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Research Article

Dynamic Analysis and Optimal Control of ISCR Rumor Propagation Model with Nonlinear Incidence and Time Delay on Complex Networks

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An Innocents-Spreaders-Calmness-Removes (ISCR) rumor propagation model is established with nonlinear incidence and time delay on complex networks in this paper. Based on the mean-field theory, the spreading dynamics of the ISCR model are discussed in detail. Firstly, the basic reproduction number $\mathbf{R}_0$ is obtained by the next generation matrix method to ensure the existence of rumor-prevailing equilibrium. Secondly, by utilizing the Routh–Hurwitz criterion and LaSalle's invariance principle, the local stability and global stability of rumor equilibria are proved. Moreover, the optimal control is presented via Pontryagin's minimum principle, which is to effectively restrain rumor diffusion. Finally, the theoretical results are verified by numerical simulations.

1. Introduction

Rumors are usually defined as unproven words and may damage personal reputation, affect financial markets, cause social panic and instability, and severely disrupt people's normal and orderly life. Social networks do build a good platform for people's communication, but it also provides an opportunity for a dishonest person to spread rumors. In the online virtual social platform, everyone has their own online virtual identity, and the virtual identity is the medium for rumors to diffuse [1]. Compared with traditional rumors, Internet rumors spread faster and wider, so people should pay more attention to them and take certain measures to deal with them when necessary. Therefore, it is vital to study the potential mechanism and control measures of rumor propagation on social networks. Based on mathematical models, the research on the mechanism of rumor propagation has received extensive attention from scholars.

Considering the similarities between the spread of rumors and epidemics, traditional rumor propagation models are mostly based on the dynamic models of infectious diseases, such as SI (Susceptible-Infected), SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed), and so on. Daley and Kendall first studied the rumor propagation dynamic analysis in 1965 and proposed a rumor spreading model named DK model [2]. After that, Maki and Thomson [3] further improved the DK model to the MT model in 1973. In addition, some scholars also applied the fractal-factional order model to the infectious disease model [4], and others discussed the backward bifurcation and optimal control of the infectious disease model [5, 6]. Based on the work of predecessors, many papers which considered other affecting factors of rumor propagation have been proposed afterwards [7–12], such as incubation [7], the proportion of wiseman in the crowd [8], debunking behavior in emergencies [9], media report [10], psychological factors and forgetting mechanism [11], different attitudes towards rumors [12], superspreaders [13], and so on.

As an effective tool, complex networks have laid a fine foundation for studying the spread of rumors on social networks. Zanette first proposed a rumor propagation model on small-world networks by utilizing complex network theory to the study of rumor propagation [14]. Li et al. had established an I2S2R (Ignorants-Spreaders 1-Spreaders 2-

Stiflers 1-Stiflers 2) rumor spreading model on homogeneous networks [15]. A novel SIR (Susceptible-Propagating-Recovery) rumor spreading model was proposed in both homogeneous and heterogeneous networks by Zhu et al. [16]. Nowadays, there are still many articles considering the rumor spreading model on complex networks [17–22]. Based on the aforementioned models, a rumor propagation model is proposed on complex networks in this paper.

It is worth emphasizing that the incidence rate plays a significant role in the spread of rumors. The incidence mainly includes bilinear incidence and nonlinear incidence. The characteristic of bilinear incidence is that the number of Spreaders increases linearly. However, the psychological changes of Innocents have a lot of influence on rumor propagation, which make bilinear incidence have some limitations, and the nonlinear incidence is used frequently in [23–25]. For example, the incidence $\mathbf{R}_0$ was proposed by Capasso and Serio [23] to represent saturation phenomena for large numbers of infectives. The incidence $g(I) = kI/(1+\alpha I)$ was utilized to explain the following phenomenon: the incidence of infectious diseases may show a downward trend during the peak of infectious diseases because some individuals take protective steps to reduce contact with others individuals [24]. The incidence $1/(1+\alpha I^h)$ was used by Ruan and Wang [25] to describe the inhibition rate that came from an increasing number of susceptible individuals. Although the nonlinear incidence was first used in disease transmission models, it can also be considered in the rumor transmission model because of the similar transmission mechanism. Based on the above discussion, the nonlinear incidence $\gamma\langle k\rangle I(t)S(t)/1+\alpha S(t)$ is used in this paper to describe the influence of the psychological changes of Innocents on rumor propagation, where γ represents the spreading ability of rumors and α represents the impact of population crowding or changes on Innocents. Obviously, this is more in line with the reality. Hence, considering the nonlinear incidence is of significance for the study of rumor propagation.

As is well known to all, people sometimes may not timely respond to rumors. When an Innocent receives a rumor, it takes some time to consider whether to spread the rumor. This is the reason why time delay exists. Therefore, this will motivate us to study rumor propagation with time delay. For instance, Jain et al. [26] analyzed the effect of delay to influence thinkers. A delayed rumor spreading model was proposed by Li and Ma [27] in emergencies. Chen et al. [28] established multiple delayed models to explore the new characteristics of rumor spreading process. Meanwhile, timeliness is the important characteristic of rumor propagation, so it is vital to study the rumor propagation model with time delay.

In light of above discussion, our main contributions are reflected as follows:

(1) A novel ISCR rumor propagation model is proposed with nonlinear incidence and time delay on complex networks. Based on the traditional SIR model, this paper adds a "Calmness" compartment, which makes the model more realistic.

(2) In real social networks, the spread of rumors depends on the degree of nodes in the network, so this paper models the spread process of rumors based on the network structure.

(3) Because individuals need a certain reaction time after exposure to rumors and the number of Spreaders is limited, this paper considers the effects of time delay and nonlinear incidence on rumor propagation. Moreover, the nonlinear incidence describes the influence of the psychological changes of Innocents on rumor propagation.

(4) By utilizing the Routh–Hurwitz criterion and LaSalle's invariance principle, the local stability and global stability of rumor equilibria are proved in detail.

(5) To reduce the density of Spreaders and control costs, an optimal control strategy with time delay is given and analyzed for the controlled system by Pontryagin's minimum principle. Meanwhile, this paper also analyzes the influence of time delay on optimal control in numerical simulation.

2. Problem Description

In this section, a novel ISCR rumor propagation model is proposed on complex networks to learn the dynamics of rumor propagation mechanism. Four states are proposed to represent the different states of individuals in the process of rumor propagation. Innocents $(I(t))$ represent those who do not perceive rumors but may be infected. Spreaders $(S(t))$ represent those who understand rumors and spread them. Calmness $(C(t))$ represents those who calm down before they stop spreading rumors. Removes $(R(t))$ represent those who perceive rumors but do not spread them. $I(t)$, $S(t)$, $C(t)$, and $R(t)$ denote the density of Innocents, Spreaders, Calmness, and Removes at time t, respectively. Moreover, we assume that $I(t) + S(t) + C(t) + R(t) = 1$.

The process of ISCR model is shown in Figure 1. According to the mean-field theory, the model can be described as follows:

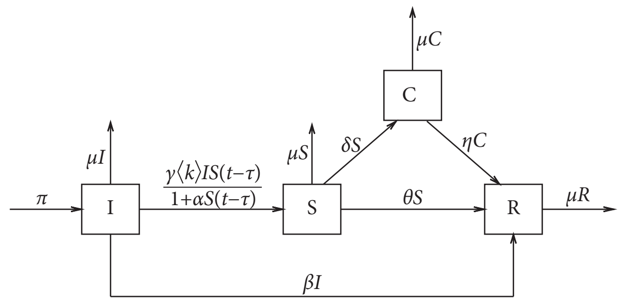


Figure 1: The process of ISCR model.

$$\begin{cases} \frac{\mathrm{d}I(t)}{\mathrm{d}t} = \pi - \frac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)} - (\beta+\mu)I(t), \\ \frac{\mathrm{d}S(t)}{\mathrm{d}t} = \frac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)} - (\delta+\theta+\mu)S(t), \\ \frac{\mathrm{d}C(t)}{\mathrm{d}t} = \delta S(t) - (\eta+\mu)C(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} = \beta I(t) + \theta S(t) + \eta C(t) - \mu R(t), \end{cases} \quad (1)$$

where $\langle k\rangle$ denotes the average degree of complex networks, $\tau \geq 0$ is the average infectious delay of the infectious rumors, π represents the recruitment rate of Innocents, γ denotes the spread rate of Spreaders, α is the saturation coefficient that measures the inhibitory or psychological effect of the general public towards rumors, β is the transfer rate from Innocents to Removes due to the immune mechanism, θ is the forgetting rate from Spreaders to Removes by the forgetting mechanism, δ is the calmness rate of Spreaders, and η is the transfer rate from Calmness to Removes. Suppose that every class has the same emigration rate μ, and the recruitment rate is equal to the emigration rate, that is, $\pi = \mu$. Assume that above parameters are positive.

The initial conditions of system (1) are taken by the following form:

$$\begin{aligned} I(t) &= \phi_I(t) \geq 0, \\ S(t) &= \phi_S(t) \geq 0, \\ C(t) &= \phi_C(t) \geq 0, \\ R(t) &= \phi_R(t) \geq 0, t \in [-\tau, 0], \end{aligned} \quad (2)$$

where $\phi \in \mathscr{C}([-\tau,0],\mathbb{R}_+)$. The Banach space $\mathscr{C}$ is nonnegative continuous, mapping the interval $[-\tau,0]$ into $\mathbb{R}_+$.

Remark 1. It is worth noting that a novel state called "Calmness" is introduced in model (1) which means rumor Spreaders may go through a calm period before becoming Removes. In the early stage of an emergency, if the mainstream media has low credibility, then some Spreaders may first tend to believe the information told by acquaintances rather than the mainstream media. However, when acquaintances know that the previous information is fabricated and tell Spreaders, then Spreaders will calm down and may gradually reduce the spread of rumors. Spreaders confirm that the information is indeed fabricated afterwards and then stop spreading rumors.

