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Research Article

Asymptotic Behavior of the Solutions of the Generalized Globally Modified Navier–Stokes Equations

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The paper is concerned with the existence and the asymptotic behavior of solutions to a class of generalized Navier–Stokes equations, which generalises the so-called globally modified Navier–Stokes equations. The existence and uniqueness of solutions are proved under different assumptions on the dissipation and modification factors. For the asymptotic behavior of solutions, we prove the existence of global attractors in proper spaces. The results generalize some results derived in our previous work Ann. Polon. Math. 122(2):101–128(2019).

1. Introduction

Well-posedness of 3D Navier–Stokes equation is one of the most challenging problems in modern mathematics [1, 2]. To understand or approximate Navier–Stokes equations, various kinds of modified models were introduced in different contexts, such as the Navier–Stokes-α models introduced by Chen et al. and Ilyin and Titi [3, 4], the Leray-α, Clark-α, and simplified Bardina models introduced by Titi et al. [5–7], and some other modified Navier–Stokes equations introduced and studied, respectively, by Caraballo and Kloeden, Constantin, Sohr, and Flandoli et al., see [8–12].

In [8], the authors proposed the global modified Navier–Stokes equations:

$$\begin{cases} \partial_t u - \nu\Delta u + F_N(\|\nabla u\|)(u\cdot\nabla)u + \nabla P = f, \\ \nabla\cdot u = 0, \end{cases} \tag{1}$$

where $F_N(r) = \min\{1, N/r\}$, $r \in \mathbb{R}^+, N \in \mathbb{R}^+$. Since the modifying factor $F_N(\|\Lambda^\beta u\|)$ decreases the singularity of the quadratic convection term $(u\cdot\nabla)u$, it allows the authors to derive the existence and uniqueness of global solutions [8]. Following [8], the existence results and the asymptotic behaviors of solutions to problem (1) were extensively studied in different contexts, see e.g., [13–21] and the review paper [22].

Recently, Dong and Song [23] studied the globally modified Navier–Stokes equations with fractional dissipation in the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + \nu\Lambda^{2\alpha} u + F_N(\|\nabla u\|)(u\cdot\nabla)u + \nabla P = 0, \\ \nabla\cdot u = 0. \end{cases} \tag{2}$$

The existence and uniqueness of global solutions was obtained under the assumption $\alpha > (3/4)$, see also [24], for the existence and uniqueness results in a bounded domains. These results review that the modifying factor $F_N(\|\Lambda^\beta u\|)$ decreases the singularity of the term $(u\cdot\nabla)u$ "too much" so that one can control the nonlinear term by using only the fractional dissipation $(-\Delta)^\alpha u, \alpha < 1$ rather than Δu in (1). This inspires us to weaken the modification term and to investigate that how the dissipation and modification terms interact with each other to determine the existence and uniqueness of the solutions. Precisely speaking, we shall consider the following modified Navier–Stokes equations in $\Omega = [0, L]^3$:

$$\begin{cases} \partial_t u + \nu\Lambda^{2\alpha}u + F_N\left(\left\|\Lambda^{\beta}u\right\|\right)(u\cdot\nabla)u + \nabla P = f, \\ \nabla\cdot u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3}$$

with periodic boundary conditions, where the constants $\nu > 0$ and $\alpha, \beta \geq 0$.

Assume that the initial data and the forcing term are mean-free functions, i.e.,

$$\int_{\Omega} u_0 \mathrm{d}x = \int_{\Omega} f \mathrm{d}x = 0. \tag{4}$$

Then, the solution is also a mean-free function, and the Poincaré inequality holds. We prove that system (3) admits at least one global weak solution when $4\alpha + 2\beta > 5, u_0, f \in \mathbb{H}$. Moreover, if $\alpha > \beta$and $4\alpha^2 - 5\alpha + 2\beta^2 \geq 0$ or $2\alpha + 4\beta > 5$, the weak solution is unique. On the contrary, we prove that when $4\alpha + 2\beta > 5, u_0 \in \mathbb{H}^s, f \in \mathbb{H}^{s-\alpha}$, and $s \geq \beta$, the system possesses a unique global strong solution and the existence of global attractor A, for the solution semigroup in $\mathbb{H}^s$ can be proved when $s \geq \beta$. Furthermore, if $s \geq \max\{1, \beta\}$, we can give explicit upper bound for the fractal dimension of the attractor A. When $\beta = 1$, we prove that the system admits at least one weak solution when $3/4 < \alpha \leq 1, u_0, f \in \mathbb{H}$, and a unique global strong solution when $3/4 < \alpha \leq 1, u_0 \in \mathbb{H}^s, f \in \mathbb{H}^{s-\alpha}$, and $s \geq 1$. The existence and uniqueness in [24] are consistent with the results in this study. If $\beta = 0$, the modifying factor $F_N(\|\Lambda^{\beta}u\|)$ is constant 1, and system (3) becomes the well-known generalized Navier–Stokes equation. The standard existence result shows that the system has global regularity when $\alpha \geq 5/4$ [25], which are consistent with our result. These results extend the previous results in [8, 23, 24] to more general settings.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries about the function spaces and several useful lemmas. Then, in Section 3, we prove the existence and uniqueness results of solutions, while in Section 4 and Section 5, we discuss the existence of a global attractor and the upper bound of its fractal dimension.

2. Preliminaries and Inequalities

Let $\Omega = [0, L]^3$. The fractional operator $\Lambda^{2\alpha} = (-\Delta)^{\alpha}$ for any $\alpha \in \mathbb{R}$ can be defined as

$$\widehat{\Lambda^{2\alpha} f}(\xi) = \sum_{\xi\in\mathbb{Z}^3} |\xi|^{2\alpha}\widehat{f}(\xi)e^{i\xi\cdot x}, \tag{5}$$

for any tempered distribution f, where $\widehat{f}(\xi)$ is the Fourier transform of $f(x)$. Especially, $\Lambda = (-\Delta)^{1/2}$. Let $\dot{C}_p^{\infty}(\Omega)$ be the space of restrictions to Ω of infinitely differentiable functions that are L-periodic in each direction and with zero mean in Ω. For $s \in \mathbb{R}$, we denote by $H^s(\Omega)$ the closure of $\dot{C}_p^{\infty}(\Omega)$ under the norm

$$\|f\|_{H^s} = \left\|\Lambda^s f\right\|_{L^2} = \left(\sum_{\xi\in Z^3} |\xi|^{2s}|\widehat{f}(\xi)|^2\right)^{1/2}, \tag{6}$$

that is, the space of periodic functions with zero mean such that $\|f\|_{H^s} < \infty$. It is obvious that $H^{s_1}(\Omega) \longrightarrow H^{s_2}(\Omega)$(compact imbedding), for any $s_1 > s_2$. Moreover, for $p \in [1, \infty]$, we denote by $H^{s,p}(\Omega)$ the space of periodic mean-free $L^p(\Omega)$ functions φ, which can be written as $\varphi = \Lambda^{-s}\psi$, with $\psi \in L^p$. This is normed by $\|\varphi\|_{H^{s,p}} = \|\Lambda^s\varphi\|_{L^p}$.

For $s \in \mathbb{R}$, we denote

$$\begin{aligned} \mathbb{H}^s &= \left\{u \in \left\|H^s(\Omega)\right\|^3, \text{div}, \quad u = 0\right\}, \\ \mathbb{H}^{s,p} &= \left\{u \in \left[H^{s,p}(\Omega)\right]^3, \text{div}, \quad u = 0\right\}. \end{aligned} \tag{7}$$

Particularly, when $s = 0$, we denote $\mathbb{H}^0$ by $\mathbb{H}$ for short. In this study, for any Banach space X, we denote its norm as $\|\cdot\|_X$; particularly, $\|\cdot\|_{L^2}$ will be abbreviated as $\|\cdot\|$.

Now, we recall the definitions of the global attractor and the fractal dimension, see [26, 27].

Definition 1. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X. A subset $A \subset X$ is called a global attractor for the semigroup if A enjoys the following properties:

(i) A is compact in X.

(ii) A is invariant, i.e., $S(t)A = A$, for any $t \geq 0$.

(iii) A attracts every bounded subset of X, i.e., $\forall B \subset X$ bounded, $\lim_{t\longrightarrow\infty}\text{dist}(S(t)B, A) = 0$, where dist is the Hausdorff semidistance between sets in X, defined as

$$\text{dist}(A, B) = \sup_{a\in A}\inf_{b\in B}\|a - b\|_X, \quad \forall A, B \subset X. \tag{8}$$

Definition 2. The fractal dimension of a compact set K in a Banach space X is defined as

$$d_f(K) = \limsup_{\epsilon\longrightarrow 0}\frac{\log N_{\epsilon}(K)}{-\log\epsilon}, \tag{9}$$

where $N_{\varepsilon}(K)$ is the minimal number of balls of radius ε in X needed to cover K.

The following inequalities may be found in [26, 28].

Lemma 1 (Young's inequality). *For any positive constants a, b, and ε and any $1 < p < \infty$, it holds that*

$$ab \leq \frac{\varepsilon}{p}a^p + \frac{p-1}{p\varepsilon^{1/(p-1)}}b^{p/(p-1)}. \tag{10}$$

Lemma 2 (Poincare's inequality). *Let Ω be a bounded Lipschitz domain in R^d and let p be a continuous seminorm on $H^1(\Omega)$, which is a norm on the constants. Then, there exists a constant c depending only on Ω such that*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega)\left\{\|\nabla u\|_{L^2(\Omega)} + p(u)\right\}, \quad \forall u \in H^1(\Omega). \tag{11}$$

Lemma 3 (Gagliardo–Nirenberg inequality). *Let $1 < p, q, r \le \infty$, $0 \le j < m$, $(j/m) \le \lambda \le 1$. For any $u \in W^{m,p}(\Omega) \cap L^q(\Omega)$, there exists a constant C such that*

$$\left\|D^j u\right\|_{L^r} \le C\left\|D^m u\right\|_{L^p}^{\lambda}\|u\|_{L^q}^{1-\lambda}, \tag{12}$$

where p, q, r, n, m, j, and λ satisfy

$$\frac{1}{r} - \frac{j}{n} = \lambda\left(\frac{1}{p} - \frac{m}{n}\right) + (1-\lambda)\frac{1}{q}. \tag{13}$$

The following product estimates play an essential role in our analysis (see [29]).

Lemma 4. *Suppose that $f, g \in \mathcal{S}$ the Schwartz class. Then, for $s > 0$, $1 < p < +\infty$, there exist a positive constant C such that*

$$\left\|\Lambda^s (fg)\right\|_{L^p} \le C\left(\|f\|_{L^{p_1}}\left\|\Lambda^s g\right\|_{L^{q_1}} + \left\|\Lambda^s f\right\|_{L^{p_2}}\|g\|_{L^{q_2}}\right), \tag{14}$$

with $q_1, p_2 \in (1, +\infty)$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}. \tag{15}$$

The following lemma will play an important role in the proof of our result. It was first proved by Romito in [30] for the case $\beta = 1$. The general case can be proved similarly.

Lemma 5. *For every $u, v \in \mathbb{H}^1$ and each $N > 0$, we have*

$$\begin{aligned}
&0 \le F_N\left(\left\|\Lambda^\beta u\right\|\right)\left\|\Lambda^\beta u\right\| \le N,\\
&\left|F_N\left(\left\|\Lambda^\beta u\right\|\right) - F_N\left(\left\|\Lambda^\beta v\right\|\right)\right| \le \frac{F_N\left(\left\|\Lambda^\beta u\right\|\right)F_N\left(\left\|\Lambda^\beta v\right\|\right)}{N}\\
&\qquad \left\|\Lambda^\beta u - \Lambda^\beta v\right\|.
\end{aligned} \tag{16}$$

3. Existence and Uniqueness Results

We now give the definition of weak solutions to system (3).

