Research Article

Longitudinal Stability Criteria for a Propeller-Driven Aircraft

Gil Iosilevskii

Faculty of Aerospace Engineering, Technion, Haifa 32000, Israel

Correspondence should be addressed to Gil Iosilevskii, igil@aerodyne.technion.ac.il

Received 1 September 2010; Accepted 30 November 2010

Academic Editor: Isaac Kaminer

Copyright © 2010 Gil Iosilevskii. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The Routh criterion is used to assess longitudinal dynamic stability of a propeller-driven aircraft. Under a few plausible assumptions on possible ranges of the pertinent stability derivatives, it reduces to a pair of simple conditions imposing a traditional aft limit (the forward of the maneuver and the neutral-speed-stability points) on the center-of-gravity position and an upper limit on the longitudinal moment of inertia. It is demonstrated that most aircraft have sufficiently small inertia to remain stable as long as their center-of-gravity is properly placed. At the same time, sailplane-like aircraft (as, e.g., long endurance UAVs), with an engine installed at the rear extremity of the aircraft, may have sufficiently high inertia to become unstable regardless of their center-of-gravity placement.

1. Introduction

Linear analysis of the longitudinal dynamics of a rigid aircraft can be found practically in any textbook on flight mechanics. In a nutshell, it can be recapitulated as follows. Assuming constant-density atmosphere, flat nonrotating Earth, zero thrust angle, and no control inputs, the equations governing small perturbations $\mathbf{x} = [u, \alpha, q, \theta]^T$ of airspeed, angle-of-attack, pitch rate and pitch angle from their respective values in a trimmed straight-and-almost-level flight can be written as

$$\mathbf{B} \cdot \dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x}, \quad (1)$$

with

$$\mathbf{A} = \begin{bmatrix} C_{m,u} & C_{m,\alpha} & C_{m,q} & 0 \\ C_{L,u} + 2C_L & C_D + C_{L,\alpha} & C_{L,q} - 2\mu & \gamma C_L \\ C_{T,u} - C_{D,u} + 2\gamma C_L & C_L - C_{D,\alpha} & 0 & -C_L \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (2)$$

$$\mathbf{B} = \begin{bmatrix} 0 & -C_{m,\dot{\alpha}} & i_y & 0 \\ 0 & -(2\mu + C_{L,\dot{\alpha}}) & 0 & 0 \\ 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad (3)$$

see equation (5.13,16) in [1]. The notation here is the same as in [1] with three exceptions: (i) an overdot replaces “ D ” to designate a derivative with respect to reduced time; (ii) the subscript “e” is omitted from unperturbed (equilibrium) values of the flight-path angle and the lift and drag coefficients; (iii) “ i_y ” and “ u ” replace “ $\hat{I}_y$ ” and “ $\Delta \hat{V}$ ” for the reduced moment of inertia and the reduced velocity.

Setting $\mathbf{x} = \mathbf{x}_0 e^{st}$ in (1), where t is the (reduced) time and $\mathbf{x}_0$ and s are obvious time-independent quantities, leads to the generalized eigenvalue problem $(s\mathbf{B} - \mathbf{A}) \cdot \mathbf{x}_0 = 0$, and, in turn, to the fourth order characteristic equation,

$$as^4 + bs^3 + cs^2 + ds + e = 0. \quad (4)$$

The coefficients in (4) can be identified with products of the elements of the matrices **A** and **B**, but except for

$$a = 2i_y\mu(2\mu + C_{L,\dot{\alpha}}), \quad (5)$$

all other coefficients are rather unwieldy (Appendix A). Of course, the aircraft is stable if all the roots of (4) are in the left half-plane.

Equation (4) has no analytical solution for s , and therefore it has to be solved either approximately [2] or numerically. Each approach has its virtues, but in many cases the roots of (4) are not really needed—what is needed is the knowledge whether the aircraft is stable or not. In principle, stability of a system (the aircraft, in this case) can be assessed without actually solving its characteristic equation—by the Routh criterion, the number of roots of (4) which are in the right-half-plane equals the number of sign alternations in the five-member sequence $S = \{a, b, f, g, e\}$, where $f = c - ad/b$ and $g = d - be/f$ [3, pages 253–255]. In practice, however, this approach is hardly ever used with (4)—the unwieldiness of the expressions for the members of S renders it rather unattractive as compared with the instant numerical solution of the generalized eigenvalue problem leading to (4).

Notwithstanding the majority approach, we suggest that for a propeller-driven aircraft, an application of the Routh criterion to (4) yields a pair of very simple conditions with immediate design implications. One of these conditions sets a traditional aft limit (the forward of the maneuver and the neutral-speed-stability points) on the center-of-gravity position; the other sets an upper limit on the longitudinal moment of inertia. The details follow.