Lemma 1. *The positive invariant set of system (1) is defined by*

$$\Omega = \left\{(I,S,C,R) \in \mathbb{R}_+^4 : 0 \leq I+S+C+R \leq 1\right\}, \quad (3)$$

with initial conditions (2).

Lemma 2. *If $I(0) \geq 0$, $S(0) \geq 0$, $C(0) \geq 0$, and $R(0) \geq 0$, the solutions $I(t)$, $S(t)$, $C(t)$, and $R(t)$ of system (1) with the initial conditions (2) are positive for all $t \geq 0$.*

Proof. If $I(0) \geq 0$, according to the first equation of system (1), one has

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \pi - \frac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)} - (\beta+\mu)I(t). \quad (4)$$

It can be rewritten as

$$\begin{aligned} &\frac{\mathrm{d}I(t)}{\mathrm{d}t}\exp\left\{\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\} \\ &\quad + I(t)\left(\frac{\gamma\langle k\rangle S(t-\tau)}{1+\alpha S(t-\tau)}+\beta+\mu\right)\exp\left\{\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\} \\ &\quad = \pi \times \exp\left\{\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\}. \end{aligned} \quad (5)$$

Thus,

$$\begin{aligned} &\frac{d}{\mathrm{d}t}\left(I(t)\exp\left\{\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\}\right) \\ &\quad = \pi \times \exp\left\{\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\}. \end{aligned} \quad (6)$$

Then, according to the variation of constant formula,

$$I(t)\exp\left\{\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\}-I(0)$$
$$=\int_0^{t-\tau}\left(\pi\times\ \exp\left\{\int_0^{u}\left(\frac{\gamma\langle k\rangle S(\lambda)}{1+\alpha S(\lambda)}+\beta+\mu\right)d\lambda\right\}\right)\mathrm{d}u. \tag{7}$$

Hence,

$$I(t)=I(0)\exp\left\{-\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\}$$
$$+\exp\left\{-\int_0^{t-\tau}\left(\frac{\gamma\langle k\rangle S(u)}{1+\alpha S(u)}+\beta+\mu\right)\mathrm{d}u\right\}$$
$$\times\left(\int_0^{t-\tau}\left(\pi\times\ \exp\left\{\int_0^{u}\left(\frac{\gamma\langle k\rangle S(\lambda)}{1+\alpha S(\lambda)}+\beta+\mu\right)\mathrm{d}\lambda\right\}\right)\mathrm{d}u\right)$$
$$>0. \tag{8}$$

Similarly, one can prove that $S(t)>0$, $C(t)>0$, and $R(t)>0$. So, the solutions $I(t)$, $S(t)$, $C(t)$, and $R(t)$ of system (1) with the initial condition (2) are positive for all $t>0$. □

3. Dynamic Analysis of the ISCR Model

In this section, the equilibria and the basic reproduction number of system (1) are calculated via utilizing the next generation matrix method [29]. Then, the stability of equilibria is discussed by using the Routh–Hurwitz criterion [30] and LaSalle's invariance principle [31].

3.1. Equilibria of Model and the Basic Reproduction Number. Let the right side of system (1) be zero, and one has

$$\begin{cases}\pi-\dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)}-(\beta+\mu)I(t)=0,\\[2ex] \dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)}-(\delta+\theta+\mu)S(t)=0,\\[2ex] \delta S(t)-(\eta+\mu)C(t)=0,\\[2ex] \beta I(t)+\theta S(t)+\eta C(t)-\mu R(t)=0.\end{cases} \tag{9}$$

Let $S(t)=0$ in system (9), and one can easily get the rumor-free equilibrium of system (1) by the following:

$$E_0=\left(I_0,0,0,R_0\right)=\left(\frac{\pi}{\beta+\mu},0,0,1-\frac{\pi}{\beta+\mu}\right). \tag{10}$$

To facilitate the calculation of the basic reproduction number of system (9), let $x=(S,I,C,R)^T$; then, system (9) can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\mathcal{F}(x)-\mathcal{V}(x), \tag{11}$$

where

$$\mathcal{F}(x)=\begin{pmatrix}\dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)}\\ 0\\ 0\\ 0\end{pmatrix},$$
$$\mathcal{V}(x)=\begin{pmatrix}(\delta+\theta+\mu)S(t)\\[1ex] \dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)}+(\beta+\mu)I(t)-\pi\\[1ex] (\eta+\mu)C(t)-\delta S(t)\\[1ex] \mu R(t)-\beta I(t)-\theta S(t)-\eta C(t)\end{pmatrix}. \tag{12}$$

Then,

$$F=D\mathcal{F}\left(E_0\right)=\begin{pmatrix}\gamma\langle k\rangle I_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$
$$V=D\mathcal{V}\left(E_0\right)=\begin{pmatrix}\delta+\theta+\mu & 0 & 0 & 0\\ \gamma\langle k\rangle I_0 & \beta+\mu & 0 & 0\\ -\delta & 0 & \eta+\mu & 0\\ -\theta & -\beta & -\eta & \mu\end{pmatrix}. \tag{13}$$

By calculation, $\mathbf{R}_0$ is obtained as follows:

$$\mathbf{R}_0=\frac{\gamma\langle k\rangle I_0}{\delta+\theta+\mu}=\frac{\gamma\langle k\rangle\pi}{(\delta+\theta+\mu)(\beta+\mu)}. \tag{14}$$

Assume that rumor-prevailing equilibrium $E^*=(I^*,S^*,C^*,R^*)$ is a solution of system (1), that is,

$$\begin{cases}\pi-\dfrac{\gamma\langle k\rangle I^*S^*}{1+\alpha S^*}-(\beta+\mu)I^*=0,\\[2ex] \dfrac{\gamma\langle k\rangle I^*S^*}{1+\alpha S^*}-(\delta+\theta+\mu)S^*=0,\\[2ex] \delta S^*-(\eta+\mu)C^*=0,\\[2ex] \beta I^*+\theta S^*+\eta C^*-\mu R^*=0.\end{cases} \tag{15}$$

From (15), one can get

$$\begin{aligned} I^* &= \frac{\delta+\theta+\mu+\alpha\pi}{\gamma\langle k\rangle+\alpha(\beta+\mu)},\\ S^* &= \frac{\gamma\langle k\rangle\pi-(\delta+\theta+\mu)(\beta+\mu)}{(\delta+\theta+\mu)[\gamma\langle k\rangle+\alpha(\beta+\mu)]}=\frac{\beta+\mu}{\gamma\langle k\rangle+\alpha(\beta+\mu)}(\mathbf{R}_0-1),\\ C^* &= \frac{\delta}{\eta+\mu}S^*=\frac{\delta(\beta+\mu)}{(\eta+\mu)(\gamma\langle k\rangle+\alpha(\beta+\mu))}(\mathbf{R}_0-1), R^*=\frac{\beta I^*+\theta S^*+\eta C^*}{\mu}. \end{aligned} \tag{16}$$

Obviously, rumor-prevailing equilibrium $E^*=(I^*, S^*, C^*, R^*)$ of system (1) exists if $\mathbf{R}_0>1$, and $I^*>0$, $S^*>0$, $C^*>0$, $R^*>0$.

Remark 2. According to the expression of $\mathbf{R}_0$, the basic reproduction number is independent of time delay, that is to say, the time delay τ is only related to the spread time of rumors and will not affect the spread scale of rumors.

3.2. Stability Analysis

Theorem 1. *Rumor-free equilibrium $E_0=((\pi/\beta+\mu), 0, 0, 1-(\pi/\beta+\mu))$ is locally asymptotically stable for all $\tau\geq 0$ if $\mathbf{R}_0<1$.*

Proof. To simplify the calculation, let $x(t)=I(t)-\pi/(\beta+\mu)$, $y(t)=S(t)$, $z(t)=C(t)$, and $w(t)=R(t)-(1-\pi/(\beta+\mu))$, and the linearized system of (1) is as follows:

$$\begin{cases} \dfrac{dx(t)}{dt}=-\dfrac{\gamma\langle k\rangle\pi}{\beta+\mu}y(t-\tau)-(\beta+\mu)x(t),\\ \dfrac{dy(t)}{dt}=\dfrac{\gamma\langle k\rangle\pi}{\beta+\mu}y(t-\tau)-(\delta+\theta+\mu)y(t),\\ \dfrac{dz(t)}{dt}=\delta y(t)-(\eta+\mu)z(t),\\ \dfrac{dw(t)}{dt}=\beta x(t)+\theta y(t)+\eta z(t)-\mu w(t). \end{cases} \tag{17}$$

The characteristic equation of system (17) is

$$\begin{vmatrix} \lambda+\beta+\mu & \dfrac{\gamma\langle k\rangle\pi}{\beta+\mu}e^{-\lambda\tau} & 0 & 0\\ 0 & \lambda-\dfrac{\gamma\langle k\rangle\pi}{\beta+\mu}e^{-\lambda\tau}+\delta+\theta+\mu & 0 & 0\\ 0 & -\delta & \lambda+\eta+\mu & 0\\ -\beta & -\theta & -\eta & \lambda+\mu \end{vmatrix}=0. \tag{18}$$

Then,

$$(\lambda+\mu)(\lambda+\beta+\mu)(\lambda+\eta+\mu)\left[\lambda+(\delta+\theta+\mu)\left(1-\mathbf{R}_0e^{-\lambda\tau}\right)\right]=0. \tag{19}$$

If $\tau=0$, the characteristic equation of system (19) is

$$(\lambda+\mu)(\lambda+\beta+\mu)(\lambda+\eta+\mu)\left[\lambda+(\delta+\theta+\mu)(1-\mathbf{R}_0)\right]=0. \tag{20}$$

One has

$$\begin{aligned} \lambda_1 &= -(\beta+\mu)<0,\\ \lambda_2 &= -(\eta+\mu)\\ \lambda_4 &= -\mu<0,\\ \lambda_3 &= (\delta+\theta+\mu)(\mathbf{R}_0-1). \end{aligned} \tag{21}$$

If $\mathbf{R}_0<1$, in light of the Routh–Hurwitz criterion [30], E_0 is locally asymptotically stable for $\tau=0$.