Definition 3. Let $u_0, f \in \mathbb{H}$. A function u is called a weak solution to system (3) if

$$\begin{aligned}
&u \in L^\infty(0, T; \mathbb{H}) \cap L^2\left(0, T; \mathbb{H}^\alpha\right),\\
&\partial_t u \in L^2\left(0, T; \mathbb{H}^{-\alpha}\right), \quad \text{for all } T > 0,
\end{aligned} \tag{17}$$

and for any function $\varphi \in \mathbb{H}^\alpha$ and any $T > 0$, it holds that

$$\begin{aligned}
&\int_\Omega u(T)\varphi \mathrm{d}x + \int_0^T \int_\Omega F_N\left(\left\|\Lambda^\beta u\right\|\right)(u \cdot \nabla)u\varphi \mathrm{d}x\mathrm{d}s\\
&\quad + \nu\int_0^T \int_\Omega \Lambda^\alpha u \Lambda^\alpha \varphi \mathrm{d}x\mathrm{d}s\\
&\quad = \int_0^T \int_\Omega f\varphi \mathrm{d}x\mathrm{d}s + \int_\Omega u_0 \varphi \mathrm{d}x.
\end{aligned} \tag{18}$$

Remark 1. Obviously, if $u(t)$ is a weak solution of system (3), then $u \in C([0, T]; \mathbb{H})$, see [26, 27].

Theorem 1. *Let α and β be two constants such that $4\alpha + 2\beta > 5$, $0 \le \alpha < 5/4$, $0 \le \beta < 3/2$. (i) If $u_0, f \in \mathbb{H}$, there exists at least one weak solution $u(t)$ to problem (3) with*

$$u \in L^\infty(0, T; \mathbb{H}) \cap L^2\left(0, T; \mathbb{H}^\alpha\right) \quad \text{for all } T > 0. \tag{19}$$

If in addition $4\alpha^2 - 5\alpha + 2\beta^2 \ge 0$ or $2\alpha + 4\beta > 5$, the weak solution is unique. (ii) On the other hand, if $u_0 \in \mathbb{H}^s$, $f \in \mathbb{H}^{s-\alpha}$, and $s \ge \beta$, then problem (3) admits a unique global solution u satisfying

$$\begin{aligned}
&u(t) \in L^\infty\left(0, T; \mathbb{H}^s\right) \cap L^2\left(0, T; \mathbb{H}^{s+\alpha}\right)\\
&\quad \cap C\left([0, T]; \mathbb{H}^s\right), \quad \text{for all } T \ge 0.
\end{aligned} \tag{20}$$

Remark 2. The standard existence result for the Navier–Stokes equations shows that system (3) possess a unique global solution, for all $\beta = 0$, when $\alpha \ge 5/4$, so we only consider the case $\alpha < 5/4$. However, when $\alpha \le 1/2$, we cannot use the dissipation term of the equations to control the nonlinear term, and the existence results is difficult to prove in this case.

Proof. Let us divide the proof into several steps.

Step 1: we prove the existence of the weak solution by the Galerkin approximation method. Let $\left\{\phi_j\right\}_{j=1}^\infty$ be an orthonormal basis of $\mathbb{H}$ consisting of eigenfunctions of the Stokes operator A and λ_j are the corresponding eigenvalues which are increasing with j. Consider the following ordinary differential system:

$$\begin{aligned}
&\frac{\mathrm{d}u_m}{\mathrm{d}t} + \nu\Lambda^{2\alpha}u_m + P_m F_N\left(\left\|\Lambda^\beta u_m\right\|\right)(u_m \cdot \nabla)u_m = P_m f,\\
&u_m(0) = P_m u_0,
\end{aligned} \tag{21}$$

where $u_m = \sum_{j=1}^{m} c_{jm}(t)\phi_j, \Lambda^{2\alpha}u_m = \sum_{j=1}^{m} \lambda_j^{\alpha} c_{jm}(t)\phi_j, \Lambda^{\beta}u_m = \sum_{j=1}^{m} \lambda_j^{\beta/2} c_{jm}(t)\phi_j$, and P_m is the orthogonal projection form $\mathbb{H}$ onto the space spanned by $\{\phi_1, \phi_2, \ldots \phi_m\}$. By the standard existence theorem for ordinary differential equations, for each m, there exists a local solution u_m to system (21) in the interval $[0, T_m)$.

Multiplying (21) by $u_m(t)$, using the Poincaré inequality $\lambda_1^{\alpha}\|u_m\|^2 \le \|\Lambda^{\alpha}u_m\|^2$, we can deduce that

$$\frac{d}{dt}\|u_m\|^2 + \nu\|\Lambda^{\alpha}u_m\|^2 \le \frac{1}{\nu\lambda_1^{\alpha}}\|f\|^2, \tag{22}$$

and integrating from 0 to t, we obtain

$$\|u_m(t)\|^2 + \nu\int_0^t \|\Lambda^{\alpha}u_m(s)\|^2 ds \le \frac{t}{\nu\lambda_1^{\alpha}}\|f\|^2 + \|u_m(0)\|^2, \quad \forall t \ge 0. \tag{23}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned} \|u_m(t)\|^2 &\le \|u_m(0)\|^2 e^{-\nu\lambda_1^{\alpha}t} + \frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}}\left(1 - e^{-\nu\lambda_1^{\alpha}t}\right) \\ &\le \|u_0\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}}, \end{aligned} \tag{24}$$

which implies that

$$u_m \text{ is bounded in } L^{\infty}(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^{\alpha}). \tag{25}$$

Let us perform the estimates for $\{\partial u_m/\partial t\}$. For any $\varphi \in \mathbb{H}^{\alpha}$, using Hölder's inequality, the product estimates (see Lemma 1), the Gagliardo–Nirenberg inequality, and Young's inequality, we deduce that

$$\begin{aligned} \left|F_N\left(\|\Lambda^{\beta}u_m\|\right)\int_{\Omega}(u_m\cdot\nabla u_m)\varphi dx\right| &\le F_N\left(\|\Lambda^{\beta}u_m\|\right)\|\Lambda^{-\alpha}(u_m\cdot\nabla u_m)\|\|\Lambda^{\alpha}\varphi\| \\ &\le F_N\left(\|\Lambda^{\beta}u_m\|\right)\|\Lambda^{1-\alpha}(u_m u_m)\|\|\Lambda^{\alpha}\varphi\| \\ &\le CF_N\left(\|\Lambda^{\beta}u_m\|\right)\|u_m\|_{L^{6/3-2\beta}}\|\Lambda^{1-\alpha}u_m\|_{L^{3/\beta}}\|\Lambda^{\alpha}\varphi\| \\ &\le C\|\Lambda^{\alpha}u_m\|\|\Lambda^{\alpha}\varphi\|. \end{aligned} \tag{26}$$

The last inequality holds since

$$\begin{aligned} &\frac{1}{2} - \frac{\beta}{3} = \frac{3-2\beta}{6}, \\ &\frac{1}{2} - \frac{\alpha}{3} \le \frac{\beta}{3} - \frac{1-\alpha}{3}. \end{aligned} \tag{27}$$

In view of (25), the sequence $\{F_N(\|\Lambda^{\beta}u_m\|)(u_m\cdot\nabla)u_m\}$ is bounded in $L^2(0, T; \mathbb{H}^{-\alpha})$. Obviously, $\{-\nu\Lambda^{2\alpha}u_m\}$ and $\{P_m f\}$ are bounded in $L^2(0, T; \mathbb{H}^{-\alpha})$. Hence, from (21), we conclude that

$$\left\{\frac{\partial u_m}{\partial t}\right\} \text{ is bounded in } L^2(0, T; \mathbb{H}^{-\alpha}). \tag{28}$$

Using the standard Aubin-Simon-type compactness results [26, 27], there exists an element

$$u \in L^2(0, T; \mathbb{H}^{\alpha}) \cap L^{\infty}(0, T; \mathbb{H}), \quad \text{for all } T > 0, \tag{29}$$

such that up to subsequences,

$$\begin{aligned} &u_m \longrightarrow u \text{ strongly in } L^2(0, T; \mathbb{H}), \\ &u_m \longrightarrow u \text{ a.e in } (0, T)\times\Omega, \\ &u_m \longrightarrow u \text{ weakly in } L^2(0, T; \mathbb{H}^{\alpha}), \\ &\frac{\partial u_m}{\partial t} \longrightarrow \frac{\partial u}{\partial t} \text{ weakly}^* \text{ in } L^2(0, T; \mathbb{H}^{-\alpha}). \end{aligned} \tag{30}$$

Now, it remains to verify that u is a weak solution to problem (3). We treat the case $\alpha > \beta$ and $\alpha \le \beta$ separately. □

Case 1. ($\alpha > \beta \ge 0$). By the Gagliardo–Nirenberg inequality and Hölder's inequality [27], we deduce that

$$\int_0^T \left\|\Lambda^{\beta}(u_m-u)\right\|^2 ds \le C\int_0^T \left\|u_m-u\right\|^{2\theta_1}\left\|\Lambda^{\alpha}(u_m-u)\right\|^{2-2\theta_1} ds$$
$$\le C\left(\int_0^T \left\|u_m-u\right\|^2 ds\right)^{\theta_1}$$
$$\left(\int_0^T \left\|\Lambda^{\alpha}(u_m-u)\right\|^2 ds\right)^{1-\theta_1}, \tag{31}$$

where $\theta_1 = \alpha-\beta/\alpha$. This, together with (25) and (30), implies that

$$u_m \longrightarrow u \text{ strongly in } L^2\left(0,T;\mathbb{H}^{\beta}\right). \tag{32}$$

Thus, up to subsequences,

$$\left\|\Lambda^{\beta}u_m\right\| \longrightarrow \left\|\Lambda^{\beta}u\right\| \text{a.e in } (0,T) \text{ for any } T>0. \tag{33}$$

And, hence,

$$F_N\left(\left\|\Lambda^{\beta}u_m\right\|\right) \longrightarrow F_N\left(\left\|\Lambda^{\beta}u\right\|\right) \text{ a.e. in } (0,T) \text{ for any } T>0. \tag{34}$$

Thanks to (30) and (34), taking $\varphi\in\mathbb{H}^{\alpha}$ as a test function in (21) and passing to the limits, we obtain that u is a weak solution to system (3). As the calculations are rather similar to those in [8], we omit the details for concision.

