Delta \xi_{6,i}^k = - \frac{\partial H}{\partial W_i^M(\Omega_k)} \]

\[ = \left[ -\zeta_{i,1}^{\delta_{i,1}} \frac{\delta_{i,1}}{2N_i(\Omega_k)} V_i^M(\Omega_k) - \zeta_{i,1}^{\delta_{i,1}} \frac{\delta_{i,1}}{2N_i(\Omega_k)} D_i^M(\Omega_k) - \zeta_{i,1}\left( 1 - \sum_{j=1}^p \frac{\delta_{j,1}}{N_j(\Omega_k)} + \frac{\delta_{i,1}}{N_i(\Omega_k)} \right) - d_k \right] \]

\[ = -\zeta_{6,i}^k \left( 1 - \sum_{j=1}^p \frac{\delta_{j,1}}{N_j(\Omega_k)} + \frac{\delta_{i,1}}{N_i(\Omega_k)} \right) - d_k \]

\[ \Delta \xi_{7,i}^k = - \frac{\partial H}{\partial M_i^M(\Omega_k)} \]

\[ = -\zeta_{7,i}^k \left( 1 - \sum_{j=1}^p \frac{\delta_{j,1}}{N_j(\Omega_k)} + \frac{\delta_{i,1}}{N_i(\Omega_k)} \right) - d_k \]

\[ \Delta \xi_{8,i}^k = - \frac{\partial H}{\partial M_i^M(\Omega_k)} \]

\[ = -\zeta_{8,i}^k \left( 1 - \sum_{j=1}^p \frac{\delta_{j,1}}{N_j(\Omega_k)} + \frac{\delta_{i,1}}{N_i(\Omega_k)} \right) - d_k \]
with transversality conditions:
\[
\begin{align*}
\zeta_{1,N} & = 0, \\
\zeta_{2,N} & = A_1, \\
\zeta_{3,N} & = 0, \\
\zeta_{4,N} & = A_2, \\
\zeta_{5,N} & = 0, \\
\zeta_{6,N} & = 0, \\
\zeta_{7,N} & = 0, \\
\zeta_{8,N} & = -A_3.
\end{align*}
\]

To obtain the optimality conditions, we take the variation with respect to control \((u_i^*, v_i^*)\) and set it equal to zero:
\[
\begin{align*}
\frac{\partial H}{\partial u_i} & = \tau_1 u_i - \zeta_{1,i+1} V_i^M(\Omega_k) - \zeta_{2,i+1} V_i^W(\Omega_k) + \zeta_{7,i+1} V_i^M(\Omega_k) \\
& \quad + \zeta_{8,i+1} V_i^W(\Omega_k) = 0, \\
\frac{\partial H}{\partial v_i} & = \tau_2 u_i - \zeta_{3,i+1} D_i^M(\Omega_k) - \zeta_{4,i+1} D_i^W(\Omega_k) + \zeta_{7,i+1} D_i^M(\Omega_k) \\
& \quad + \zeta_{8,i+1} D_i^W(\Omega_k) = 0.
\end{align*}
\]

Then, we obtain the optimal control:
\[
\begin{align*}
u_i^* & = \frac{V_i^M(\Omega_k)(\zeta_{1,i+1} - \zeta_{7,i+1}) + V_i^W(\Omega_k)(\zeta_{2,i+1} - \zeta_{8,i+1})}{\tau_1}, \\
v_i^* & = \frac{D_i^M(\Omega_k)(\zeta_{3,i+1} - \zeta_{7,i+1}) + D_i^W(\Omega_k)(\zeta_{4,i+1} - \zeta_{8,i+1})}{\tau_2}.
\end{align*}
\]

By the bounds in \(\mathcal{U}_{ad}\), it is easy to obtain \((u_i^*, v_i^*)\) in the following form:
\[
\begin{align*}
u_i^* & = \min\left\{ \max\left\{ \frac{V_i^M(\Omega_k)(\zeta_{1,i+1} - \zeta_{7,i+1}) + V_i^W(\Omega_k)(\zeta_{2,i+1} - \zeta_{8,i+1})}{\tau_1}, u_{\text{min}}, u_{\text{max}} \right\}, \right. \\
v_i^* & = \min\left\{ \max\left\{ \frac{D_i^M(\Omega_k)(\zeta_{3,i+1} - \zeta_{7,i+1}) + D_i^W(\Omega_k)(\zeta_{4,i+1} - \zeta_{8,i+1})}{\tau_2}, u_{\text{min}}, u_{\text{max}} \right\}, \right.
\end{align*}
\]

for \(i = 0, \ldots, N - 1\).