Now, if $\tau>0$, one only needs to consider the following formula:

$$\lambda+(\delta+\theta+\mu)\left(1-\mathbf{R}_0e^{-\lambda\tau}\right)=0. \tag{22}$$

Assume that (22) has a purely imaginary root $\lambda=i\omega$, with $\omega>0$. Then, separating real and imaginary parts gives

$$\begin{cases} (\delta+\theta+\mu)\mathbf{R}_0\cos\omega\tau=\delta+\theta+\mu,\\ (\delta+\theta+\mu)\mathbf{R}_0\sin\omega\tau=-\omega. \end{cases} \tag{23}$$

Equation (23) is squared and then added to obtain

$$(\delta+\theta+\mu)^2\mathbf{R}_0^2=(\delta+\theta+\mu)^2+\omega^2. \tag{24}$$

Then,

$$\omega^2=(\delta+\theta+\mu)^2\left(\mathbf{R}_0^2-1\right). \tag{25}$$

Therefore, equation (22) has no purely imaginary root if $\mathbf{R}_0<1$. Hence, rumor-free equilibrium E_0 is locally asymptotically stable for any $\tau\geq 0$ if $\mathbf{R}_0<1$. □

Remark 3. If $\mathbf{R}_0<1$, rumor-free equilibrium E_0 is unstable for any $\tau\geq 0$.

Remark 4 (see [30]). Consider the following linear delay differential equation:

$$\dot{x}(t)=Ax(t)+Bx(t-r), \tag{26}$$

where $x\in R^n$, $A,B\in R^{n\times n}$, and $r>0$. The solution of equation (26) is $e^{\lambda c}(c\in R^n, c\neq 0)$. Also, the characteristic equation of system (26) is $h(\lambda)=\lambda-A-Be^{-\lambda r}=0$.

Theorem 2. *Rumor-prevailing equilibrium* $E^* = (I^*, S^*, C^*, R^*)$ *of system (1) is locally asymptotically stable for all* $\tau \geq 0$ *if* $\mathbf{R}_0 > 1$.

Proof. Let $x(t) = I(t) - I^*$, $y(t) = S(t) - S^*$, $z(t) = C(t) - C^*$, and $w(t) = R(t) - R^*$, and the linearized system of (1) takes the following form:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -(\beta+\mu)x(t) - \dfrac{\gamma\langle k\rangle S^*}{1+\alpha S^*}x(t) - \dfrac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}y(t-\tau), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = -(\delta+\theta+\mu)y(t) + \dfrac{\gamma\langle k\rangle S^*}{1+\alpha S^*}x(t) + \dfrac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}y(t-\tau), \\ \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = \delta y(t) - (\eta+\mu)z(t), \\ \dfrac{\mathrm{d}w(t)}{\mathrm{d}t} = \beta x(t) + \theta y(t) + \eta z(t) - \mu w(t). \end{cases} \tag{27}$$

The characteristic equation of system (27) is

$$\begin{vmatrix} \lambda+\beta+\mu+\dfrac{\gamma\langle k\rangle S^*}{1+\alpha S^*} & \dfrac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}e^{-\lambda\tau} & 0 & 0 \\ -\dfrac{\gamma\langle k\rangle S^*}{1+\alpha S^*} & \lambda-\dfrac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}e^{-\lambda\tau}+\delta+\theta+\mu & 0 & 0 \\ 0 & -\delta & \lambda+\eta+\mu & 0 \\ -\beta & -\theta & -\eta & \lambda+\mu \end{vmatrix} = 0. \tag{28}$$

Then,

$$(\lambda+\eta+\mu)(\lambda+\mu)\left[\lambda^2 + p_1\lambda + p_0 - (q_1\lambda + q_0)e^{-\lambda\tau}\right] = 0, \tag{29}$$

where

$$p_1 = \delta+\theta+\mu+\beta+\mu+\frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}, q_1 = \frac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2},$$
$$p_0 = (\delta+\theta+\mu)\left(\beta+\mu+\frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}\right), q_0 = \frac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}(\beta+\mu). \tag{30}$$

If $\tau = 0$, equation (29) can be rewritten as

$$(\lambda+\eta+\mu)(\lambda+\mu)\left[\lambda^2 + (p_1-q_1)\lambda + p_0 - q_0\right] = 0. \tag{31}$$

Obviously, $\lambda_1 = -(\eta+\mu) < 0$, $\lambda_4 = -\mu < 0$. Furthermore, one only need to consider the following equation:

$$\lambda^2 + (p_1-q_1)\lambda + p_0 - q_0 = 0, \tag{32}$$

where

$$
\begin{aligned}
p_1 - q_1 &= \delta + \theta + \mu + \beta + \mu + \frac{\gamma\langle k\rangle S^*}{1+\alpha S^*} - \frac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2} \\
&= \delta + \theta + \mu + \beta + \mu + \frac{\gamma\langle k\rangle}{1+\alpha S^*}\left(S^* - \frac{I^*}{1+\alpha S^*}\right) \\
&= \delta + \theta + \mu + \beta + \mu + \frac{(\delta+\theta+\mu)[\gamma\langle k\rangle + \alpha(\beta+\mu)]}{\delta+\theta+\mu+\alpha\pi} \\
&\quad \times \left[\frac{\beta+\mu}{\gamma\langle k\rangle + \alpha(\beta+\mu)}(\mathbf{R}_0 - 1) - \frac{\delta+\theta+\mu}{\gamma\langle k\rangle}\right] \\
&= \frac{(\delta+\theta+\mu)(\beta+\mu)}{\delta+\theta+\mu+\alpha\pi}(\mathbf{R}_0 - 1) + \frac{\alpha(\delta+\theta+\mu)^2(\beta+\mu)(\mathbf{R}_0-1)}{\gamma\langle k\rangle(\delta+\theta+\mu+\alpha\pi)} + \beta + \mu.
\end{aligned} \tag{33}
$$

If $\mathbf{R}_0 > 1$, $p_1 - q_1 > 0$. Then,

$$
\begin{aligned}
p_0 - q_0 &= (\delta+\theta+\mu)\left(\beta + \mu + \frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}\right) - \frac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}(\beta+\mu) \\
&= \frac{\gamma\langle k\rangle}{1+\alpha S^*}\left[S^*(\delta+\theta+\mu) - \frac{I^*}{1+\alpha S^*}(\beta+\mu)\right] + (\beta+\mu)(\delta+\theta+\mu) \\
&= \frac{(\delta+\theta+\mu)[\gamma\langle k\rangle + \alpha(\beta+\mu)]}{\delta+\theta+\mu+\alpha\pi}\left[\frac{(\beta+\mu)(\delta+\theta+\mu)}{\gamma\langle k\rangle + \alpha(\beta+\mu)}(\mathbf{R}_0 - 1) - \frac{\delta+\theta+\mu}{\gamma\langle k\rangle}(\beta+\mu)\right] \\
&\quad + (\beta+\mu)(\delta+\theta+\mu) = \frac{(\delta+\theta+\mu)^2(\beta+\mu)}{\delta+\theta+\mu+\alpha\pi}(\mathbf{R}_0 - 1) + \frac{\alpha(\beta+\mu)(\delta+\theta+\mu)^2}{\gamma\langle k\rangle(\delta+\theta+\mu+\alpha\pi)}(\mathbf{R}_0 - 1).
\end{aligned} \tag{34}
$$

If $\mathbf{R}_0 > 1$, $p_0 - q_0 > 0$. Then, one has $\lambda_2 < 0$, $\lambda_3 < 0$. Therefore, rumor-prevailing equilibrium E^* is locally asymptotically stable for $\tau = 0$ if $\mathbf{R}_0 > 1$.

Let

$$F(\lambda) = \lambda^2 + p_1\lambda + p_0 - (q_1\lambda + q_0)e^{-\lambda\tau}. \tag{35}$$

Now, if $\tau > 0$, assume that (35) has a purely imaginary root $\lambda = i\omega$, with $\omega > 0$. Then,

$$-\omega^2 + p_1 i\omega + p_0 - (q_1 i\omega + q_0)(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{36}$$

Separating real and imaginary parts gives

$$
\begin{cases}
p_0 - \omega^2 = q_1\omega\sin\omega\tau + q_0\cos\omega\tau, \\
p_1\omega^2 = q_1\omega\cos\omega\tau - q_0\sin\omega\tau.
\end{cases} \tag{37}
$$

Equation (37) is squared and then added to obtain

$$\omega^4 + \left(p_1^2 - q_1^2 - 2p_0\right)\omega^2 + p_0^2 - q_0^2 = 0. \tag{38}$$

Let $\rho = \omega^2$, and equation (38) can be rewritten as

$$\rho^2 + \left(p_1^2 - q_1^2 - 2p_0\right)\rho + p_0^2 - q_0^2 = 0, \tag{39}$$

where

$$\begin{aligned}p_1^2-q_1^2-2p_0&=\left(\delta+\theta+\mu+\beta+\mu+\frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}\right)^2-\left(\frac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}\right)^2\\&\quad-2(\delta+\theta+\mu)\left(\beta+\mu+\frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}\right)\\&=(\delta+\theta+\mu)^2+(\beta+\mu)^2+2(\beta+\mu)\frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}\\&\quad+\frac{\gamma^2\langle k\rangle^2}{(1+\alpha S^*)^2}\left[(S^*)^2-\left(\frac{I^*}{1+\alpha S^*}\right)^2\right].\end{aligned}\tag{40}$$