Case 2. ($\beta\ge\alpha$). Assume that there exist a positive integer N_0 such that $N_0\alpha\le\beta<(N_0+1)\alpha$, without lose of generality, we set $N_0=1$. After multiplying equation (21) with $\Lambda^{2\alpha}u_m$ and integrating, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{\alpha}u_m\right\|^2 + \nu\left\|\Lambda^{2\alpha}u_m\right\|^2$$
$$\le F_N\left(\left\|\Lambda^{\beta}u_m\right\|\right)\left\|\Lambda(u_mu_m)\right\|\left\|\Lambda^{2\alpha}u_m\right\| + \frac{1}{\nu}\|f\|^2 + \frac{\nu}{4}\left\|\Lambda^{2\alpha}u_m\right\|^2. \tag{35}$$

We can estimate the first term of the right side as follows. Since $\beta+2\alpha>5/2, \alpha>1/2$, using the product estimates (see Lemma 1), the Gagliardo–Nirenberg inequality, and Young's inequality, we have the estimate

$$\begin{aligned}&F_N\left(\left\|\Lambda^{\beta}u_m\right\|\right)\left\|\Lambda(u_mu_m)\right\|\left\|\Lambda^{2\alpha}u_m\right\|\\&\le CF_N\left(\left\|\Lambda^{\beta}u_m\right\|\right)\left\|u_m\right\|_{L^{6/3-2\beta}}\left\|\Lambda u_m\right\|_{L^{3/\beta}}\left\|\Lambda^{2\alpha}u_m\right\|\\&\le C\left\|u_m\right\|^{\theta_4}\left\|\Lambda^{2\alpha}u_m\right\|^{2-\theta_4}\\&\le\frac{\nu}{4}\left\|\Lambda^{2\alpha}u_m\right\|^2 + C\left\|u_m\right\|^2,\end{aligned} \tag{36}$$

where $\theta_4=\min\{1,(4\alpha+2\beta-5/4\alpha)\}$. Combining with (24), (35), and (36), we obtain

$$\frac{d}{dt}\left\|\Lambda^{\alpha}u_m\right\|^2 + \nu\left\|\Lambda^{2\alpha}u_m\right\|^2 \le \frac{2}{\nu}\|f\|^2 + C\left\|u_m\right\|^2$$
$$\le\left(\frac{2}{\nu}+\frac{C}{\nu^2\lambda_1^{2\alpha}}\right)\|f\|^2 + C\left\|u_0\right\|^2. \tag{37}$$

For any $t\ge\tau\ge0$, integrating (22) between t and $t+\tau$ and using (24), we obtain

$$\nu\int_t^{t+\tau}\left\|\Lambda^{\alpha}u_m\right\|^2 ds \le \left\|u_0\right\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{\alpha}}\left(\tau+\frac{1}{\nu\lambda_1^{\alpha}}\right). \tag{38}$$

Set

$$\begin{aligned}a^2 &= \frac{2}{\nu\tau}\left\{\left\|u_0\right\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{\alpha}}\left(\tau+\frac{1}{\nu\lambda_1^{\alpha}}\right)\right\},\\ \Omega_m &= \left\{s\in[t,t+\tau]\colon \left\|\Lambda^{\alpha}u_m\right\|\ge a\right\},\end{aligned} \tag{39}$$

and denote by $|\Omega_m|$ the Lebesgue measure of Ω_m. We have

$$\begin{aligned}a^2\left|\Omega_m\right| &\le \int_{\Omega_m}\left\|\Lambda^{\alpha}u_m\right\|^2 ds \le \int_t^{t+\tau}\left\|\Lambda^{\alpha}u_m\right\|^2 ds\\&\le\frac{1}{\nu}\left\{\left\|u_0\right\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{\alpha}}\left(\tau+\frac{1}{\nu\lambda_1^{\alpha}}\right)\right\} = \frac{\tau a^2}{2},\end{aligned} \tag{40}$$

which implies that $\left\|\Omega_m\right\|\le\tau/2$. Therefore, for any given $\varepsilon>0$, there exist a $t_0\in(0,\varepsilon)$ such that

$$\left\|\Lambda^{\alpha}u_m(t_0)\right\|^2 \le \frac{2}{\nu\varepsilon}\left\{\left\|u_0\right\|^2 + \frac{\|f\|^2}{\nu\lambda_1^{\alpha}}\left(\varepsilon+\frac{1}{\nu\lambda_1^{\alpha}}\right)\right\}. \tag{41}$$

By using the Gronwall inequality, we obtain, for all $t\ge\varepsilon$,

$$\left\|\Lambda^{\alpha}u_m(t)\right\|^2 \le \left\|\Lambda^{\alpha}u_m(t_0)\right\|^2 e^{-\nu\lambda_1^{\alpha}(t-t_0)} + \left(\frac{2}{\nu^2\lambda_1^{\alpha}}+\frac{C}{\nu^3\lambda_1^{3\alpha}}\right)\|f\|^2$$
$$+\frac{C}{\nu\lambda_1^{\alpha}}\left\|u_0\right\|^2. \tag{42}$$

Integrating (37) from ε to T and taking (42) into consideration, we deduce that

$$\begin{aligned}&\left\|\Lambda^{\alpha}u_m(t)\right\|^2 + \int_{\varepsilon}^T\left\|\Lambda^{2\alpha}u_m(t)\right\|^2 ds\\&\le\int_{\varepsilon}^T\left(\frac{1}{\nu}+\frac{C}{\nu^2\lambda_1^{2\alpha}}\right)\|f\|^2 + C\left\|u_0\right\|^2 ds + \left\|\Lambda^{\alpha}u_m(\varepsilon)\right\|^2\\&\le\left\{\left(\frac{2}{\nu}+\frac{C}{\nu^2\lambda_1^{2\alpha}}\right)\|f\|^2 + C\left\|u_0\right\|^2\right\}(T-\varepsilon)\\&\quad+\frac{2}{\nu\varepsilon}\left\{\left\|u_0\right\|^2+\frac{\|f\|^2}{\nu\lambda_1^{\alpha}}\left(\varepsilon+\frac{1}{\nu\lambda_1^{\alpha}}\right)\right\}\\&\quad+\frac{2}{\nu\lambda_1^{\alpha}}\left\{C(N,\nu,\alpha,\beta)\left(\left\|u_0\right\|^2+\frac{\|f\|^2}{(\nu\lambda_1^{\alpha})^2}\right)+\frac{2\|f\|^2}{\nu}\right\},\end{aligned} \tag{43}$$

for all $\varepsilon \leq t \leq T$. Thus, we have

$$u_m \text{ is bounded in } L^{\infty}\left(\varepsilon, T; \mathbb{H}^{\alpha}\right) \cap L^2\left(\varepsilon, T; \mathbb{H}^{2\alpha}\right), \text{ for any } T > \varepsilon. \tag{44}$$

Taking $\Lambda^{4\alpha}u_m, \Lambda^{6\alpha}u_m, \ldots, \Lambda^{N_0\alpha}u_m$ as test functions and performing similar analysis, we may prove that

$$u_m \text{ is bounded in } L^{\infty}\left(\varepsilon, T; \mathbb{H}^{N_0\alpha}\right) \cap L^2\left(\varepsilon, T; \mathbb{H}^{(N_0+1)\alpha}\right). \tag{45}$$

Denotes $\alpha_1 = (N_0+1)\alpha$; since $\beta < \alpha_1$, we deduce that

$$\begin{aligned} \int_\epsilon^T \left\|\Lambda^{\beta}(u_m-u)\right\|^2 \mathrm{d}s &\leq C\int_\epsilon^T \left\|u_m-u\right\|^{2\delta_1}\left\|\Lambda^{\alpha_1}(u_m-u)\right\|^{2-2\delta_1}\mathrm{d}s \\ &\leq C\left(\int_\epsilon^T \left\|u_m-u\right\|^2 \mathrm{d}s\right)^{\delta_1} \\ &\quad \left(\int_\epsilon^T \left\|\Lambda^{\alpha_1}(u_m-u)\right\|^2 \mathrm{d}s\right)^{1-\delta_1}, \end{aligned} \tag{46}$$

with $\delta_1 = (\alpha_1 - \beta/\alpha_1)$. Therefore,

$$u_m \longrightarrow u \text{ strongly in } L^2\left(\varepsilon, T; \mathbb{H}^{\beta}\right). \tag{47}$$

Thus, up to subsequences,

$$\left\|\Lambda^{\beta}u_m\right\| \longrightarrow \left\|\Lambda^{\beta}u\right\| \text{ a.e in } (\varepsilon, T) \text{ for any } T > \varepsilon > 0. \tag{48}$$

By the standard diagonal process, we can extract a subsequence of $\{u_m\}$ (still labeled by $\{u_m\}$) such that

$$\left\|\Lambda^{\beta}u_m\right\| \longrightarrow \left\|\Lambda^{\beta}u\right\| \text{ a.e. in } (0, T) \text{ for any } T > 0. \tag{49}$$

Then, using (30) and (34), we can take limits in (21) as in Case 1 to obtain that u is a solution of (3).

Step 2: now, we prove that if $4\alpha^2 - 5\alpha + 2\beta^2 \geq 0$ or $2\alpha + 4\beta > 5$, the weak solution is unique. Let u and v be two solutions to system (3) corresponding to the initial condition u_0, v_0, respectively. Set $w = u - v$ and let $\mathscr{P}$ be the Helmholtz-Leray project operator [27]. It is easy to check that w satisfies

$$\begin{aligned} &w_t + \nu\Lambda^{2\alpha}w + F_N\left(\left\|\Lambda^{\beta}u\right\|\right)\mathscr{P}(v\cdot\nabla)w + F_N\left(\left\|\Lambda^{\beta}u\right\|\right)\mathscr{P}(w\cdot\nabla)u \\ &\quad + \left\{F_N\left(\left\|\Lambda^{\beta}u\right\|\right) - F_N\left(\left\|\Lambda^{\beta}v\right\|\right)\right\}\mathscr{P}(v\cdot\nabla)v = 0. \end{aligned} \tag{50}$$

Multiplying (50) by w and integrating, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu\left\|\Lambda^{\alpha}w\right\|^2 &\leq F_N\left(\left\|\Lambda^{\beta}u\right\|\right)\int_0^T w\cdot\nabla uw\mathrm{d}x| \\ &\quad + \left|\left(F_N\left(\left\|\Lambda^{\beta}u\right\|\right) - F_N\left(\left\|\Lambda^{\beta}v\right\|\right)\right)\int_0^T v\cdot\nabla vw\mathrm{d}x\right|, \end{aligned} \tag{51}$$

where we have used the fact $\int_0^T v\cdot\nabla ww\mathrm{d}x = 0$. Since $4\alpha + 2\beta > 5$ and $\alpha > 1/2, \beta \geq 0$, we have $\alpha + \beta > 1$. Using Hölder's inequality and the product estimates (see Lemma 1) and the Gagliardo–Nirenberg inequality, we have the first term of the right side

$$\begin{aligned} F_N\left(\left\|\Lambda^{\beta}u\right\|\right)|\int_0^T w\cdot\nabla uw\mathrm{d}x| &\leq F_N\left(\left\|\Lambda^{\beta}u\right\|\right)\left\|\Lambda^{\beta}u\right\|\left\|\Lambda^{1-\beta}(ww)\right\| \\ &\leq C\left\|\Lambda^{1-\beta}w\right\|_{L^{3/\alpha}}\|w\|_{L^{6/3\,2\alpha}} \\ &\leq C\|w\|^{\theta_2}\left\|\Lambda^{\alpha}w\right\|^{2-\theta_2} \\ &\leq C\|w\|^2 + \frac{\nu}{4}\left\|\Lambda^{\alpha}w\right\|^2, \end{aligned} \tag{52}$$

where $\theta_2 = \min\{1, 4\alpha + 2\beta - 5/2\alpha\}$. For the second term of the right side in (51), using Lemma 2 and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned} \left|F_N\left(\left\|\Lambda^{\beta}u\right\|\right) - F_N\left(\left\|\Lambda^{\beta}v\right\|\right)\right| &\leq \frac{F_N\left(\left\|\Lambda^{\beta}u\right\|\right)F_N\left(\left\|\Lambda^{\beta}v\right\|\right)}{N}\left\|\Lambda^{\beta}w\right\| \\ &\leq \frac{F_N\left(\left\|\Lambda^{\beta}u\right\|\right)F_N\left(\left\|\Lambda^{\beta}v\right\|\right)}{N}\left\|\Lambda^{\beta}w\right\| \\ &\leq \frac{F_N\left(\left\|\Lambda^{\beta}u\right\|\right)F_N\left(\left\|\Lambda^{\beta}v\right\|\right)}{N} \\ &\quad \|w\|^{\theta_1}\left\|\Lambda^{\alpha}w\right\|^{1-\theta_1}, \end{aligned} \tag{53}$$

with $\theta_1 = \alpha - \beta/\alpha$. When $2\alpha + 4\beta \geq 5$, we can always find $(p, q) = ((3/\beta), (6/3 - 2\beta))$ such that $(1/p) + (1/q) = (1/2)$ and

$$\begin{aligned} \frac{1}{p} - \frac{1-\alpha}{3} &\geq \frac{1}{2} - \frac{\beta}{3}, \\ \frac{1}{q} &= \frac{1}{2} - \frac{\beta}{3}. \end{aligned} \tag{54}$$