4. Discussion

In this section, we provide numerical simulations to demonstrate our theoretical results in the case when the studied domain represents the assembly of \(p\) regions (cities, towns, etc.). The code is written and compiled in MATLAB using the data cited in Table 2. The optimality systems are solved using an iterative method where, at instant \(i\), the states \(V_i^M, V_i^W, \ldots, M_i^W\) with an initial guess are obtained based on a progressive scheme in time, and their adjoint variables \(\zeta_{l,i+1}, l = 1, 2, \ldots, 8\) are obtained based on a regressive scheme in time because of the transversality conditions. Afterward, we update the optimal control values (30) and (31) using the values of state and costate variables obtained in the previous steps. Finally, we execute the previous steps until a tolerance criterion is reached. In order to show the importance of our work and without loss of generality, we consider here that \(p = 4\). An area with 4 regions was considered: two regions \(\Omega_1\) and \(\Omega_2\) represent two urban areas and the regions \(\Omega_3\) and \(\Omega_4\) represent the two rural areas. The High Commissioner for Planning (HCP, public establishment) [14] gave various population statistics according to the regions (urban and rural). The values defined in Table 2 are inspired by data mentioned on page 2. Generally, the HCP gives the average sum of marriage, divorce, and widow rates between different population areas \(\alpha_{k,j}, \beta_{k,j}, \ldots, \omega_{k}\) and, precisely, one of the main objects of our work is to estimate as close as possible the values of the parameters \(\alpha_{k,j}, \beta_{k,j}, \ldots, \omega_{k}\) defined in Table 2.

4.1. Simulations without Controls. In this section, Figure 1 depicts dynamics of the states \(V_i^M, V_i^W, \ldots, M_i^W\) in regions \(\Omega_1, \Omega_2, \Omega_3\), and \(\Omega_4\), respectively, in the case when there is yet no control strategy, and we note that in all these figures presented here, simulations give us an idea about the evolution of marital status in each region whether rural or urban and the impact of flows of populations between regions on the dynamics of the marital status of each one.

The study of the evolution of the dynamics of the marital status was spread out over a period of 20 years. In Figure 1, we note that the evolution of the population in regions \(\Omega_1\) and \(\Omega_4\) is almost the same and similarly for regions \(\Omega_2\) and \(\Omega_3\).
V^M = 4500, V^W = 4600, D^M = 400
D^W = 600, W^M = 200, W^W = 300
M^M = 2500, M^W = 2000
λ_j = 250, λ_{12} = 250, d_1 = 0.03

(\alpha_1^j)_{1\leq j \leq 4} = (0.07, 0.05, 0.06, 0.05)
(\alpha_2^j)_{1\leq j \leq 4} = (0.06, 0.07, 0.06, 0.08)
(\beta_1^j)_{1\leq j \leq 4} = (0.04, 0.02, 0.03, 0.03)
(\beta_2^j)_{1\leq j \leq 4} = (0.03, 0.04, 0.04, 0.05)
(\gamma_1^j)_{1\leq j \leq 4} = (0.02, 0.02, 0.02, 0.03)
(\gamma_2^j)_{1\leq j \leq 4} = (0.02, 0.02, 0.02, 0.03)
(\delta_1^j)_{1\leq j \leq 4} = (0.01, 0.02, 0.01, 0.01)
(\delta_2^j)_{1\leq j \leq 4} = (0.01, 0.01, 0.01, 0.01)
(\eta_1^j)_{1\leq j \leq 4} = (0.02, 0.02, 0.02, 0.03)

Ω_1:  (\theta_1^j)_{1\leq j \leq 4} = (0.02, 0.03, 0.02, 0.03)
(\theta_2^j)_{1\leq j \leq 4} = (0.02, 0.02, 0.01, 0.02)
(\theta_3^j)_{1\leq j \leq 4} = (0.01, 0.01, 0.01, 0.01)
(\theta_4^j)_{1\leq j \leq 4} = (0.01, 0.02, 0.01, 0.01)

λ_3 = (0.003, 0.001, 0.001, 0.004)
ρ_3 = (0.01, 0.01, 0.01, 0.01, 0.01)

V^M = 5500, V^W = 5600, D^M = 900
D^W = 820, W^M = 400, W^W = 300
M^M = 3500, M^W = 3000
λ_{31} = 250, λ_{32} = 260, d_3 = 0.04

(\alpha_1^j)_{1\leq j \leq 4} = (0.05, 0.06, 0.05, 0.05)
(\alpha_2^j)_{1\leq j \leq 4} = (0.07, 0.06, 0.07, 0.07)
(\beta_1^j)_{1\leq j \leq 4} = (0.03, 0.04, 0.03, 0.04)
(\beta_2^j)_{1\leq j \leq 4} = (0.03, 0.02, 0.04, 0.03)
(\gamma_1^j)_{1\leq j \leq 4} = (0.02, 0.02, 0.02, 0.01)
(\gamma_2^j)_{1\leq j \leq 4} = (0.02, 0.01, 0.02, 0.02)
(\delta_1^j)_{1\leq j \leq 4} = (0.01, 0.01, 0.01, 0.02)
(\delta_2^j)_{1\leq j \leq 4} = (0.01, 0.02, 0.01, 0.01)
(\eta_1^j)_{1\leq j \leq 4} = (0.04, 0.03, 0.03, 0.02)

Ω_2:  (\theta_1^j)_{1\leq j \leq 4} = (0.03, 0.02, 0.02, 0.03)
(\theta_2^j)_{1\leq j \leq 4} = (0.01, 0.02, 0.02, 0.03)
(\theta_3^j)_{1\leq j \leq 4} = (0.01, 0.02, 0.01, 0.01)
(\theta_4^j)_{1\leq j \leq 4} = (0.01, 0.01, 0.01, 0.01)

λ_4 = (0.002, 0.001, 0.001, 0.004)
ρ_4 = (0.01, 0.01, 0.01, 0.01, 0.01)

Table 2: Parameters values associated with discrete-time systems (9)–(16) and with optimal control problem (18).