Let

$$\begin{aligned}f&=(\delta+\theta+\mu)^2+\frac{\gamma^2\langle k\rangle^2}{(1+\alpha S^*)^2}\left[(S^*)^2-\left(\frac{I^*}{1+\alpha S^*}\right)^2\right]\\&=(\delta+\theta+\mu)^2+\frac{(\beta+\mu)^2(\delta+\theta+\mu)^2}{(\delta+\theta+\mu+\alpha\pi)^2}(\mathbf{R}_0-1)^2-\frac{[\gamma\langle k\rangle+\alpha(\beta+\mu)]^2(\delta+\theta+\mu)^4}{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2}\\&=(\delta+\theta+\mu)^2\left(\frac{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2-[\gamma\langle k\rangle+\alpha(\beta+\mu)]^2(\delta+\theta+\mu)^2}{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2}\right)\\&\quad+\frac{(\beta+\mu)^2(\delta+\theta+\mu)^2}{(\delta+\theta+\mu+\alpha\pi)^2}(\mathbf{R}_0-1)^2=(\delta+\theta+\mu)^4\left(\frac{2\alpha\gamma\langle k\rangle(\beta+\mu)(\mathbf{R}_0-1)+\alpha^2(\beta+\mu)^2\left(\mathbf{R}_0^2-1\right)}{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2}\right)\\&\quad+\frac{(\beta+\mu)^2(\delta+\theta+\mu)^2}{(\delta+\theta+\mu+\alpha\pi)^2}(\mathbf{R}_0-1)^2.\end{aligned}\tag{41}$$

If $\mathbf{R}_0>1,\ p_1^2-q_1^2-2p_0>0$. Then,

$$\begin{aligned}p_0^2-q_0^2&=(\delta+\theta+\mu)^2\left(\beta+\mu+\frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}\right)^2-\left(\frac{\gamma\langle k\rangle I^*}{(1+\alpha S^*)^2}\right)^2(\beta+\mu)^2\\&=(\delta+\theta+\mu)^2(\beta+\mu)^2+2(\delta+\theta+\mu)(\beta+\mu)\frac{\gamma\langle k\rangle S^*}{1+\alpha S^*}\\&\quad+\frac{\gamma^2\langle k\rangle^2}{(1+\alpha S^*)^2}\left[(S^*)^2(\delta+\theta+\mu)^2-\left(\frac{I^*}{1+\alpha S^*}\right)^2(\beta+\mu)^2\right].\end{aligned}\tag{42}$$

Let

$$\begin{aligned}
m &= (\delta+\theta+\mu)^2(\beta+\mu)^2 + \frac{\gamma^2\langle k\rangle^2}{(1+\alpha S^*)^2}\left[(S^*)^2(\delta+\theta+\mu)^2 - \left(\frac{I^*}{1+\alpha S^*}\right)^2(\beta+\mu)^2\right] \\
&= (\delta+\theta+\mu)^2(\beta+\mu)^2 + \frac{[\gamma\langle k\rangle + \alpha(\beta+\mu)]^2(\delta+\theta+\mu)^2}{(\delta+\theta+\mu+\alpha\pi)^2} \\
&\quad\times\left[\frac{(\delta+\theta+\mu)^2(\beta+\mu)^2}{[\gamma\langle k\rangle+\alpha(\beta+\mu)]^2}(\mathbf{R}_0-1)^2 - \frac{(\delta+\theta+\mu)^2(\beta+\mu)^2}{\gamma^2\langle k\rangle^2}\right] \\
&= (\delta+\theta+\mu)^2(\beta+\mu)^2 - \frac{(\delta+\theta+\mu)^4(\beta+\mu)^2[\gamma\langle k\rangle+\alpha(\beta+\mu)]^2}{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2} \\
&\quad+\frac{(\delta+\theta+\mu)^4(\beta+\mu)^2}{(\delta+\theta+\mu+\alpha\pi)^2}(\mathbf{R}_0-1)^2 \\
&= \frac{(\delta+\theta+\mu)^4(\beta+\mu)^2}{(\delta+\theta+\mu+\alpha\pi)^2}(\mathbf{R}_0-1)^2 + (\delta+\theta+\mu)^2\left(\frac{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2(\beta+\mu)^2}{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2}\right. \\
&\quad\left.-\frac{(\delta+\theta+\mu)^2(\beta+\mu)^2[\gamma\langle k\rangle+\alpha(\beta+\mu)]^2}{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2}\right) \\
&= (\delta+\theta+\mu)^4\left(\frac{2\alpha\gamma\langle k\rangle(\beta+\mu)^3(\mathbf{R}_0-1)+\alpha^2(\beta+\mu)^4(\mathbf{R}_0^2-1)}{\gamma^2\langle k\rangle^2(\delta+\theta+\mu+\alpha\pi)^2}\right) \\
&\quad+\frac{(\delta+\theta+\mu)^4(\beta+\mu)^2}{(\delta+\theta+\mu+\alpha\pi)^2}(\mathbf{R}_0-1)^2.
\end{aligned} \tag{43}$$

If $\mathbf{R}_0 > 1$, $p_0^2 - q_0^2 > 0$. Then, (35) has no purely imaginary roots if $\mathbf{R}_0 > 1$. Hence, rumor-prevailing equilibrium E^* is locally asymptotically stable for any $\tau \geq 0$ if $\mathbf{R}_0 > 1$. □

Theorem 3. *If $\mathbf{R}_0 < 1$, rumor-free equilibrium $E_0 = (\pi/(\beta+\mu), 0, 0, 1-\pi/(\beta+\mu))$ is globally asymptotically stable for all $\tau \geq 0$.*

Proof. Construct the following Lyapunov function:

$$V(t) = S(t) + \int_{t-\tau}^{t} \frac{\gamma\langle k\rangle I_0 S(u)}{1+\alpha S(u)}\,\mathrm{d}u. \tag{44}$$

Differentiate $V(t)$ along the solution E_0 of system (1) as follows:

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \frac{\mathrm{d}S(t)}{\mathrm{d}t} + \frac{\gamma\langle k\rangle I_0 S(t)}{1+\alpha S(t)} - \frac{\gamma\langle k\rangle I_0 S(t-\tau)}{1+\alpha S(t-\tau)} \\
&= \frac{\gamma\langle k\rangle I(t) S(t-\tau)}{1+\alpha S(t-\tau)} - (\delta+\theta+\mu)S(t) + \frac{\gamma\langle k\rangle I_0 S(t)}{1+\alpha S(t)} - \frac{\gamma\langle k\rangle I_0 S(t-\tau)}{1+\alpha S(t-\tau)} \\
&\leq \langle k\rangle I_0 S(t) - (\delta+\theta+\mu)S(t) = \frac{\gamma\langle k\rangle \pi S(t)}{\beta+\mu} - (\delta+\theta+\mu)S(t) \\
&= (\delta+\theta+\mu)S(t)(\mathbf{R}_0-1).
\end{aligned} \tag{45}$$

Therefore, $\mathbf{R}_0 < 1$ ensures that $\mathrm{d}V(t)/\mathrm{d}t \leq 0$ and $\mathrm{d}V(t)/\mathrm{d}t = 0$ if and only if $I(t) = I_0$, $S(t) = 0$, $C(t) = 0$, $R(t) = R_0$. Adopting LaSalle's invariance principle [31], we have $\lim_{t\longrightarrow\infty} I(t) = I_0$, $\lim_{t\longrightarrow\infty} S(t) = 0$, $\lim_{t\longrightarrow\infty} C(t) = 0$ and $\lim_{t\longrightarrow\infty} R(t) = R_0$, that is, E_0 is globally asymptotically stable for all $\tau \geq 0$ if $\mathbf{R}_0 < 1$. □

Remark 5. Rumor-free equilibrium is globally asymptotically stable for all $\tau \geq 0$ if $\mathbf{R}_0 < 1$, that is to say, rumor will gradually die out with the growth of time, and the density of Spreaders will gradually approach zero.

Theorem 4. *If $\mathbf{R}_0 > 1$, rumor-prevailing equilibrium $E^* = (I^*, S^*, C^*, R^*)$ is globally asymptotically stable for all $\tau \geq 0$.*

Proof. Define the Lyapunov function as

$$W(t) = W_1(t) + W_2(t) + \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*} W_3(t), \tag{46}$$

where

$$\begin{aligned}
W_1(t) &= I(t) - I^* - I^* \ln\left(\frac{I(t)}{I^*}\right),\\
W_2(t) &= S(t) - S^* - S^* \ln\left(\frac{S(t)}{S^*}\right),\\
W_3(t) &= \int_{t-\tau}^{t} \left[\frac{S(u)}{S^*} - 1 - \ln\left(\frac{S(u)}{S^*}\right)\right] \mathrm{d}u, \quad x - 1 - \ln x > 0\,(x > 0).
\end{aligned} \tag{47}$$

Differentiate $W_1(t)$ along the solutions of system (1) as follows:

$$\begin{aligned}
\frac{\mathrm{d}W_1(t)}{\mathrm{d}t} &= \left(1 - \frac{I^*}{I(t)}\right)\frac{\mathrm{d}I(t)}{\mathrm{d}t}\\
&= \left(1 - \frac{I^*}{I(t)}\right)\left[\pi - \frac{\gamma\langle k\rangle I(t) S(t-\tau)}{1+\alpha S(t-\tau)} - (\beta+\mu) I(t)\right]\\
&= \left(1 - \frac{I^*}{I(t)}\right)\left(\frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*} + (\beta+\mu) I^* - \frac{\gamma\langle k\rangle I(t) S(t-\tau)}{1+\alpha S(t-\tau)} - (\beta+\mu) I(t)\right)\\
&= -(\beta+\mu)\frac{(I(t)-I^*)^2}{I(t)} + \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*}\left(1 - \frac{(1+\alpha S^*) I(t) S(t-\tau)}{I^* S^* [1+\alpha S(t-\tau)]}\right)\\
&\quad \times \left(1 - \frac{I^*}{I(t)}\right).
\end{aligned} \tag{48}$$