Thanks to the product estimates (see Lemma 1) and the fractional Sobolev inequality, we have

$$\begin{aligned} \left|\int_0^T v\cdot\nabla vw\mathrm{d}x\right| &= \left|\int_\Omega v\cdot\nabla uw\mathrm{d}x\right| \\ &\leq C\left\|\Lambda^{\alpha}w\right\|\left(\left\|\Lambda^{1-\alpha}u\right\|_{L^p}\|v\|_{L^q} + \left\|\Lambda^{1-\alpha}v\right\|_{L^p}\|u\|_{L^q}\right) \\ &\leq C\left\|\Lambda^{\alpha}w\right\|\left(\left\|\Lambda^{\beta}u\right\|\left\|\Lambda^{\beta}v\right\| + \left\|\Lambda^{\beta}v\right\|\left\|\Lambda^{\beta}u\right\|\right). \end{aligned} \tag{55}$$

Combining (53) and (55) and using Young inequality, we obtain

$$\begin{aligned}
&\left|\left(F_N\left(\left\|\Lambda^\beta u\right\|\right)-F_N\left(\left\|\Lambda^\beta v\right\|\right)\right)\int_0^T v\cdot\nabla v w \mathrm{d}x\right| \\
&\leq C\frac{F_N\left(\left\|\Lambda^\beta u\right\|\right)F_N\left(\left\|\Lambda^\beta v\right\|\right)}{N}\|w\|^{\theta_1}\left\|\Lambda^\alpha w\right\|^{2-\theta_1}\left(\left\|\Lambda^\beta u\right\|\left\|\Lambda^\beta v\right\|+\left\|\Lambda^\beta v\right\|\left\|\Lambda^\beta u\right\|\right) \\
&\leq C\|w\|^2+\frac{\nu}{4}\left\|\Lambda^\alpha w\right\|^2.
\end{aligned} \tag{56}$$

Hence, if $2\alpha+4\beta\geq 5$, from (56), (52), and (51), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+\left\|\Lambda^\alpha w\right\|^2\leq C\|w\|^2, \tag{57}$$

from which the uniqueness result follows easily.

On the contrary, if $2\alpha+4\beta<5$, setting $p=(6/3-2\beta)$ and $q=3/\beta$, we have

$$\begin{aligned}
&\left|\int_0^T v\cdot\nabla v w \mathrm{d}x\right|=\left|\int_\Omega v\cdot\nabla u w \mathrm{d}x\right| \\
&\leq C\left\|\Lambda^\alpha w\right\|\left(\left\|\Lambda^{1-\alpha}u\right\|_{L^{3/\beta}}\|v\|_{L^{6/(3-2\beta)}}+\left\|\Lambda^{1-\alpha}v\right\|_{L^{3/\beta}}\|u\|_{L^{6/(3-2\beta)}}\right) \\
&\leq C\left\|\Lambda^\alpha w\right\|\left(\left\|\Lambda^\alpha u\right\|^{\theta_3}\left\|\Lambda^\beta u\right\|^{1-\theta_3}\left\|\Lambda^\beta v\right\|+\left\|\Lambda^\alpha v\right\|^{\theta_3}\left\|\Lambda^\beta v\right\|^{1-\theta_3}\left\|\Lambda^\beta u\right\|\right) \\
&\leq C\left\|\Lambda^\alpha w\right\|\left(\left\|\Lambda^\alpha u\right\|^{\theta_3}+\left\|\Lambda^\alpha v\right\|^{\theta_3}\right)\left(\left\|\Lambda^\beta u\right\|^{1-\theta_3}\left\|\Lambda^\beta v\right\|+\left\|\Lambda^\beta v\right\|^{1-\theta_3}\left\|\Lambda^\beta u\right\|\right),
\end{aligned} \tag{58}$$

where θ_3 satisfies

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{\alpha}{3}\right)\theta_3+\left(\frac{1}{2}-\frac{\beta}{3}\right)(1-\theta_3)=\frac{\beta}{3} \\
&-\frac{1-\alpha}{3}\left(\text{i.e. }\theta_3=\frac{-4\beta-2\alpha+5}{2\alpha-2\beta}>0\right).
\end{aligned} \tag{59}$$

Note that when $4\alpha+2\beta>5$ and $\alpha>\beta\geq 0$, we have $\theta_1=(\alpha-\beta/\alpha)\geq\theta_3=(5-2\alpha-4\beta/2\alpha-2\beta)$ iff $4\alpha^2-5\alpha+2\beta^2\geq 0$. Combining (53) and (58), we have

$$\begin{aligned}
&\left|\left(F_N\left(\left\|\Lambda^\beta u\right\|\right)-F_N\left(\left\|\Lambda^\beta v\right\|\right)\right)\int_0^T v\cdot\nabla v w \mathrm{d}x\right| \\
&\leq C\|w\|^{\theta_1}\left\|\Lambda^\alpha w\right\|^{2-\theta_1}\left(\left\|\Lambda^\alpha u\right\|^{\theta_3}+\left\|\Lambda^\alpha v\right\|^{\theta_3}\right) \\
&\leq\frac{\nu}{4}\left\|\Lambda^\alpha w\right\|^2+C\left(\left\|\Lambda^\alpha u\right\|^{2\theta_3/\theta_1}+\left\|\Lambda^\alpha v\right\|^{(2\theta_3/\theta_1)/\theta_1}\right)\|w\|^2,
\end{aligned} \tag{60}$$

which, combined with (51) and (52), implies that when $2\alpha+4\beta<5$ and $4\alpha^2-5\alpha+2\beta^2\geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+\left\|\Lambda^\alpha w\right\|^2\leq C\|w\|^2\left(\left\|\Lambda^\alpha u\right\|^2+\left\|\Lambda^\alpha v\right\|^2+1\right). \tag{61}$$

By using Gronwall's inequality, we obtain

$$\|w(t)\|^2\leq\|w(0)\|^2\exp\left\{C\int_0^t\left\|\Lambda^\alpha u\right\|^2+\left\|\Lambda^\alpha v\right\|^2+1\mathrm{d}s\right\}. \tag{62}$$

The uniqueness result follows easily.

Step 3: we now prove the second part of the theorem. If $u_0\in\mathbb{H}^s$, $f\in\mathbb{H}^{s-\alpha}$, and $s\geq\beta$, we multiply (21) by $\Lambda^{2s}u_m$ to deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\left\|\Lambda^s u_m\right\|^2+\nu\left\|\Lambda^{s+\alpha}u_m\right\|^2 \\
&\leq\|f\|_{\mathbb{H}^{s-\alpha}}\left\|\Lambda^{s+\alpha}u_m\right\|+F_N\left(\left\|\Lambda^\beta u_m\right\|\right)\left|\int_\Omega(u_m\cdot\nabla u_m)\Lambda^{2s}u_m \mathrm{d}x\right| \\
&\leq\frac{1}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}+\frac{\nu}{4}\left\|\Lambda^{s+\alpha}u_m\right\|^2+F_N\left(\left\|\Lambda^\beta u_m\right\|\right) \\
&\left\|\Lambda^{s+1-\alpha}(u_m u_m)\right\|\left\|\Lambda^{s+\alpha}u_m\right\|.
\end{aligned} \tag{63}$$

Using the product estimates and the Gagliardo–Nirenberg inequality, we deduce that

$$
\begin{aligned}
&F_N\left(\left\|\Lambda^\beta u_m\right\|\right)\left\|\Lambda^{s+1-\alpha}(u_m u_m)\right\|\left\|\Lambda^{s+\alpha}u_m\right\| \\
&\leq CF_N\left(\left\|\Lambda^\beta u_m\right\|\right)\left\|u_m\right\|_{L^{6/(3-2\beta)}}\left\|\Lambda^{s+1-\alpha}u_m\right\|_{L^{3/\beta}}\left\|\Lambda^{s+\alpha}u_m\right\| \\
&\leq CN\left\|u_m\right\|^{\theta_5}\left\|\Lambda^{s+\alpha}u_m\right\|^{2-\theta_5} \\
&\leq \frac{1}{2}C_0(N,\nu,\alpha,\beta)\left\|u_m\right\|^2+\frac{\nu}{4}\left\|\Lambda^{s+\alpha}u_m\right\|^2,
\end{aligned} \tag{64}
$$

where $\theta_5 = 4\alpha + 2\beta - 5/2s + 2\alpha$ and $C_0(N,\nu,\alpha,\beta) = (CN)^{2/\theta_5}(\nu/4 - 2\theta_5)^{\theta_5 - 2/\theta_5}\theta_5$. This, combined with (63), yields that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^s u_m\right\|^2+\nu\left\|\Lambda^{s+\alpha}u_m\right\|^2 \leq C_0(N,\nu,\alpha,\beta)\left\|u_m\right\|^2+\frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}. \tag{65}
$$

Hence, for all $t \geq \tau \geq 0$,

$$
\begin{aligned}
&\left\|\Lambda^s u_m(t)\right\|^2+\nu\int_\tau^t\left\|\Lambda^{s+\alpha}u_m\right\|^2\mathrm{d}\varsigma \\
&\leq \left\|\Lambda^s u(\tau)\right\|^2+C_0(N,\nu,\alpha,\beta)(t-\tau)\left(\left\|u_0\right\|^2+\frac{\|f\|^2}{(\nu\lambda_1^\alpha)^2}\right) \\
&\quad+\frac{2(t-\tau)}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}.
\end{aligned} \tag{66}
$$

Thus, $\{u_m\}$ is bounded in $L^\infty(0,T;\mathbb{H}^s)\cap L^2(0,T;\mathbb{H}^{s+\alpha})$, $\forall T>0$. Passing to the limit, we obtain (20) immediately.

Next, we prove the uniqueness result. Let u and v be two solutions in $L^\infty(0,T;\mathbb{H}^s)\cap L^2(0,T;\mathbb{H}^{s+\alpha})$. Taking the inner product in (50) with $\Lambda^{2s}w$, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^s w\right\|^2+\nu\left\|\Lambda^{s+\alpha}w\right\|^2 \leq F_N\left(\left\|\Lambda^\beta u\right\|\right)\left|\int_\Omega(u\cdot\nabla w)\Lambda^{2s}w\mathrm{d}x\right| \\
&\quad+F_N\left(\left\|\Lambda^\beta u\right\|\right)\left|\int_\Omega(w\cdot\nabla v)\Lambda^{2s}w\mathrm{d}x\right| \\
&\quad+\left|\left\{F_N\left(\left\|\Lambda^\beta u\right\|\right)-F_N\left(\left\|\Lambda^\beta v\right\|\right)\right\}\int_\Omega(v\cdot\nabla v)\Lambda^{2s}w\mathrm{d}x\right| \\
&\doteq I_1+I_2+I_3.
\end{aligned} \tag{67}
$$