2. Approximate Stability Criterion

As a first step, we replace the moment derivatives $C_{m,\bullet}$ in (2) and (3) (or, to the same end, in (A.1)–(A.5) of Appendix A), all referred to the center of gravity, with combinations

$$C_{m,\bullet} = C_{M,\bullet} - C_{L,\bullet}(x_{cg} - x_n) \quad (6)$$

of their neutral-point-referred counterparts, $C_{M,\bullet}$, and lift coefficient derivatives, $C_{L,\bullet}$. Here, x_{cg} and x_n are the center-of-gravity and neutral-point positions, and “ $\bullet$ ” stands for “ α ,” “ $\dot{\alpha}$,” “ q ,” and “ u .” Among the neutral-point-referred moment derivatives,

$$C_{M,\alpha} = 0, \quad (7)$$

$$C_{M,q} = 2\mu(x_m - x_n), \quad (8)$$

$$C_{M,u} = -2C_L(x_s - x_n), \quad (9)$$

by definitions of the neutral (x_n), maneuver (x_m), and neutral-speed-stability (x_s) points; the last pair follows equations (6.10, 8) and (6.4, 26) of [1] by (6).

These substitutions do simplify the expressions for $b, \dots, e$, but essentially do not reduce the number of terms comprising them. Hence, as a next step, we sort these terms

by their magnitudes and retain a few dominant ones only. For a typical subsonic aircraft in an almost level flight, invariably

$C_D, C_{D,u}, C_{D,\alpha}, C_{T,u}, C_{L,u}, C_{M,u}, \gamma$ are of the order 10^{-1} ,

C_L is of the order 10^0 ,

$C_{L,\alpha}, C_{L,\dot{\alpha}}, C_{L,q}, C_{M,\dot{\alpha}}$ and $C_{M,q}$ are of the order 10^1 ,

μ is of the order 10^2 ,

i_y is of the order 10^2 or 10^3 .

(10)

The word “order” can be understood here (and below) in its intuitive sense; nevertheless, we shall remove possible ambiguity and define the order $O(a)$ of a quantity a as $10^{\text{round}(\log_{10}|a|)}$. Thus, 4 and 30 are both of the order 10^1 .

Sorting out the terms in (A.2)–(A.5) by (10), and discarding all those that are of the order 10^{-2} (or smaller) as compared with the leading ones, leaves

$$b = 2\mu(i_y(C_{L,\alpha} + O(10^{-1})) - (2\mu - C_{L,q})(C_{M,q} + C_{M,\dot{\alpha}}) + 2\mu(C_{L,q} + C_{L,\dot{\alpha}})(x_{cg} - x_m)), \quad (11)$$

$$c = 4\mu^2 C_{L,\alpha}(x_{cg} - x'_m)(1 + O(10^{-3})), \quad (12)$$

$$d = -C_L((C_{M,q} + C_{M,\dot{\alpha}})(2C_L + C_{L,u}) + \gamma C_{L,\alpha} C_{M,q} + O(10^{-1})) + 2C_L(x_{cg} - x_s)((C_{L,q} + C_{L,\dot{\alpha}})C_L + 2\mu C_{D,\alpha} + O(1)) - 2\mu(x_{cg} - x_m) \times (C_{L,\alpha}(C_{T,u} - C_{D,u} + 3\gamma C_L) + 2C_{D,\alpha} C_L + O(10^{-2})) + O(10^{-2}), \quad (13)$$

$$e = 2C_L^2 C_{L,\alpha}(x_{cg} - x'_s)(1 + O(10^{-2})), \quad (14)$$

where

$$x'_m = x_m + (x_m - x_n)O(10^{-2}i_y/\mu), \quad (15)$$

$$x'_s = x_s + (x_s - x_n)O(10^{-2}). \quad (16)$$

Since $x_m - x_n$ and $x_s - x_n$ are of the order 10^{-1} by (8)–(10), therefore $x'_s = x_s + O(10^{-3})$ and $x'_m = x_m + O(10^{-3}i_y/\mu)$. Full expression for x'_s can be found in (A.7)–(A.9).