Differentiating $W_2(t)$ along the solutions of system (1), one has

$$\begin{aligned}
\frac{\mathrm{d}W_2(t)}{\mathrm{d}t} &= \left(1 - \frac{S^*}{S(t)}\right)\frac{\mathrm{d}S(t)}{\mathrm{d}t}\\
&= \left(1 - \frac{S^*}{S(t)}\right)\left[\frac{\gamma\langle k\rangle I(t) S(t-\tau)}{1+\alpha S(t-\tau)} - (\delta+\theta+\mu) S(t)\right]\\
&= \left(1 - \frac{S^*}{S(t)}\right)\left(\frac{\gamma\langle k\rangle I(t) S(t-\tau)}{1+\alpha S(t-\tau)} - \frac{\gamma\langle k\rangle I^* S(t)}{S^*} \times \frac{S^*}{1+\alpha S^*}\right)\\
&= \left(1 - \frac{S^*}{S(t)}\right)\frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*}\left[\frac{(1+\alpha S^*) I(t) S(t-\tau)}{I^* S^* [1+\alpha S(t-\tau)]} - \frac{S(t)}{S^*}\right]\\
&= \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*}\left[\frac{(1+\alpha S^*) I(t) S(t-\tau)}{I^* S^* [1+\alpha S(t-\tau)]} - \frac{S^*}{S(t)}\frac{(1+\alpha S^*) I(t) S(t-\tau)}{I^* S^* [1+\alpha S(t-\tau)]}\right.\\
&\quad \left. - \frac{S(t)}{S^*} + 1\right].
\end{aligned} \tag{49}$$

Differentiate $W_3(t)$ along the solutions of system (1) as follows:

$$\begin{aligned}\frac{dW_3(t)}{dt} &= \frac{S(t)}{S^*} - 1 - \ln\left(\frac{S(t)}{S^*}\right) - \frac{S(t-\tau)}{S^*} + 1 + \ln\left(\frac{S(t-\tau)}{S^*}\right)\\ &= \frac{S(t)}{S^*} - \frac{S(t-\tau)}{S^*} + \ln\left(\frac{S(t-\tau)}{S(t)}\right).\end{aligned} \quad (50)$$

Hence,

$$\begin{aligned}\frac{dW(t)}{dt} &= \frac{dW_1(t)}{dt} + \frac{dW_2(t)}{dt} + \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*}\frac{dW_3(t)}{dt}\\
&= -(\beta+\mu)\frac{(I(t)-I^*)^2}{I(t)} + \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*} \times \left[\left(1 - \frac{(1+\alpha S^*)I(t)S(t-\tau)}{I^*S^*[1+\alpha S(t-\tau)]}\right)\right.\\
&\quad \times\left(1-\frac{I^*}{I(t)}\right) + \frac{(1+\alpha S^*)I(t)S(t-\tau)}{I^*S^*[1+\alpha S(t-\tau)]} - \frac{S^*}{S(t)} \times \frac{(1+\alpha S^*)I(t)S(t-\tau)}{I^*S^*[1+\alpha S(t-\tau)]}\\
&\quad \left. - \frac{S(t)}{S^*} + 1 + \frac{S(t)}{S^*} - \frac{S(t-\tau)}{S^*} + \ln\left(\frac{S(t-\tau)}{S(t)}\right)\right]\\
&= -(\beta+\mu)\frac{(I(t)-I^*)^2}{I(t)} + \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*}\left[2 - \frac{I^*}{I(t)} + \frac{(1+\alpha S^*)S(t-\tau)}{S^*[1+\alpha S(t-\tau)]}\right.\\
&\quad \left. - \frac{S(t-\tau)}{S^*} - \frac{S^*(1+\alpha S^*)I(t)S(t-\tau)}{S(t)I^*S^*[1+\alpha S(t-\tau)]} + \ln\left(\frac{S(t-\tau)}{S(t)}\right)\right]\\
&= -(\beta+\mu)\frac{(I(t)-I^*)^2}{I(t)} - \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*}\left[\frac{I^*}{I(t)} - 1 - \ln\left(\frac{I^*}{I(t)}\right)\right.\\
&\quad + \frac{S^*(1+\alpha S^*)I(t)S(t-\tau)}{S(t)I^*S^*[1+\alpha S(t-\tau)]} - 1 - \ln\left(\frac{S^*(1+\alpha S^*)I(t)S(t-\tau)}{S(t)I^*S^*[1+\alpha S(t-\tau)]}\right)\\
&\quad \left. + \frac{1+\alpha S(t-\tau)}{1+\alpha S^*} - 1 - \ln\left(\frac{1+\alpha S(t-\tau)}{1+\alpha S^*}\right)\right]\\
&\quad - \frac{\gamma\langle k\rangle I^* S^*}{1+\alpha S^*}\left[1 - \frac{1+\alpha S(t-\tau)}{1+\alpha S^*} - \frac{(1+\alpha S^*)S(t-\tau)}{S^*[1+\alpha S(t-\tau)]} + \frac{S(t-\tau)}{S^*}\right].\end{aligned} \quad (51)$$

Let

$$\begin{aligned}Y &= 1 - \frac{1+\alpha S(t-\tau)}{1+\alpha S^*} - \frac{(1+\alpha S^*)S(t-\tau)}{S^*[1+\alpha S(t-\tau)]} + \frac{S(t-\tau)}{S^*}\\ &= \frac{\alpha[S^* - S(t-\tau)]^2}{S^*(1+\alpha S^*)[1+\alpha S(t-\tau)]} \geq 0.\end{aligned} \quad (52)$$

Because $x - 1 - \ln x > 0\,(x > 0)$,

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= -(\beta+\mu)\frac{(I(t)-I^*)^2}{I(t)} - \frac{\gamma\langle k\rangle I^*S^*}{1+\alpha S^*}\left[\frac{I^*}{I(t)} - 1 - \ln\left(\frac{I^*}{I(t)}\right)\right.\\ &\quad + \frac{S^*(1+\alpha S^*)I(t)S(t-\tau)}{S(t)I^*S^*[1+\alpha S(t-\tau)]} - 1 - \ln\left(\frac{S^*(1+\alpha S^*)I(t)S(t-\tau)}{S(t)I^*S^*[1+\alpha S(t-\tau)]}\right)\\ &\quad \left. + \frac{1+\alpha S(t-\tau)}{1+\alpha S^*} - 1 - \ln\left(\frac{1+\alpha S(t-\tau)}{1+\alpha S^*}\right)\right] - \frac{\gamma\langle k\rangle I^*S^*}{1+\alpha S^*}\times\frac{\alpha[S^*-S(t-\tau)]^2}{S^*(1+\alpha S^*)[1+\alpha S(t-\tau)]}\\ &\geq 0.\end{aligned} \tag{53}$$

Obviously, $\mathrm{d}W(t)/\mathrm{d}t \leq 0$ and $\mathrm{d}W(t)/\mathrm{d}t = 0$ if and only if $I(t) = I^*$, $S(t) = S^*$, $C(t) = C^*$, $R(t) = R^*$. Then, one has $\lim_{t\longrightarrow\infty} I(t) = I^*$, $\lim_{t\longrightarrow\infty} S(t) = S^*$, $\lim_{t\longrightarrow\infty} C(t) = C^*$, and $\lim_{t\longrightarrow\infty} R(t) = R^*$, that is, E^* is globally asymptotically stable for all $\tau \geq 0$ if $\mathbf{R}_0 > 1$. □

Remark 6. If $\mathbf{R}_0 > 1$ for all $\tau \geq 0$, rumor-prevailing equilibrium is globally asymptotically stable, that means rumors will spread steadily over time and will not die out in the end.

Remark 7. For the construction of the Lyapunov function of the delayed model in this paper, we must first ensure the positive definiteness of the function. Secondly, uncertain limit integral functions are added to the Lyapunov function to offset the previous delayed term. Finally, the algebraic method is used to ensure that the derivative of the Lyapunov function is negative. Meanwhile, Holling-type II functional response is added in this paper, which makes the analysis of the delayed model more difficult.

4. Optimal Control

In this section, optimized control strategies are proposed so as to control the spread of rumors and reduce the control cost of social platforms, and the optimal control solution is found by utilizing Pontryagin's minimum principle [32]. Next, a control variable $u(t)$ is introduced to represent the function of control strategy for $S(t)$. Then, an admissible control set is defined:

$$U = \left\{u(t) \in L^2(0,T)\colon\ 0 \leq t \leq T; 0 \leq u(t) \leq 1\right\}. \tag{54}$$

The controlled system can be obtained by

$$\begin{cases}\dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \pi - \dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)} - (\beta+\mu)I(t),\\[2mm] \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)} - (\delta+\theta+\mu)S(t) - u(t)S(t),\\[2mm] \dfrac{\mathrm{d}C(t)}{\mathrm{d}t} = \delta S(t) - (\eta+\mu)C(t),\\[2mm] \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \beta I(t) + \theta S(t) + u(t)S(t) + \eta C(t) - \mu R(t),\end{cases} \tag{55}$$

with initial conditions (2). Then, an objective function is considered as

$$J(t) = \int_0^t \left[S(t) + \frac{A}{2}u^2(t)\right]\mathrm{d}t, \tag{56}$$

where $A\,(A > 0)$ is a weight coefficient to keep the density of $S(t)$ in balance and control the cost of $u(t)$.