For I_1, using Hölder's inequality, the product estimates, Gagliardo–Nirenberg inequality, and Young's inequality, we deduce that

$$
\begin{aligned}
I_1 &\leq F_N\left(\left\|\Lambda^\beta u\right\|\right)\left\|\Lambda^{s+1-\alpha}(uw)\right\|\left\|\Lambda^{s+\alpha}w\right\| \\
&\leq CF_N\left(\left\|\Lambda^\beta u\right\|\right)\left(\|u\|_{L^{6/3-2\beta}}\left\|\Lambda^{s+1-\alpha}w\right\|_{L^{3/\beta}}\right. \\
&\quad\left.+\|w\|_{L^{6/3-2s}}\left\|\Lambda^{s+1-\alpha}u\right\|_{L^{3/s}}\right)\left\|\Lambda^{s+\alpha}w\right\| \\
&\leq C\left(\|w\|^{\theta_2}_{H^s}\|w\|^{2-\theta_2}_{H^{s+\alpha}}+\|w\|_{H^s}\|u\|_{H^{s+\alpha}}\|w\|_{H^{s+\alpha}}\right) \\
&\leq C\left(1+\|u\|^2_{H^{s+\alpha}}\right)\|w\|^2_{H^s}+\frac{\nu}{6}\left\|\Lambda^{s+\alpha}w\right\|^2,
\end{aligned} \tag{68}
$$

where $\theta_2 = \min\{1, 2\beta + 4\alpha - 5/2\alpha\}$ (here, we may assume that $s < 3/2$. If $s = 3/2$, we can choose (p,q) satisfies $(1/p) + (1/q) = (1/p)$ and $2 < q < 6/5 - 4\alpha$, and the Sobolev inequality implies that $\|w\|_{L^p}\|\Lambda^{s+1-\alpha}u\|_{L^q} \leq C\|w\|_{H^{3/2}}\|u\|_{H^{s+\alpha}}$. If $s > 3/2$, we may choose $(p,q) = (\infty, 2)$ to get $\|w\|_{L^\infty}\|\Lambda^{s+1-\alpha}u\|_{L^2} \leq C\|w\|_{H^s}\|u\|_{H^{s+\alpha}}$). Similarly, we have

$$
I_2 \leq C\left(1+\|v\|^2_{H^{s+\alpha}}\right)\|w\|^2_{H^s}+\frac{\nu}{6}\left\|\Lambda^{s+\alpha}w\right\|^2. \tag{69}
$$

Moreover,

$$
\begin{aligned}
I_3 &\leq \left|F_N\left(\left\|\Lambda^\beta u\right\|\right)-F_N\left(\left\|\Lambda^\beta v\right\|\right)\right|\left\|\Lambda^{s+1-\alpha}(vv)\right\|\left\|\Lambda^{s+\alpha}w\right\| \\
&\leq C_3\frac{F_N\left(\left\|\Lambda^\beta u\right\|\right)F_N\left(\left\|\Lambda^\beta v\right\|\right)}{N}\left\|\Lambda^\beta w\right\|\|v\|_{L^{6/3-2\beta}} \\
&\quad\left\|\Lambda^{s+1-\alpha}v\right\|_{L^{3/\beta}}\left\|\Lambda^{s+\alpha}w\right\| \\
&\leq C\left\|\Lambda^s w\right\|\left\|\Lambda^{s+\alpha}v\right\|\left\|\Lambda^{s+\alpha}w\right\| \\
&\leq \frac{\nu}{6}\left\|\Lambda^{s+\alpha}w\right\|^2+C\left\|\Lambda^{s+\alpha}v\right\|^2\left\|\Lambda^s w\right\|^2.
\end{aligned} \tag{70}
$$

By (67)–(70), we obtain that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^s w\right\|^2 \leq C\left(\left\|\Lambda^{s+\alpha}u\right\|^2+\left\|\Lambda^{s+\alpha}v\right\|^2+1\right)\left\|\Lambda^s w\right\|^2. \tag{71}
$$

Gronwall's inequality then implies that

$$
\left\|\Lambda^s w\right\|^2 \leq \exp\left\{C\int_0^t\left(\left\|\Lambda^{s+\alpha}u\right\|^2+\left\|\Lambda^{s+\alpha}v\right\|^2+1\right)\mathrm{d}\tau\right\}\left\|\Lambda^s w(0)\right\|^2, \tag{72}
$$

from which the uniqueness of the solution and the continuity of the solution semigroup in $\mathbb{H}^s$ follow immediately.

Finally, let us verify that $u \in C([0,T];\mathbb{H}^s)$. Note that (66) implies that

$$u \in L^2(0,T;\mathbb{H}^{s+\alpha}), \quad \forall T>0, \text{ i.e., } \Lambda^s u \in L^2(0,T;\mathbb{H}^{\alpha}), \forall T>0. \tag{73}$$

Hence, according to the standard Sobolev embedding result [26, 27], we need only to show that

$$\Lambda^s u_t \in L^2(0,T;\mathbb{H}^{-\alpha}). \tag{74}$$

Indeed, for any $\varphi \in \mathbb{H}^\alpha$, we have

$$\begin{aligned}\langle \Lambda^s u_t, \varphi\rangle &= -F_N\left(\left\|\Lambda^\beta u\right\|\right)\langle \Lambda^s(u\cdot\nabla u),\varphi\rangle - \langle \Lambda^{s+2\alpha}u,\varphi\rangle \\ &\quad + \langle \Lambda^s f,\varphi\rangle.\end{aligned} \tag{75}$$

Therefore,

$$\begin{aligned}\left|\langle \Lambda^s u_t,\varphi\rangle\right| \le \Big\{ & F_N\left(\left\|\Lambda^\beta u\right\|\right)\left\|\Lambda^{s-\alpha}(u\cdot\nabla u)\right\| + \left\|\Lambda^{s+\alpha}u\right\| \\ & + \left\|\Lambda^{s-\alpha}f\right\|\Big\}\left\|\Lambda^\alpha\varphi\right\|,\end{aligned} \tag{76}$$

which implies that

$$\left\|\Lambda^s u_t\right\|_{\mathbb{H}^{-\alpha}} \le F_N\left(\left\|\Lambda^\beta u\right\|\right)\left\|\Lambda^{s-\alpha}(u\cdot\nabla u)\right\| + \left\|\Lambda^{s+\alpha}u\right\| + \left\|\Lambda^{s-\alpha}f\right\|. \tag{77}$$

Applying the product estimates and the imbedding of fractional Sobolev spaces, we have

$$\begin{aligned}F_N\left(\left\|\Lambda^\beta u\right\|\right)\left\|\Lambda^{s-\alpha}(u\cdot\nabla)u\right\| &\le CF_N\left(\left\|\Lambda^\beta u\right\|\right)\|u\|_{L^{6/3-2\beta}} \\ &\quad \left\|\Lambda^{s+1-\alpha}u\right\|_{L^{3/\beta}} \le CN\left\|\Lambda^{s+\alpha}u\right\|.\end{aligned} \tag{78}$$

Therefore,

$$\left\|\Lambda^s u_t\right\|_{\mathbb{H}^{-\alpha}} \le C\left(\left\|\Lambda^{s+\alpha}u\right\| + \left\|\Lambda^{s-\alpha}f\right\|\right). \tag{79}$$

Combining (73) and the assumption $f \in \mathbb{H}^{s-\alpha}$, we know that $\Lambda^s u_t \in L^2(0,T;\mathbb{H}^{-\alpha})$. The proof is thus complete.

4. Long Time Behaviors

4.1. Attractor for Strong Solution. In this section, we prove the existence of a global attractor for system (3).

Theorem 2. *Assume that* $4\alpha+2\beta>5, 0\le\alpha<5/4, 0\le\beta<3/2$, *and* $f\in\mathbb{H}^{s-\alpha}, u_0\in\mathbb{H}^s, s\ge\beta$. *Then, system (3) generates a continuous semigroup* $\{S(t)\}_{t\ge0}$ *in* $\mathbb{H}^s$, *and the semigroup possesses a global attractor A, which is compact, invariant, and connected in* $\mathbb{H}^s$ *and attracts all the bounded subsets of* $\mathbb{H}^s$ *in the* $\mathbb{H}^s$*-norm. Moreover, if* $s\ge\max\{\beta,1\}$, $f\in\mathbb{H}^{s-1+\alpha}$, *the global attractor is bounded in* $\mathbb{H}^{s-1+2\alpha}$.

Proof. Thanks to Theorem 1, we know that the semigroup is continuous. It remains to prove the existence of an absorbing set and the compactness of the semigroup in $\mathbb{H}^s$.

Absorbing set: let $u(t)$ be the solution of system (1). Similar to (24), we have

$$\|u(t)\|^2 \le \|u(0)\|^2 e^{-\nu\lambda_1^\alpha t} + \frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}}. \tag{80}$$

From the above inequality, we can deduce that there exists a $T_0 = t(\|u_0\|)$ such that

$$\|u(t)\|^2 \le 2\frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}}, \quad \forall t>T_0. \tag{81}$$

Multiplying (3) by $\Lambda^{2s}u$ and integrating, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\Lambda^s u\right\|^2 + \nu\left\|\Lambda^{s+\alpha}u\right\|^2 &\le \|f\|_{\mathbb{H}^{s-\alpha}}\left\|\Lambda^{s+\alpha}u\right\| + F_N\left(\left\|\Lambda^\beta u\right\|\right) \\ &\quad \left|\iint_\Omega (u\cdot\nabla u)\Lambda^{2s}u\,dx\right|,\end{aligned} \tag{82}$$

for all $t>0$. Similar to (65), we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^s u\right\|^2 + \nu\left\|\Lambda^{s+\alpha}u\right\|^2 &\le C_0(N,\nu,\alpha,\beta)\|u\|^2 \\ &\quad + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}, \quad \text{for all } t>0,\end{aligned} \tag{83}$$

where $C_0(N,\nu,\alpha,\beta) = (CN)^{2/\theta_5}(\nu/4-2\theta_5)^{\theta_5-2/\theta_5}\theta_5, \theta_5 = 4\alpha+2\beta-5/2s+2\alpha$. Integrating the both sides from 0 to T_0 and taking (24) into consideration, we obtain

$$\begin{aligned}\left\|\Lambda^s u(T_0)\right\|^2 &\le \left\|\Lambda^s u_0\right\|^2 + \int_0^{T_0}\left(C_0(N,\nu,\alpha,\beta)\|u\|^2 + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}\right)d\tau \\ &\le \left\|\Lambda^s u_0\right\|^2 + C_0(N,\nu,\alpha,\beta)T_0\left(\left\|u_0\right\|^2 + \frac{\|f\|^2}{(\nu\lambda_1^\alpha)^2}\right) \\ &\quad + \frac{2T_0}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}} \doteq \mathscr{K}.\end{aligned} \tag{84}$$

Using the Poincaré inequality and (81), we obtain from (83) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^s u\right\|^2 + \nu\lambda_1^\alpha\left\|\Lambda^s u\right\|^2 &\le C_0(N,\nu,\alpha,\beta)\frac{\|f\|^2}{(\nu\lambda_1^\alpha)^2} \\ &\quad + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}, \quad \text{for all } t\ge t_{u_0}.\end{aligned} \tag{85}$$

Gronwall's inequality then implies that

$$\begin{aligned}\left\|\Lambda^s u(t)\right\|^2 &\le \left\|\Lambda^s u(T_0)\right\|^2 e^{-\nu\lambda_1^\alpha(t-T_0)} \\ &\quad + \frac{1}{\nu\lambda_1^\alpha}\left(C_0(N,\nu,\alpha,\beta)\frac{\|f\|^2}{(\nu\lambda_1^\alpha)^2} + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}\right).\end{aligned} \tag{86}$$

Thanks to (96), we know that if

$$t \ge \max\left\{T_0, \frac{1}{\nu\lambda_1^\alpha}\ln\mathscr{K} + T_0\right\}, \tag{87}$$

then

$$\|\Lambda^s u(t)\|^2 \le \frac{1}{\nu\lambda_1^\alpha}\left(C_0(N,\nu,\alpha,\beta)\frac{\|f\|^2}{(\nu\lambda_1^\alpha)^2} + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s-\alpha}}\right) + 1 = \rho_0^2. \tag{88}$$

Therefore, there is an absorbing set B_1 for the semigroup $\{S(t)\}_{t\ge 0}$ in $\mathbb{H}^s$.