\( \tau_1 = 2.5 \times 10^5, \tau_2 = 3 \times 10^5 \)

A_1 = 0.0005, A_2 = 0.0004, A_3 = 0.0002
Figure 1: Time evolution of marital status of family dynamics VMDW without controls.

Figure 2: Continued.
In the urban area $\Omega_1$, the number of single men is 4500 at time $i = 0$, and the number of married men is about 2000 and 400 cases of divorce for men and the same numbers for women, with a birth rate of 250 per year and 0.03% mortality. The number of virgins declined a little and stabilized after 10 years to 3750. By cons, for the number of married men and women, there is a slight evolution that reaches after 12 years the number of 3200 married people and finally, for the number of divorces, it remains almost stable throughout the period and remains around 600 divorces.
In rural Ω, the number of unmarried men at time $i = 0$ is 1050, the number of married men is about 500 and 100 divorces for men and the same numbers for women, with a birth rate of 60 per year and 0.03% mortality. These numbers will be evolved, so the number of virgins has decreased to reach about 700 virgins. The number of married women will experience considerable growth to reach around 1400 for men and 1100 for women. The number of women divorces will witness a slight growth of about 300 divorces and stability for men of almost 100.

### 4.2. Simulations with Controls

Considering the critical level of control, we give optimal control sufficient to reduce the number of virgin and divorced individuals and to increase the married individuals in $\Omega_1$ and we also studied the effect of controls on other neighboring regions $\Omega_2$, $\Omega_3$, and $\Omega_4$.

In the following, we can see that the optimal control function has a very desirable effect upon the population of virgin and divorced people which decreases while the married population increases in a consistent way during the length of the process. The time evolution of the respective populations with control is displayed in Figure 2.

Figures 3–5 allow us to compare changes in the number of virgins, divorced, and married individuals before and after the introduction of control.

In a population of virgin men of 4500 individuals (Figure 3(a)) by applying the control law, this number...
decreases rapidly to reach after 3 years the number of 1700 virgins and then goes back up to reach out at the end of the companion awareness 3000 virgin men. This result is similar to virgin women as shown in Figure 3(b).

Figure 3(c) also shows the effect of applying the control law by indicating that the number of divorced men decreases more rapidly at the beginning of treatment to reach 100 male divorces. Then, we notice that the number increases and we can justify it by the fact that the number of divorced people is proportional to the number of married people. Since the number of married individuals increases along the course of treatment, it has been assumed that divorced men remain in $\Omega_1$. Unlike divorced women who can return to their region of origin, therefore the number of divorced women that was 600 women decreases rapidly to reach 150 divorced women after 4 years and then increases to 280 divorced women at the end of the companion as shown in Figure 3(d).

Figures 4(c) and 4(d) show that the number of married men and women grows after 2 years and reaches a maximum value of about 2500 for married men and 2000 for married women; then it declines to 3800 for men and 4250 for women at the end of companion. This decline is due to the fact that the number of divorced men and women increases, but it is still quite high as far as the uncontrolled number of married men and women. In Figures 4(a) and 4(b), we show the evolution of widowed men and women; the number increases slightly concerning the number of widowed individuals in a dynamics of the marital system without control strategies.

We also note in Figures 1 and 2 and contrary to the evolution of the dynamics of marital status in $\Omega_1$ that the number of virgin and divorced people slightly increases in the regions $\Omega_2$, $\Omega_3$, and $\Omega_4$ and the number of married people decreases slightly.

Finally, Figure 5 displays the time evolution of the optimal control ($u^*, v^*$).

5. Conclusions

In this paper, we consider a discrete-time marital status model; we treat the modelling and control of a system that described the case of a multiregion model. Optimal control is investigated to reduce the number of virgins and the divorced population and to increase the married ones. The first control introduced is supposing the benefits of an awareness campaign to educate virgin men and women about the benefits of marriage for individuals and the society, and the second control characterizes the legal procedures, administrative complications, and the heavy financial and social consequences of divorces. A discrete version of Pontryagin et al.’s maximum principle is done to analyze the optimal control problem and numerical simulation is given to illustrate the obtained results.

Data Availability

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[17] I. Waldron, C. C. Weiss, and M. E. Hughes, “Marital status effects on health: are there differences between never married