Lemma 3 (see [33]). *The controlled system (56) with any initial conditions has a unique solution.Proof.* The controlled system (56) can be rewritten as

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = BX(t) + F\left(X(t), X_\tau(t)\right) + C(u, X(t)), \tag{57}$$

where

$$X(t)=\begin{pmatrix} I(t)\\ S(t)\\ C(t)\\ R(t)\end{pmatrix},$$

$$B=\begin{pmatrix} -(\beta+\mu) & 0 & 0 & 0\\ 0 & -(\delta+\theta+\mu) & 0 & 0\\ 0 & \delta & -(\eta+\mu) & 0\\ \beta & \theta & \eta & -\mu \end{pmatrix},$$

$$F(X(t),X_\tau(t))=\begin{pmatrix} \pi-\dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)}\\ \dfrac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)}\\ 0\\ 0\end{pmatrix},$$

$$C(u,X(t))=\begin{pmatrix} 0\\ -u(t)S(t)\\ 0\\ u(t)S(t)\end{pmatrix}. \tag{58}$$

and $X_\tau(t)=X(t-\tau)$. System (57) is a nonlinear system with a bounded coefficient. Set

$$G(X(t),X_\tau(t))=BX(t)+F(X(t),X_\tau(t)). \tag{59}$$

A simple calculation shows that

$$\begin{aligned}\left|F(X_1(t),(X_1)_\tau(t))-F(X_2(t),(X_2)_\tau(t))\right| &\le M_1\left|X_1(t)-X_2(t)\right|\\ &\quad+M_2\left|(X_1)_\tau(t))-(X_2)_\tau(t))\right|,\end{aligned} \tag{60}$$

where M_1 and M_2 are some positive constants, independent of state variables $I(t)$, $S(t)$, $C(t)$, and $R(t)\le N(t)$, and

$$\begin{aligned}\left|X_1(t)-X_2(t)\right| &= \left|I_1(t)-I_2(t)\right|+\left|S_1(t)-S_2(t)\right|+\left|C_1(t)-C_2(t)\right|+\left|R_1(t)-R_2(t)\right|,\\ \left|(X_1)_\tau(t))-(X_2)_\tau(t))\right| &= \left|(I_1)_\tau(t))-(I_2)_\tau(t))\right|+\left|(S_1)_\tau(t))-(S_2)_\tau(t))\right|\\ &\quad+\left|(C_1)_\tau(t))-(C_2)_\tau(t))\right|+\left|(R_1)_\tau(t))-(R_2)_\tau(t))\right|.\end{aligned} \tag{61}$$

Here $(I_i)_\tau(t)=I_i(t-\tau)$, $(S_i)_\tau(t)=S_i(t-\tau)$, $(C_i)_\tau(t)=C_i(t-\tau)$, and $(R_i)_\tau(t)=R_i(t-\tau)$, for $i=1,2$. Therefore, it is easy to show that

$$\begin{aligned}\left|G(X_1(t),(X_1)_\tau(t))-G(X_2(t),(X_2)_\tau(t))\right| &\le L\left|X_1(t)-X_2(t)\right|\\ &\quad+\left|(X_1)_\tau(t))-(X_2)_\tau(t))\right|\end{aligned} \tag{62}$$

where $L=\max\{M_1,M_2,\|B\|\}<\infty$. Thus, it follows that the function G is uniformly Lipschitz continuous. The solution of system (55) exists from (62). And the solution of system (55) takes into account the constraints on the controls $u(t)$ and the restrictions on the non-negativeness of the state variables.

In order to find an optimal solution, first we find the Lagrangian function and Hamiltonian function for the optimal control system (55). The Lagrangian function of the problem is taken as

$$L(S(t),u(t))=S(t)+\frac{A}{2}u^2(t), \tag{63}$$

and we define the Hamiltonian function $H(t)$ as

$$\begin{aligned} H(t) &= L(S(t),u(t)) + \lambda(t) f(I(t),S(t),C(t),R(t)) \\ &= S(t) + \frac{A}{2}u(t)^2 + \lambda_1(t)\left(\pi - \frac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)} - (\beta+\mu)I(t)\right) \\ &\quad + \lambda_2(t)\left(\frac{\gamma\langle k\rangle I(t)S(t-\tau)}{1+\alpha S(t-\tau)} - u(t)S(t) - (\delta+\theta+\mu)S(t)\right) + \lambda_3(t)(\delta S(t) \\ &\quad - (\eta+\mu)C(t)) + \lambda_4(t)(\beta I(t) + \theta S(t) + u(t)S(t) + \eta C(t) - \mu R(t)), \end{aligned} \tag{64}$$

where $\lambda_i(t)$, $(i = 1,2,3,4)$, represent the adjoint variables to be determined appropriately. □

Lemma 4. *There exists an optimal control $u^*(t) = (i^*(t), s^*(t), c^*(t), r^*(t))$ such that*

$$J(u^*(t)) = \min(J(u(t))), \tag{65}$$

for system (55) under the initial conditions (2).

Proof. In fact, the following conditions are satisfied. (i) The set of control and corresponding state variables is not empty. (ii) The control space U is convex and closed by definition. (iii) Each right hand side of the state system is continuous and is bounded by a sum of the bounded control and the state. Furthermore, it can be written as a linear function of the control variate $u(t)$ with coefficients depending on time and the state. (iv) $S(t) + (A/2)u^2(t)$ is convex on the control set U and is bounded below. Thus, according to [34], there exists an optimal control $u^*(t)$. This completes the proof. □

Theorem 5. *Let $(i^*(t), s^*(t), c^*(t), r^*(t))$ be the optimal state solutions with associated optimal control variable $u^*(t)$ for the optimal control system (55). Then, adjoint variables $\lambda_i(t)(i = 1,2,3,4)$ satisfy*

$$\begin{cases} \dfrac{d\lambda_1(t)}{dt} = \lambda_1(t) \times \left(\dfrac{\gamma\langle k\rangle s^*(t)}{1+\alpha s^*(t)} + \beta + \mu\right) - \lambda_2(t) \times \dfrac{\gamma\langle k\rangle s^*(t)}{1+\alpha s^*(t)} - \lambda_4(t)\beta, \\ \dfrac{d\lambda_2(t) = -1 + \lambda_2(t)(u(t)+\delta+\theta+\mu) - \lambda_3(t)\delta - \lambda_4(t)(\theta+u)}{dt} \\ +\lambda_2(t)(t+\tau)\dfrac{\gamma\langle k\rangle i^*(t)}{(1+\alpha s^*(t))^2}(\lambda_1(t) - \lambda_2(t)), \\ \dfrac{d\lambda_3(t)}{dt} = \lambda_3(t)(\eta+\mu) - \lambda_4(t)\eta, \\ \dfrac{d\lambda_4(t)}{dt} = \lambda_4(t)\mu, \end{cases} \tag{66}$$

with transversality conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0. \tag{67}$$

Furthermore, the optimal control

$$u^*(t) = \max\left\{\min\left\{\frac{(\lambda_2(t) - \lambda_4(t))s^*(t)}{\sigma}, 1\right\}, 0\right\}, \tag{68}$$

can be found.

Proof. Differentiating the Hamiltonian function (64) with respect to $I(t)$, $S(t)$, $C(t)$, and $R(t)$ and substituting $I(t) = i^*(t)$, $S(t) = s^*(t)$, $S(t-\tau) = s^*(t)$, $C(t) = c^*(t)$, $R(t) = r^*(t)$, and $u(t) = u^*(t)$ into equations, one has

$$\begin{cases} \dfrac{d\lambda_1(t)}{dt} = -\dfrac{\partial H(t)}{\partial I(t)} = \lambda_1(t)\times\left(\dfrac{\gamma\langle k\rangle s^*(t)}{1+\alpha s^*(t)}+\beta+\mu\right) - \lambda_2(t)\times\dfrac{\gamma\langle k\rangle s^*(t)}{1+\alpha s^*(t)} - \lambda_4(t)\beta, \\ \dfrac{d\lambda_2(t)}{dt} = -\dfrac{\partial H(t)}{\partial S(t)} - \lambda_2(t+\tau)\dfrac{\partial H(t)}{\partial S(t-\tau)} = -1+\lambda_2(t)(u(t)+\delta+\theta+\mu) - \lambda_3(t)\delta \\ -\lambda_4(t)(\theta+u) + \lambda_2(t+\tau)\dfrac{\gamma\langle k\rangle i^*(t)}{(1+\alpha s^*(t))^2}(\lambda_1(t)-\lambda_2(t)), \\ \dfrac{d\lambda_3(t)}{dt} = -\dfrac{\partial H(t)}{\partial C(t)} = \lambda_3(t)(\eta+\mu) - \lambda_4(t)\eta, \\ \dfrac{d\lambda_4(t)}{dt} = -\dfrac{\partial H(t)}{\partial R(t)} = \lambda_4(t)\mu. \end{cases} \quad (69)$$

By the optimal conditions, one has

$$\frac{\partial H(t)}{\partial u(t)} = \sigma u^*(t) - \lambda_2(t)s^*(t) + \lambda_4(t)s^*(t) = 0 \Rightarrow u^*(t) = \frac{(\lambda_2(t)-\lambda_4(t))s^*(t)}{\sigma}. \quad (70)$$

Considering the range of control variable and the property of Hamiltonian function, one has

$$u^*(t) = \begin{cases} 0, & \dfrac{(\lambda_2(t)-\lambda_4(t))s^*(t)}{\sigma} \le 0, \\ \dfrac{(\lambda_2(t)-\lambda_4(t))s^*(t)}{\sigma}, & 0 < \dfrac{(\lambda_2(t)-\lambda_4(t))s^*(t)}{\sigma} \le 1, \\ 1, & \dfrac{(\lambda_2(t)-\lambda_4(t))s^*(t)}{\sigma} > 1. \end{cases} \quad (71)$$

So, the optimal control $u^*(t)$ can be obtained as

$$u^*(t) = \max\left\{\min\left\{\frac{(\lambda_2(t)-\lambda_4(t))s^*(t)}{\sigma}, 1\right\}, 0\right\}. \quad (72)$$

□

5. Numerical Simulations

In this section, the theoretical results are verified by some numerical simulations. We choose the average degree $\langle k\rangle = 3.2799$ in this paper.