Compactness of the semigroup: we show that, for any bounded sequence $\{v_0^n\}$ in $\mathbb{H}^s$ any $t>0$, the sequence $\{S(t)v_0^n\} = \{v^n(t)\}$ has a convergent subsequence in $\mathbb{H}^s$. Similar to (66) and (79), we can prove that

$$\begin{aligned}&\{\Lambda^s v^n(t)\} \text{is bounded in } L^2(0,1;\mathbb{H}^\alpha),\\ &\{\Lambda^s v_t^n(t)\} \text{is bounded in } \mathrm{L}^2(0,1;\mathbb{H}^{-\alpha}).\end{aligned} \tag{89}$$

Using the Aubin-Simon type compactness results [26, 27], there exists an element v with

$$\begin{aligned}\Lambda^s v &\in L^2(0,1;\mathbb{H}^\alpha),\\ \Lambda^s v_t &\in L^2(0,1;\mathbb{H}^{-\alpha}),\end{aligned} \tag{90}$$

such that up to subsequences,

$$\Lambda^s v^n(t) \longrightarrow \Lambda^s v(t) \text{ strongly in } L^2(0,1;\mathbb{H}), \tag{91}$$

i.e.,

$$v^n(t) \longrightarrow v(t) \text{ strongly in } L^2(0,1;\mathbb{H}^s). \tag{92}$$

In particular, there exists a $\tau \in (0,1)$ such that

$$v^n(\tau) \longrightarrow v(\tau), \quad \text{in } \mathbb{H}^s. \tag{93}$$

Recall that the map $S(t)$: $\mathbb{H}^s \longrightarrow \mathbb{H}^s$ is continuous, and we obtain that

$$\begin{aligned}S(t)v_0^n &= S(t-\tau)S(\tau)v_0^n = S(t-\tau)v^n(\tau)\\ &\longrightarrow S(t-\tau)v(\tau) \text{ in } \mathbb{H}^s, \quad \text{for all } t\ge 1.\end{aligned} \tag{94}$$

Thus, the semigroup $S(t)$ is compact, for any $t \ge 1$. Thanks to the standard existence results on global attractors, we may obtain a global attractor in $\mathbb{H}^s$ for the solution semigroup $S(t)$.

Regularity of the attractor: now, we prove that A is bounded in $\mathbb{H}^{s_0+\alpha}$ if $f \in \mathbb{H}^{s_0}$, $s_0 = s-1+\alpha$. Take the inner product of (3) with $\Lambda^{2s_0+2\alpha}$, and we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{s_0+\alpha}u\|^2 + \nu\|\Lambda^{s_0+2\alpha}u\|^2\\ &\le \|f\|_{\mathbb{H}^{s_0}}\|\Lambda^{s_0+2\alpha}u\| + F_N\left(\|\Lambda^\beta u\|\right)\\ &\left|\int_\Omega (u\cdot\nabla u)\Lambda^{2s_0+2\alpha}u\mathrm{d}x\right|, \quad \forall t>0.\end{aligned} \tag{95}$$

Similar to (65) and (83), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{s_0+\alpha}u\|^2 + \nu\|\Lambda^{s_0+2\alpha}u\|^2 \le C_1(N,\nu,\alpha,\beta)\|u\|^2 + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s_0}}\\ &\le C_1(N,\nu,\alpha,\beta)\left(\|u_0\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}}\right) + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s_0}}, \quad \forall t>0.\end{aligned} \tag{96}$$

If $s_0 \le s$, we have $u_0 \in \mathbb{H}^{s_0}$. By standard regularity result, we know that $u(t)$ is bounded in $\mathrm{L}^\infty(0,1;\mathbb{H}^{s_0}) \cap \mathrm{L}^2(0,1;\mathbb{H}^{s_0+\alpha})$. Thus, there exists a time $t_0 \in [0,1]$ and a positive constant M_1 such that $\|\Lambda^{s_0+\alpha}u(t_0)\|^2 < M_1$. By using Gronwall's inequality, we deduce from (96) that

$$\begin{aligned}\|\Lambda^{s_0+\alpha}u(t)\|^2 &\le M_1 e^{-\nu\lambda_1^\alpha(t-t_0)} + \frac{1}{\nu\lambda_1^\alpha}\Bigg\{C_0(N,\nu,\alpha,\beta)\\ &\cdot\left(\|u_0\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}}\right) + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s_0}}\Bigg\}, \quad \forall t>t_0.\end{aligned} \tag{97}$$

If $s_0 > s$, since $u_0 \in \mathbb{H}^s$ and $f \in \mathbb{H}^{s_0}$, we know that $u(t)$ is bounded in $L^\infty(0,1;\mathbb{H}^s) \cap L^2(0,1;\mathbb{H}^{s+\alpha})$. Since $s_0 < s+\alpha$, we conclude that there exist a time $t_1 \in [0,1]$ and a positive constant M_2 such that $\|\Lambda^{s_0}u(t_1)\|^2 < M_2$. Let $v(t) = S(t)v_0 = S(t)S(t_1)u_0 = u(t+t_1)$. We know that $v(t)$ is bounded in $L^\infty(0,1;\mathbb{H}^{s_0}) \cap L^2(0,1;\mathbb{H}^{s_0+\alpha})$. Then, there exists a time $t_2 \in [0,1]$ and a positive constant M_2, which does not depend on v, such that $\|\Lambda^{s_0+\alpha}v(t_2)\|^2 < M_2$, i.e., $\|\Lambda^{s_0+\alpha}u(t_2+t_1)\|^2 < M_2$. Denote $t_3 = t_2+t_1, t_3 \in (0,2)$; then, $u(t_3) \in \mathbb{H}^{s_0+\alpha}$. Similar to (97), we have

$$\begin{aligned}\|\Lambda^{s_0+\alpha}u(t)\|^2 &\le M_3 e^{-\nu\lambda_1^\alpha(t-t_3)} + \frac{1}{\nu\lambda_1^\alpha}\Bigg\{C_0(N,\nu,\alpha,\beta)\\ &\cdot\left(\|u_0\|^2 + \frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}}\right) + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s_0}}\Bigg\}.\end{aligned} \tag{98}$$

Since the attractor A is invariant, for any $t>0$ and any $\chi \in A$, there exists a $u_0 \in A$ such that $S(t)u_0 = S(t-t_0)S(t_0)u_0 = u(t) = \chi$ (or $S(t)u_0 = S(t-t_3)S(t_3)u_0 = u(t) = \chi$). We assume t is large enough and takes (81) into consideration to obtain

$$\begin{aligned}\|\Lambda^{s_0+\alpha}\chi\|^2 &= \|\Lambda^{s_0+\alpha}u(t)\|^2\\ &\le \frac{2}{\nu\lambda_1^\alpha}\left\{3C_1(N,\nu,\alpha,\beta)\frac{\|f\|^2}{\nu^2\lambda_1^{2\alpha}} + \frac{2}{\nu}\|f\|^2_{\mathbb{H}^{s_0}}\right\} = C_A^2.\end{aligned} \tag{99}$$

Therefore, A is bounded in $\mathbb{H}^{s_0+\alpha}$ by C_A, which may depend on $\|f\|, \|f\|_{\mathbb{H}}^{s-\alpha}, N, \alpha, \beta$, and ν and the bound in $\mathbb{H}^s$ of the attractor A. □

5. Finite Dimensionality of the Attractor

In this section, we provide the upper bound for the fractal dimension of the attractor derived in Section 4.

Theorem 3. *Assume that* $4\alpha + 2\beta > 5, 0 \le \alpha < 5/4, 0 \le \beta < 3/2$, $u_0 \in \mathbb{H}^s$, *and* $f \in \mathbb{H}^{s-1+\alpha}, s \ge \max\{1, \beta\}$. *Then, the fractal dimension of the global attractor A derived in Theorem 2 is finite.*

To prove Theorem 3, we use the following abstract results derived in [31–33].

Lemma 6. *Let* H_0 *be a separable Hilbert space and let M be a bounded closed set in* H_0. *Assume that there exists a mapping* S_0: $M \mapsto H_0$ *such that* $M \subseteq S_0 M$:

(i) S_0 *is Lipschitz on M, i.e., there exists* $L > 0$ *such that*

$$\left\| \|S_0 v_1 - S_0 v_2\| \right\|_{H_0} \le L \|v_1 - v_2\|_{H_0}, \quad v_1, v_2 \in M. \tag{100}$$

(ii) There exist finite dimension orthoprojectors P_1 *and* P_2 *on* H_0 *such that*

$$\|S_0 v_1 - S_0 v_2\|_{H_0} \le \eta \|v_1 - v_2\|_{H_0} + K\Big(\|P_1 (v_1 - v_2)\|_{H_0} + \|P_2 (v_1 - v_2)\|_{H_0} \Big), \quad \forall\, v_1, v_2 \in M, \tag{101}$$

where $0 < \eta < 1$ *and* $K > 0$ *are constants.*
Then,

$$\dim_f M \le (\dim P_1 + \dim P_2) \ln\left\{1 + \frac{8\sqrt{2}\,(1+L)K}{1-\eta}\right\} \left(\ln \frac{2}{1+\eta}\right)^{-1}. \tag{102}$$

Denote $Z_m = \operatorname{span}\{(e_j - k_j k/|k|^2) e^{ik\cdot x}\colon\ j = 1, 2, 3, |k| = \sqrt{|k_1|^2 + |k_2|^2 + |k_3|^2} \le m\}$, where $k = (k_1, k_2, k_3) \in \mathbb{Z}^3, k \ne 0$, and e_1, e_2, and e_3 represent the canonical basis of $\mathbb{R}^3$. Let P_m: $L^2(\Omega) \mapsto Z_m$ be the projection operator. Similar to Lemma 3.4 in [34] (see also Lemma 2.12 in [31]), we have the following lemma.

Lemma 7. *Let* $\eta \ge 0$ *and* $\iota > 0$. *For any* $\varepsilon > 0$, *there exists a positive integer* $m(\varepsilon)$ *such that for* $m \ge m(\varepsilon)$, *and we have*

$$\|\varphi\|_{H^\eta} \le \varepsilon \|\varphi\|_{H^{\eta+\iota}} + \|P_m \varphi\|_{H^\eta}, \quad \forall \varphi \in H^{\eta+\iota}, \tag{103}$$

where $m(\varepsilon) = [\varepsilon^{-(1/t)}]$, *the integer part of number* $\varepsilon^{-(1/t)}$.

Thanks to Lemma 8 in [24], we have the following.

Lemma 8. *The projection operator* P_m: $L^2(\Omega) \mapsto Z_m$ *has a finite range with*

$$\dim P_m \le 8\big(4m^3 + 6m^2 + 8m + 3\big). \tag{104}$$

Lemma 9. *Assume that* $4\alpha + 2\beta > 5, 0 \le \alpha < 5/4, 0 \le \beta < 3/2$, $u_0 \in \mathbb{H}^s$, *and* $f \in \mathbb{H}^{s_0}, s \ge \min\{1, \beta\}$. *Let* $\mathscr{A}$ *be the global attractor of system (3) derived in Theorem 2 for the smooth solution. Let* $u(t)$ *and* $v(t)$ *be two solutions of system (3) corresponding to the initial data* $u_0, v_0 \in \mathscr{A}$, *respectively. Let* $w(t) = u(t) - v(t)$, *and let* ϑ *be a positive constant such that* $\max\{s, 3/2\} < s - 1 + 2\alpha - \vartheta$. *Then, for any* $s_1 \in [s - 1 + 2\alpha - \vartheta, s - 1 + 2\alpha]$, *we have*

$$\|\Lambda^{s_1} w(t)\|^2 \le \exp\left\{ C(N, \alpha, \beta, \nu) \int_0^t \Big(\|\Lambda^{s+\alpha} u\|^2 + \|\Lambda^{s+\alpha} v\|^2 + 1 \Big) d\tau \right\} \|\Lambda^{s_1} w(0)\|^2, \tag{105}$$

for some positive constant.