5.1. Stability of Rumor-Free Equilibrium

Case 1. In model (1), let $\tau = 2$, $\gamma = 0.02$, $\pi = 0.013$, $\delta = 0.02$, $\theta = 0.08$, $\mu = 0.013$, $\beta = 0.025$, $\alpha = 0.08$, and $\eta = 0.06$. By simple calculation, the basic reproduction number $\mathbf{R}_0 = 0.1986 < 1$. From Theorem 1, rumor-free equilibrium E_0 of model (1) is locally asymptotically stable which is verified by Figure 2(a). Figure 2(b) describes the asymptotic stability of equilibrium E_0.

5.2. Stability of Rumor-Prevailing Equilibrium

Case 2. Choose $\tau = 2$, $\gamma = 0.3$, $\pi = 0.02$, $\delta = 0.015$, $\theta = 0.01$, $\mu = 0.02$, $\beta = 0.02$, $\alpha = 0.2$, and $\eta = 0.04$ in model (1). By simple calculation, $\mathbf{R}_0 = 10.9330 > 1$. The local stability of rumor-prevailing equilibrium E^* is examined as shown in Figure 3(a), and the asymptotic stability of rumor-prevailing equilibrium E^* is examined as shown in Figure 3(b).

5.3. The Influence of τ on Rumor Propagation. Choose $\tau = 0, 2, 4, 6, 8$, and other parameters are fixed as Case 2. The global stability of rumor-prevailing equilibrium E^* can be obtained when $\mathbf{R}_0 > 1$, and rumors always exist. From Figure 4(a), we can find that time delay will suppress the peak of Spreaders' density. As the time delay increases, the maximum density of Spreaders gradually decreases. Therefore, it is appropriate to consider time delay in the spread of rumors.

5.4. The Influence of α on Rumor Propagation. Let $\alpha = 0, 0.2, 0.4, 0.6, 0.8$, and other parameters are fixed as Case 2. The influence of α on rumor propagation can be observed from Figure 4(b). With the increase of t, when α becomes larger, the density of Spreaders will be reduced, which indicates that psychological factors have a positive impact on rumor propagation.

5.5. The Influence of γ on Rumor Propagation. Choose $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$, and other parameters are fixed as Case 2. From Figure 5(a), the influence of γ is apparent.

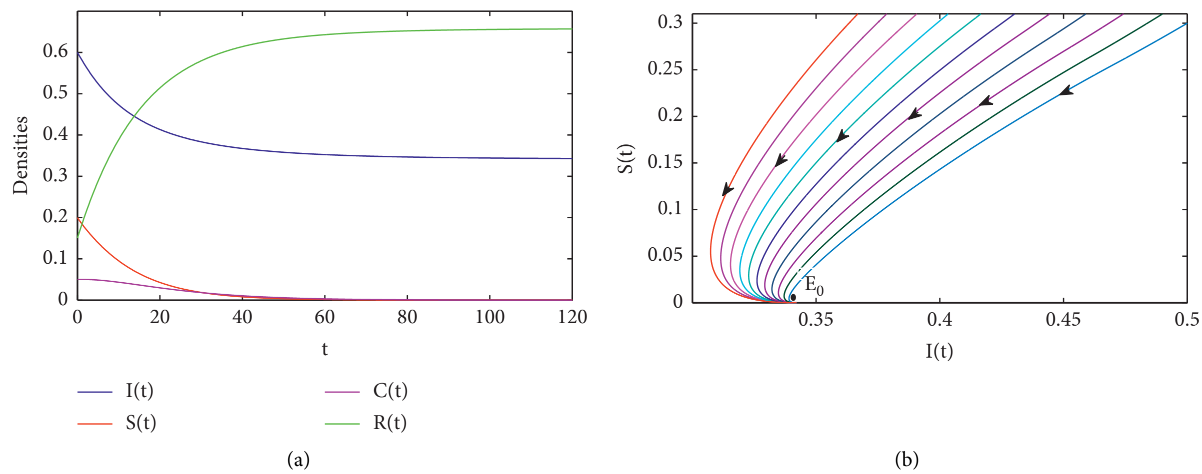


FIGURE 2: The stability of rumor-free equilibrium E_0 when $\mathscr{R}_0 < 1$.

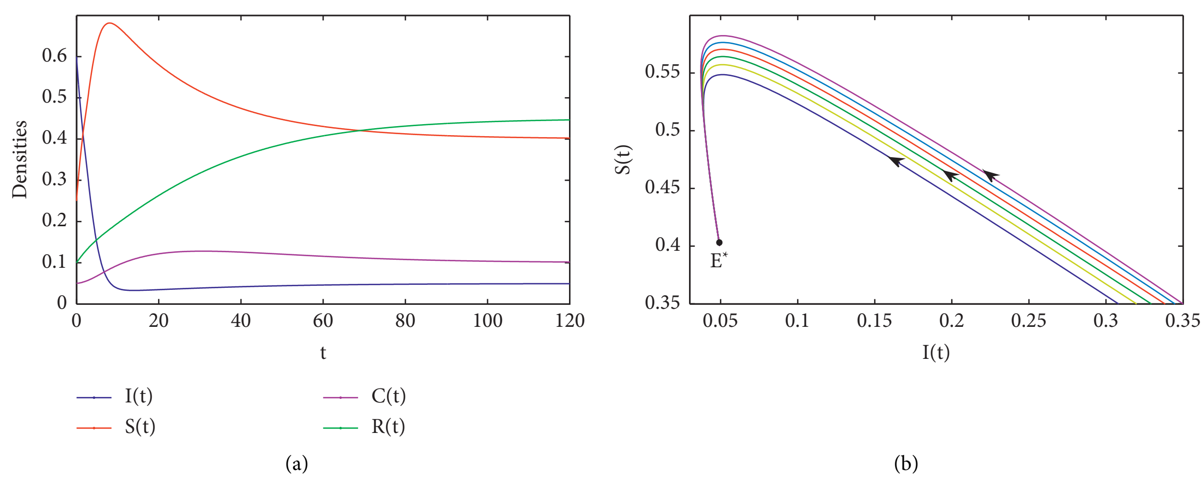


FIGURE 3: The stability of rumor-prevailing equilibrium E^* when $\mathscr{R}_0 > 1$.

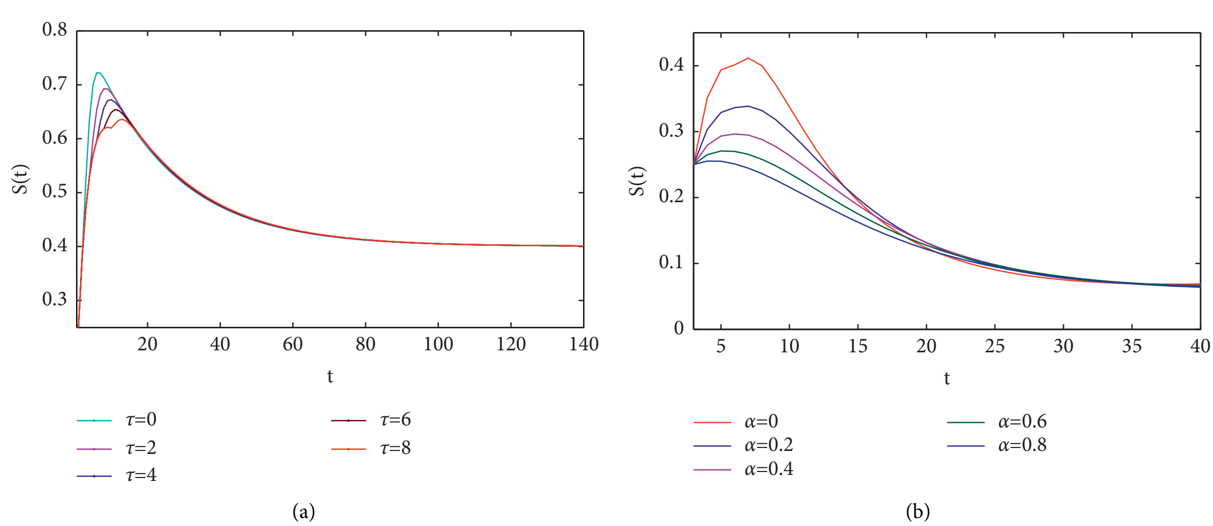


FIGURE 4: (a) The density of $S(t)$ with different values of τ, $\mathscr{R}_0 > 1$. (b) The density of $S(t)$ with different values of α, $\mathscr{R}_0 > 1$.

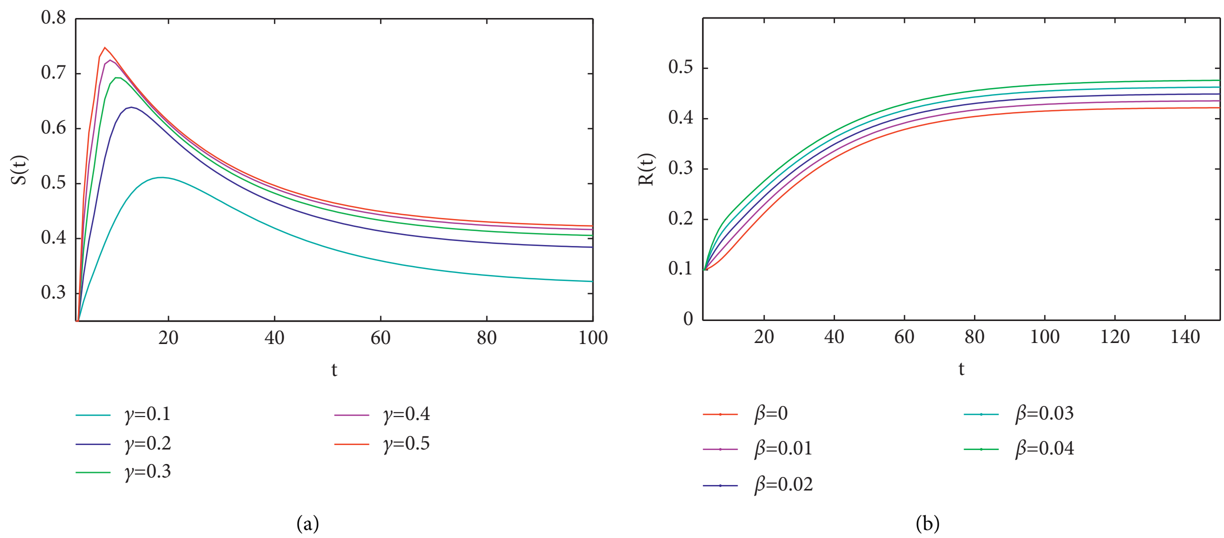


Figure 5: (a) The density of $S(t)$ with different values of γ, $\mathscr{R}_0 > 1$. (b) The density of $R(t)$ with different values of β, $\mathscr{R}_0 > 1$.

Infection rate of rumors will affect the ultimate extent of Spreaders, which provides a novel channel to control rumors. For instance, in the early stage of rumor propagation, forbidding Spreaders in online social networks can reduce the infection rate of rumors.

5.6. The Influence of β on Rumor Propagation. Let $\beta = 0, 0.01, 0.02, 0.03, 0.04$, and other parameters are fixed as Case 2. The positive influence of immune mechanism can be shown by Figure 5(b) on rumor propagation. The density of Removes is affected by different immunization rate. Increasing the density of Removes can be achieved by increasing the immunization rate. For example, more popularization of science videos or articles should be published by some well-known online social networking platforms such as Weibo, WeChat, and so on to improve the level of public scientific knowledge.

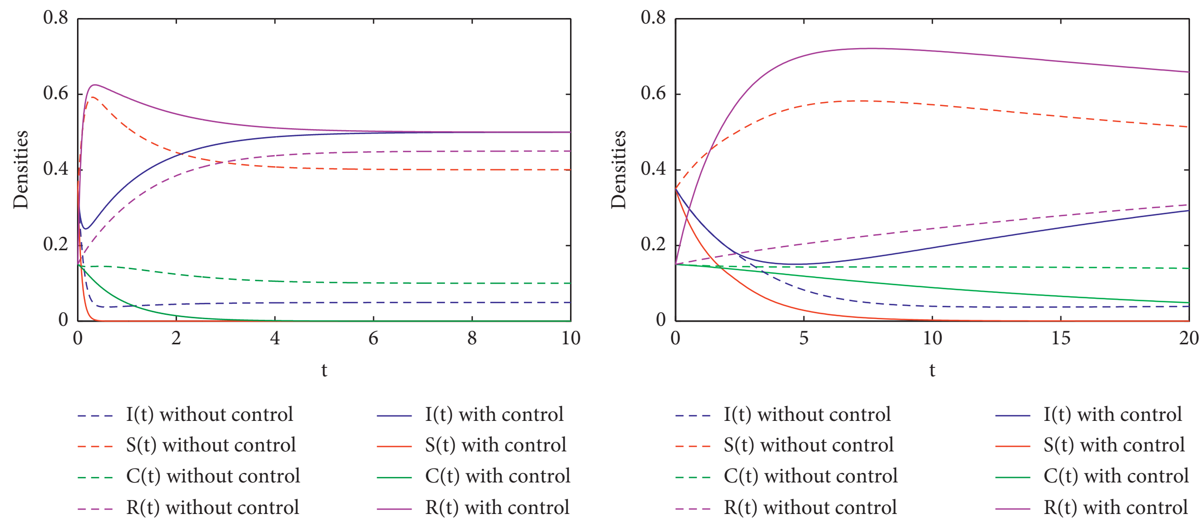
5.7. Effect of Optimal Control. In this part, the optimal control can effectively lessen the density of Spreaders and extend the region of rumors by numerical simulations. Next, we discuss the influence of time delay on optimal control and give some suggestions to control rumor propagation.

5.7.1. Without Time Delay. Choose $A = 4$ in the objective function $J(t)$, and other parameters are fixed as Case 2 besides $\tau = 0$. The density of individuals with and without optimal control is shown in Figure 6(a). If time delay is equal to zero, the spread speed of rumors is accelerated, and the control time should also be advanced. In daily life, our country and government should spread the harmfulness of rumors through various channels, strengthen publicity, and strengthen the education of the people. When rumors appear, the government and relevant departments should quickly formulate emergency plans to ensure social security. They will do a good job with the goal of "the most true information, the fastest speed, and the best effect" to reduce the harm to the society caused by the spread of rumors.

5.7.2. With Time Delay. Choose $A = 4$ in the objective function $J(t)$, and other parameters are fixed as Case 2. The density of individuals with and without optimal control is shown in Figure 6(b). It is obvious that control strategy that we proposed controls rumors successfully. At the beginning of the rumor, a downward trend is shown from the density of the Spreaders, and the density of Removes is increasing rapidly under the control strategy.

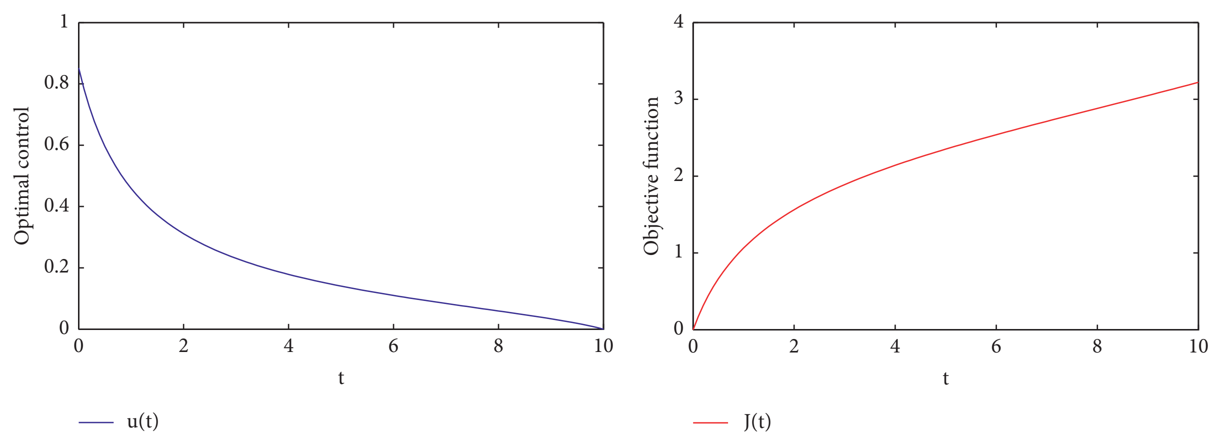
Next, the path of optimal control $u(t)$ and control cost $J(t)$ is shown in Figure 7. Choose $t = 10$, and the optimal control $u(t)$ is gradually decreased to 0 with time as shown in Figure 7(a). Moreover, the manifestation of objective function $J(t)$ is shown in Figure 7(b). Obviously, with the decrease of control force, the control cost gradually increases for a certain time.

5.8. General Comparison. Figure 8 clearly shows the impact of time delay on optimal control. Obviously, time delay only affects the control time and does not affect the final result of control. When the time delay is equal to zero, the density of Spreaders decreases rapidly under control, but the density of Spreaders decreases very slowly when time delay exists. Therefore, in the initial stage of rumor propagation, the government and relevant media industries should refute the rumor in time to shorten the public's response time when facing the rumor, so as to achieve the purpose of controlling the rumor.



(a) (b)

Figure 6: (a) The impact of optimal control without time delay. (b) The impact of optimal control with time delay.



(a) (b)

Figure 7: (a) The path of optimal control $u(t)$. (b) The path of control costs $J(t)$.

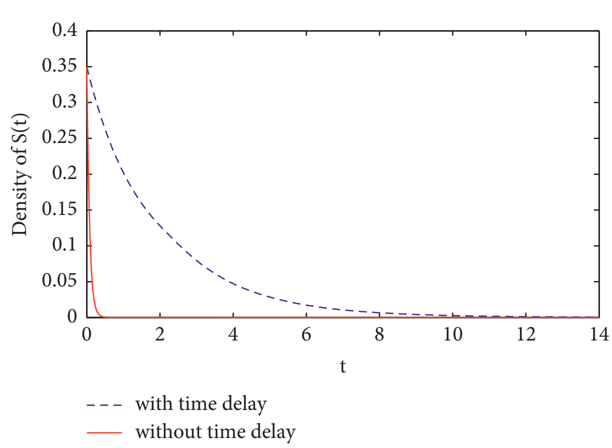


Figure 8: The impact of time delay with optimal control.

6. Conclusion

In this paper, an ISCR rumor propagation model with nonlinear incidence and time delay is presented on complex networks. According to the mean-field theory, the ISCR model is discussed in detail. Firstly, the basic reproduction number $\mathbf{R}_0 = \gamma\langle k\rangle\pi/(\delta+\theta+\mu)(\beta+\mu)$ is calculated by utilizing the next generation matrix theory. Secondly, the locally asymptotic stability of rumor-free (prevailing) equilibrium is verified by using the Routh-Hurwitz criterion and the globally asymptotic stability of equilibria is confirmed by using LaSalle's invariance principle under $\mathbf{R}_0 < 1\,(>1)$. Because time delay τ only affects the spread time of rumors, it does not affect the final spread scale and state of rumors, and the following results are given:

(i) If $\tau = 0$, rumor-free equilibrium is locally asymptotically stable under $\mathbf{R}_0 < 1$ and rumor-prevailing equilibrium under $\mathbf{R}_0 > 1$ is globally asymptotically stable.

(ii) If $\tau > 0$, the locally and globally asymptotic stability of rumor-free (prevailing) equilibrium is satisfied under $\mathbf{R}_0 < 1\,(>1)$.

Then, according to Pontryagin's minimum principle, the optimal control is presented to minimize the density of Spreaders and control costs. Finally, the theoretical results of this paper are verified via some numerical simulations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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