Proof. Take the inner product of (50) with $\Lambda^{2s_1} w$, and we know that w satisfies, for any $s - 1 + 2\alpha - \vartheta \le s_1 \le s - 1 + 2\alpha$:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Lambda^{s_1} w\|^2 + \nu \|\Lambda^{s_1+\alpha} w\|^2 &\le F_N\Big(\|\Lambda^\beta u\|\Big) \left| \int_\Omega (u\cdot\nabla w) \Lambda^{2s_1} w \, dx \right| \\ &\quad + F_N\Big(\|\Lambda^\beta u\|\Big) \left| \int_\Omega (w\cdot\nabla v) \Lambda^{2s_1} w \, dx \right| \\ &\quad + \left| \Big\{ F_N\Big(\|\Lambda^\beta u\|\Big) - F_N\Big(\|\Lambda^\beta v\|\Big) \Big\} \int_\Omega (v\cdot\nabla v) \Lambda^{2s_1} w \, dx \right| \\ &\doteq L_1 + L_2 + L_3. \end{aligned} \tag{106}$$

Since A is bounded in $\mathbb{H}^{s_0+\alpha}$, we have

$$\|\Lambda^{s_0+\alpha} u(t)\| \le C_A, \ \|\Lambda^{s_0+\alpha} v(t)\| \le C_A, \forall t \ge 0. \tag{107}$$

For L_1, using Hölder's inequality, the product estimates, Gagliardo–Nirenberg inequality, and Young's inequality, we deduce that

$$
\begin{aligned}
L_1 &\le F_N\left(\left\|\Lambda^\beta u\right\|\right)\left\|\Lambda^{s_1+1-\alpha}(uw)\right\|\left\|\Lambda^{s_1+\alpha}w\right\| \\
&\le C\left\{F_N\left(\left\|\Lambda^\beta u\right\|\right)\left(\|u\|_{L^\infty}\left\|\Lambda^{s_1+1-\alpha}w\right\|+\|w\|_{L^\infty}\left\|\Lambda^{s_1+1-\alpha}u\right\|\right)\left\|\Lambda^{s_1+\alpha}w\right\|\right\} \\
&\le C_1\left\|\Lambda^{s_0+\alpha}u\right\|\left\|\Lambda^{s_1}w\right\|^{\theta_6}\left\|\Lambda^{s_1+\alpha}w\right\|^{2-\theta_6}+C_2\left\|\Lambda^{s+\alpha}u\right\|\left\|\Lambda^{s_1}w\right\|\left\|\Lambda^{s_1+\alpha}w\right\| \\
&\le C_1C_{\mathscr{A}}\left\|\Lambda^{s_1}w\right\|^{\theta_6}\left\|\Lambda^{s_1+\alpha}w\right\|^{2-\theta_6}+C_2\left\|\Lambda^{s+\alpha}u\right\|\left\|\Lambda^{s_1}w\right\|\left\|\Lambda^{s_1+\alpha}w\right\| \\
&\le \frac{\nu}{6}\left\|\Lambda^{s_1+\alpha}w\right\|^2+C_1(N,\alpha,,\beta,\nu)\left(1+\left\|\Lambda^{s+\alpha}u\right\|^2\right)\left\|\Lambda^{s_1}w\right\|^2,
\end{aligned}
\tag{108}
$$

where $\theta_6=\min\{2\alpha-1/\alpha,1\}$. Similarly, we have

$$
L_2\le\frac{\nu}{6}\left\|\Lambda^{s_1+\alpha}w\right\|^2+C_1(N,\alpha,\beta,\nu)\left(1+\left\|\Lambda^{s+\alpha}v\right\|^2\right)\left\|\Lambda^{s_1}w\right\|^2. \tag{109}
$$

Finally, for L_3, we use Hölder's inequality, Lemma 2, product estimates, and imbedding of fractional Sobolev spaces to deduce that

$$
\begin{aligned}
L_3 &\le \left|F_N\left(\left\|\Lambda^\beta u\right\|\right)-F_N\left(\left\|\Lambda^\beta v\right\|\right)\right|\left\|\Lambda^{s_1+1-\alpha}(vv)\right\|\left\|\Lambda^{s_1+\alpha}w\right\| \\
&\le C_3\frac{F_N\left(\left\|\Lambda^\beta u\right\|\right)F_N\left(\left\|\Lambda^\beta v\right\|\right)}{N}\left\|\Lambda^\beta w\right\|\|v\|_{L^\infty}\left\|\Lambda^{s_1+1-\alpha}v\right\|\left\|\Lambda^{s_1+\alpha}w\right\| \\
&\le \frac{C_3}{N}\left\|\Lambda^{s_1}w\right\|\left\|\Lambda^{s_0+\alpha}v\right\|\left\|\Lambda^{s+\alpha}v\right\|\left\|\Lambda^{s_1+\alpha}w\right\| \\
&\le \frac{C_3C_{\mathscr{A}}}{N}\left\|\Lambda^{s_1}w\right\|\left\|\Lambda^{s+\alpha}v\right\|\left\|\Lambda^{s_1+\alpha}w\right\| \\
&\le \frac{\nu}{6}\left\|\Lambda^{s_1+\alpha}w\right\|^2+\frac{3C_3^2C_{\mathscr{A}}^2}{2N^2\nu}\left\|\Lambda^{s+\alpha}v\right\|^2\left\|\Lambda^{s_1}w\right\|^2.
\end{aligned}
\tag{110}
$$

Combining (106)–(110), we obtain that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^{s_1}w\right\|^2\le C(N,\alpha,\beta,\nu)\left(\left\|\Lambda^{s+\alpha}u\right\|^2+\left\|\Lambda^{s+\alpha}v\right\|^2+1\right)\left\|\Lambda^{s_1}w\right\|^2, \tag{111}
$$

with

$$
C(N,\alpha,\beta,\nu)=\max\left\{(C_1C_{\mathscr{A}})^{2/\theta_6}\left(\frac{\nu}{12-6\theta_6}\right)^{\theta_6-2/\theta_6}\frac{\theta_6}{2},\frac{3C_2^2}{\nu},\frac{3C_3^2C_{\mathscr{A}}^2}{2N^2\nu}\right\}. \tag{112}
$$

Gronwall's inequality then implies (105):

$$
\left\|\Lambda^{s_1}w(t)\right\|^2\le\exp\left\{C(N,\alpha,\beta,\nu)\int_0^t\left(\left\|\Lambda^{s+\alpha}u\right\|^2+\left\|\Lambda^{s+\alpha}v\right\|^2+1\right)\mathrm{d}\tau\right\}\left\|\Lambda^{s_1}w(0)\right\|^2\doteq\widetilde{C}(t)\left\|\Lambda^{s_1}w(0)\right\|^2. \tag{113}
$$

Obviously, $\widetilde{C}(t)$ is a monotone function with respect to t, and it is finite for finite time t. □

Lemma 10. *Assume that $4\alpha+2\beta>5$, $\alpha>1/2$, $\beta\ge0$, $u_0\in\mathbb{H}^s$, and $f\in\mathbb{H}^{s_0}$, $s\ge\max\{1,\beta\}$. Let u and v be two solutions of system (3) with initial condition $u_0,v_0\in\mathscr{A}$, respectively. Setting $w=u-v$, then there exists a positive constant C such that $\forall t\ge0$:*

$$
\begin{aligned}
\|w(t)\|_{\mathbb{H}^\gamma}&\le Ce^{-(\nu\lambda_1^\alpha t/2)}\|w(0)\|_{\mathbb{H}^\gamma} \\
&\quad+\nu^{-\epsilon_0}C\left(C_{\mathscr{A}}^2+2C_{\mathscr{A}}\right)\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}},
\end{aligned}
\tag{114}
$$

where $\epsilon_0=1/2\min\{2\alpha-1/2\alpha+1,2\vartheta\}$, $\gamma=s-1+2\alpha$, $\widetilde{C}(t)$ is from (113), and $C_{\mathscr{A}}$ is from (99).

Proof. Similar to (50), we know that w satisfies

$$
\begin{aligned}
&w_t+\nu\Lambda^{2\alpha}w+F_N\left(\left\|\Lambda^\beta u\right\|\right)\mathscr{P}(u\cdot\nabla)w+F_N\left(\left\|\Lambda^\beta u\right\|\right)\mathscr{P}(w\cdot\nabla)v \\
&\quad+\left\{F_N\left(\left\|\Lambda^\beta u\right\|\right)-F_N\left(\left\|\Lambda^\beta v\right\|\right)\right\}\mathscr{P}(v\cdot\nabla)v=0.
\end{aligned}
\tag{115}
$$

By using the Duhamel principle [35], the solution of (114) can be given by

$$w(t) = e^{-\nu(-\Delta)^{\alpha}t}w(0) - \int_0^t e^{-\nu(-\Delta)^{\alpha}(t-\tau)}\Big\{F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big)\mathcal{P}(u\cdot\nabla)w + F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big)\mathcal{P}(w\cdot\nabla)v + \Big(F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big) - F_N\Big(\big\|\Lambda^{\beta}v\big\|\Big)\Big)\mathcal{P}(v\cdot\nabla)v\Big\}\mathrm{d}\tau. \quad (116)$$

And moreover, we have the following estimate of the semigroup $e^{-(-\Delta)^{\alpha}t}$:

$$\left\|e^{-(-\Delta)^{\alpha}t}\right\|_{\mathbb{H}^{\beta''}\mathbb{H}^{\beta'}} \le Ce^{-(\lambda_1^{\alpha}t/2)}t^{-(\beta'-\beta''/2\alpha)}, \quad \beta' \ge \beta'' \ge 0, t > 0. \quad (117)$$

Denote

$$\Psi(\tau) = \left\|F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big)u\cdot\nabla w + F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big)w\cdot\nabla v + \Big\{F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big) - F_N\Big(\big\|\Lambda^{\beta}v\big\|\Big)\Big\}v\cdot\nabla v\right\|_{\mathbb{H}^{\gamma-2(1-\varepsilon)\alpha}}, \quad (118)$$

and then,

$$\|w(t)\|_{\mathbb{H}^{\gamma}} \le Ce^{-\nu\lambda_1^{\alpha}t/2}\|w(0)\|_{\mathbb{H}^{\gamma}} + C\int_0^t e^{-\nu\lambda_1^{\alpha}(t-\tau)/2}(t-\tau)^{-1+\epsilon}\Psi(\tau)\mathrm{d}\tau, \quad (119)$$

and $0<\epsilon<1$ will be determined later. Setting $\sigma = \gamma - 2(1-\epsilon)\alpha$, we have

$$\Psi(\tau) \le F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big)\|u\cdot\nabla w\|_{\mathbb{H}^{\sigma}} + F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big)\|w\cdot\nabla v\|_{\mathbb{H}^{\sigma}} + \Big|F_N\Big(\big\|\Lambda^{\beta}u\big\|\Big) - F_N\Big(\big\|\Lambda^{\beta}v\big\|\Big)\Big|\|v\cdot\nabla v\|_{\mathbb{H}^{\sigma}} \doteq S_1 + S_2 + S_3. \quad (120)$$

Next, we estimate S_1, S_2, and S_3 one by one. By the product estimates of Sobolev spaces, we have

$$S_1 \le CF_N\Big(\big\|\Lambda^{\beta}u\big\|\Big)\big(\|u\|_{L^{p_1}}\|\nabla w\|_{\mathbb{H}^{\sigma,q_1}} + \|u\|_{\mathbb{H}^{\sigma,p_2}}\|\nabla w\|_{L^{q_2}}\big), \quad (121)$$

for positive integers p_i, q_i satisfying $(1/p_i) + (1/q_i) = (1/2), i = 1, 2$. Let $\varepsilon < 2\alpha - 1/2\alpha + 1$; since $4\alpha + 2\beta > 5$, $\alpha > 1/2$ and $s \ge \max\{1, \beta\}$, we have

$$\frac{2\gamma + 4\alpha - 5}{4\alpha + 2} \ge \frac{2(\beta - 1 + 2\alpha) + 4\alpha - 5}{4\alpha + 2} > \frac{2\alpha - 1}{2\alpha + 1} > \varepsilon > 0. \quad (122)$$

Hence, it is easy to check that there exists a pair of positive integers (p_1, q_1) such that $(1/p_1) + (1/q_1) = (1/2)$, and

$$\frac{1}{p_1} \ge \frac{1}{2} - \frac{\gamma}{3}, \qquad \frac{1}{q_1} \ge \frac{1}{2} - \frac{\gamma - \varepsilon - 1 - \sigma}{3}, \qquad \sigma + 1 \le \gamma - \varepsilon. \quad (123)$$

By the Sobolev inequality, we have

$$\|u\|_{L^{p_1}} \le C\|u\|_{\mathbb{H}^{\gamma}}, \qquad \|\nabla w\|_{\mathbb{H}^{\sigma,q_1}} \le C\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}, \quad (124)$$

and thus,

$$\|u\|_{L^{p_1}}\|\nabla w\|_{\mathbb{H}^{\sigma,q_1}} \le C\|u\|_{\mathbb{H}^{\gamma}}\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}. \quad (125)$$

On the contrary, since

$$\gamma - 1 = s - 1 + 2\alpha - 1 \ge 2\alpha - 1 > \frac{2\alpha - 1}{2\alpha + 1} > \varepsilon > 0, \quad (126)$$

rhen we can check that there exists a pair of positive integers (p_2, q_2) such that $(1/p_2) + (1/q_2) = (1/2)$, and

$$\frac{1}{p_2} \ge \frac{1}{2} - \frac{\gamma - \sigma}{3}, \qquad \frac{1}{q_2} \ge \frac{1}{2} - \frac{\gamma - \varepsilon - 1}{3}, \qquad \gamma - \varepsilon > 1. \quad (127)$$

By the Sobolev inequality, we have

$$\|u\|_{\mathbb{H}^{\sigma,p_2}} \le C\|u\|_{\mathbb{H}^{\gamma}}, \qquad \|\nabla w\|_{L^{q_2}} \le C\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}, \quad (128)$$

and hence,

$$\|u\|_{\mathbb{H}^{\sigma,p_2}}\|\nabla w\|_{L^{q_2}} \le C\|u\|_{\mathbb{H}^{\gamma}}\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}. \quad (129)$$

Combining (121)–(129), we obtain

$$S_1 \le C\|u\|_{\mathbb{H}^{\gamma}}\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}, \quad (130)$$

when $\varepsilon \in (0, 2\alpha - 1/2\alpha + 1)$.

Similarly, for S_2, we have

$$S_2 \le C\|v\|_{\mathbb{H}^{\gamma}}\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}. \quad (131)$$

Since $\varepsilon \in (0, 2\alpha - 1/2\alpha + 1), s \ge \max\{1, \beta\}$, we have that $\beta < \gamma - \varepsilon$. For S_3, using Lemma 2, the product estimate, and the Sobolev inequality, it is easy to deduce that

$$S_3 \leq C \frac{F_N\left(\left\|\Lambda^\beta u\right\|\right) F_N\left(\left\|\Lambda^\beta v\right\|\right)}{N}\left\|\Lambda^\beta w\right\|\|v\|_{\mathbb{H}^\gamma}\|v\|_{\mathbb{H}^{\gamma-\varepsilon}} \leq C\|v\|_{\mathbb{H}^\gamma}^2\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}. \tag{132}$$

Now, combining (99) and (130)–(132), we obtain

$$\begin{aligned}(S_1+S_2+S_3) &\leq C\left(\|u\|_{\mathbb{H}^\gamma}+\|v\|_{\mathbb{H}^\gamma}+\|v\|_{\mathbb{H}^\gamma}^2\right)\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}\\ &\leq C\left(C_A^2+2C_A\right)\|w\|_{\mathbb{H}^{\gamma-\varepsilon}}.\end{aligned} \tag{133}$$

Now, set $\varepsilon_0 = 1/2\min\{2\alpha - 1/2\alpha + 1, 2\vartheta\}$. Taking (132) into (119) and using (99) and (105) (note that $s - 1 + 2\alpha - \vartheta \leq \gamma - \varepsilon_0 \leq s - 1 + 2\alpha$), we obtain that

$$\begin{aligned}\|w(t)\|_{\mathbb{H}^\gamma} &\leq C\exp\left\{-\frac{\nu\lambda_1^\alpha t}{2}\right\}\|w(0)\|_{\mathbb{H}^\gamma} + C\left(C_{\mathrm{A}}^2+2C_{\mathrm{A}}\right)\int_0^t \exp\left\{\frac{-\nu\lambda_1^\alpha(t-\tau)}{2}\right\}(t-\tau)^{-1+\epsilon_0}\|w(\tau)\|_{\mathbb{H}^{\gamma-\epsilon_0}}d\tau\\ &\leq C\exp\left\{-\frac{\nu\lambda_1^\alpha t}{2}\right\}\|w(0)\|_{\mathbb{H}^\gamma} + C\left(C_{\mathrm{A}}^2+2C_{\mathrm{A}}\right)\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}}\int_0^t \exp\left\{\frac{-\nu\lambda_1^\alpha(t-\tau)}{2}\right\}(t-\tau)^{-1+\epsilon_0}d\tau\\ &\leq C\exp\left\{-\frac{\nu\lambda_1^\alpha t}{2}\right\}\|w(0)\|_{\mathbb{H}^\gamma} + \Gamma(\epsilon_0)\nu^{-\epsilon_0}C\left(C_{\mathrm{A}}^2+2C_{\mathrm{A}}\right)\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}},\end{aligned} \tag{134}$$

where $\Gamma(\cdot)$ is the Gamma function. This completes the proof of Lemma 10. □

Proof. of Theorem 3. Combining with Lemmas 6–10, we can give an upper bound on the fractal dimension of the attractor derived in Theorem 2 as follows. Let w be a solution of (115). It follows from Lemma 7 and Lemma 10 that

$$\begin{aligned}\|w(t)\|_{\mathbb{H}^\gamma} &\leq C\exp\left\{-\frac{\nu\lambda_1^\alpha t}{2}\right\}\|w(0)\|_{\mathbb{H}^\gamma} + C\nu^{-\epsilon_0}\left(C_{\mathrm{A}}^2+2C_{\mathrm{A}}\right)\widetilde{C}(t)\|w(0)\|_{\mathbb{H}^{\gamma-\epsilon_0}},\\ &\leq \left(C\exp\left\{-\frac{\nu\lambda_1^\alpha t}{2}\right\} + \varepsilon\nu^{-\epsilon_0}C\left(C_{\mathrm{A}}^2+2C_{\mathrm{A}}\right)\widetilde{C}(t)\right)\|w(0)\|_{\mathbb{H}^\gamma}\\ &\quad + \nu^{-\epsilon_0}C\left(C_{\mathrm{A}}^2+2C_{\mathrm{A}}\right)\widetilde{C}(t)\left\|P_m w(0)\right\|_{\mathbb{H}^{\gamma-\epsilon_0}},\end{aligned} \tag{135}$$

with $m \geq m(\varepsilon) = [\varepsilon^{-(1/\varepsilon_0)}]$ and $\varepsilon_0 = 1/2\min\{2\alpha - 1/2\alpha + 1, 2\vartheta\}$. After some elementary calculations, we can choose

$$\begin{aligned}t_0 &= \frac{2\ln 4C}{\nu\lambda_1^\alpha},\\ \varepsilon &= \frac{1}{4}\nu^{\varepsilon_0}C^{-1}\left(C_A^2+2C_A\right)^{-1}\widetilde{C}(t_0)^{-1},\end{aligned} \tag{136}$$

such that

$$\begin{aligned}C\exp\left\{-\frac{\nu\lambda_1^\alpha t_0}{2}\right\} &= \frac{1}{4},\\ \varepsilon\nu^{-\varepsilon_0}C\left(C_A^2+2C_A\right)\widetilde{C}(t_0) &= \frac{1}{4},\end{aligned} \tag{137}$$

with

$$\widetilde{C}(t_0) = \exp\left\{C_2(N,\alpha,\beta,\nu)\int_0^{t_0}\left(\left\|\Lambda^{s+\alpha}u\right\|^2 + \left\|\Lambda^{s+\alpha}v\right\|^2 + 1\right)d\tau\right\}. \tag{138}$$

Combining with (135) and (137), we have

$$\left\|w(t_0)\right\|_{\mathbb{H}^\gamma} \leq \frac{1}{2}\|w_0\|_{\mathbb{H}^\gamma} + \nu^{-\varepsilon_0}C\left(C_A^2+2C_A\right)\widetilde{C}(t_0)\left\|P_m w(0)\right\|_{\mathbb{H}^{\gamma-\epsilon_0}}. \tag{139}$$

Meanwhile, it follows from Lemma 9 that

$$\begin{aligned}&\left\|\Lambda^{s_1}w(t_0)\right\|^2\\ &\leq \exp\left\{C(N,\alpha,\beta,\nu)\int_0^{t_0}\left(\left\|\Lambda^{s+\alpha}u\right\|^2 + \left\|\Lambda^{s+\alpha}v\right\|^2 + 1\right)d\tau\right\}\\ &\quad \left\|\Lambda^{s_1}w(0)\right\|^2 \doteq \widetilde{C}(t_0)\left\|\Lambda^{s_1}w(0)\right\|^2.\end{aligned} \tag{140}$$

Define $S_0 = S(t_0)$: $w(0) \mapsto w(t_0)$, and we can check that S_0 satisfies conditions (i) and (ii) of Lemma 6, and

$$\begin{aligned}
M &= A, \\
H_0 &= \mathbb{H}^{\gamma} = \mathbb{H}^{s-1+2\alpha}, \\
S_0 &= S(t_0)\colon w(0) \mapsto w(t_0), \\
t_0 &= \frac{2 \ln 4C}{\nu \lambda_1^{\alpha}}, \\
\eta &= \frac{1}{2}, \\
K &= \nu^{-\varepsilon_0} C\left(C_{\mathscr{A}}^2 + 2C_{\mathscr{A}}\right)\widetilde{C}(t_0), \\
L &= \widetilde{C}(t_0), \\
m &= \left[C\nu^{-1}\left(C_{\mathscr{A}}^2 + 2C_{\mathscr{A}}\right)^{1/\varepsilon_0}\widetilde{C}(t_0)^{1/\varepsilon_0}\right], \\
\varepsilon_0 &= \frac{1}{2}\min\left\{\frac{2\alpha-1}{2\alpha+1}, 2\vartheta\right\}, \\
\dim(P_2) &= 0, \\
P_1 &= P_m, \\
\dim(P_m) &\leq 8\left(4m^3 + 6m^2 + 8m + 3\right),
\end{aligned} \tag{141}$$

where C_A is from (99), $\widetilde{C}(t_0)$ is from (140), ϑ is the constant in the Lemma 9, C is a universal constant, and $[x]$ denotes the integer part of the number x.

Thus, we have

$$\begin{aligned}
\dim_f A \leq{}& 8\left(4m^3 + 6m^2 + 8m + 3\right) \\
& \ln\left\{1 + 16\sqrt{2}\left(1 + \widetilde{C}(t_0)\right)\nu^{-\varepsilon_0} C\left(C_A^2 + 2C_A\right)\widetilde{C}(t_0)\right\}\left(\ln\frac{4}{3}\right)^{-1}.
\end{aligned} \tag{142}$$

□

6. Conclusions

The authors established the existence of weak solutions under proper assumptions on α and β. The existence of finite-dimensional global attractors is also obtained. It is possible to study the global modified MHD equations by using similar ideas herein.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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