It can be verified by direct substitution that the ratio b/a is of the order 10^{-1} , whereas the ratios c/a , d/a , and e/a are of the orders $10(x_{cg} - x'_m)i_y^{-1}$, $10^{-3}i_y^{-1}$, and $10^{-3}i_y^{-1}(x_{cg} - x'_s)$, respectively. Hence, as long as

$$x_{cg} - x_m \text{ is of the order } 10^{-1}, \quad (17)$$

the third and the fourth terms in the Routh sequence, $f = c - ad/b$ and $g = d - be/f$, can be approximated with c and $d - be/c$; the error of these approximations should be of the

order of 10^{-2} relative to the respective leading terms. Explicit expression for c is found in (12); explicit (approximate) expression for g follows from (8) and (11)–(16). It can be regrouped in various forms; the following two,

$$g = C_{L,\alpha} C_L^2 \left(i_2 \bar{X} + (i_1 + i_3 - \bar{i}_y) + (i_0 - i_3 + \bar{i}_y) \bar{X}^{-1} \right) \quad (18a)$$

and

$$g = C_{L,\alpha} C_L^2 \frac{\bar{X} - 1}{\bar{X}} \times \left(\frac{i_2 (\bar{X} - 1)^2 + (2i_2 + i_1 + i_3) (\bar{X} - 1) + (i_0 + i_1 + i_2)}{\bar{X} - 1} - \bar{i}_y \right), \quad (18b)$$

will serve the needs of the following discussion. Here,

$$\bar{X} = \frac{x_{cg} - x_m}{x_s - x_m}, \quad (19)$$

$$i_0 = 4\mu k_q \frac{x_n - x_m}{C_{L,\alpha}} = -\frac{2k_q C_{M,q}}{C_{L,\alpha}}, \quad (20)$$

$$i_1 = \frac{2\mu}{C_{L,\alpha} C_L} \left((k_q C_{L,u} + \gamma C_{L,\alpha}) (x_n - x_m) - 2C_{D,\alpha} (x_s - x_m) \right), \quad (21)$$

$$i_2 = \frac{2\mu}{C_L^2} (C_{D,u} - C_{T,u} - 3\gamma C_L) (x_s - x_m), \quad (22)$$

$$i_3 = i_0 \frac{C_{L,q}}{2\mu} = -\frac{k_q C_{M,q} C_{L,q}}{\mu C_{L,\alpha}}, \quad (23)$$

$$\bar{i}_y = \frac{i_y}{\mu}, \quad (24)$$

and in which

$$k_q = 1 + \frac{C_{M,\dot{\alpha}}}{C_{M,q}}. \quad (25)$$

By the Routh criterion, the number of roots in the right-half-plane equals the number of the sign alternations in S . Since, invariably, $C_{M,q} < 0$, $C_{M,\dot{\alpha}} < 0$, and $C_{L,\alpha} > 0$, the first two members of S , a , and b , are positive by (5), (11), (10) and (17). Hence, the aircraft is stable only if the remaining members, f , g , and e , are positive as well. The first of these conditions yields $x_{cg} > x'_m$ (see (12)), or, approximately,

$$x_{cg} > x_m. \quad (26a)$$

In view of (17) this condition is almost redundant. We shall assume that it is satisfied unconditionally. The second of these conditions,

$$g > 0, \quad (26b)$$

will be discussed in the following sections. The last of these conditions yields $x_{cg} > x'_s$ (see (14)), or, approximately,

$$x_{cg} > x_s. \quad (26c)$$

It suggests that the center of gravity should be placed forward of the neutral speed stability point. There is nothing new in this result [1, page 335], but it is satisfying to obtain it here. In the analysis of (26b) we shall assume that this condition is satisfied as well. In conjunction with (26a) and (19) it implies that either $\bar{X} > 1$ or $\bar{X} < 0$.

Note that failing (26b), but not (26a) and (26c), implies two sign alternations in the Routh sequence and hence two divergent (phugoid) roots; failing (26c) only, or both (26c) and (26b), implies one divergent root.

3. Condition (26b)

The consequences of (26b) can be inferred from either (18a) or (18b), depending on the variable that is considered given. For example, the conjunction of (26b) and (18b) sets an upper limit on $\bar{i}_y$,

$$\bar{i}_y < \bar{J}(\bar{X}), \quad (27)$$

where

$$\bar{J}(\bar{X}) = i_2 (\bar{X} - 1) + (2i_2 + i_1 + i_3) + \frac{i_0 + i_2 + i_1}{\bar{X} - 1}. \quad (28)$$

Conversely, given $\bar{i}_y$, the conjunction of (26b) and (18a) sets limits on $\bar{X}$:

$$\text{if } i_2 > 0 \text{ and } \bar{i}_y \geq \bar{J}_0, \text{ then } \bar{X}_-(\bar{i}_y) < \bar{X} < 0 \text{ or} \quad (29a)$$

$$0 < \bar{X} < \bar{X}_-(\bar{i}_y) \text{ or } 0 < \bar{X}_+(\bar{i}_y) < \bar{X};$$

$$\text{if } i_2 > 0 \text{ and } \bar{i}_y < \bar{J}_0, \text{ then } 0 < \bar{X}; \quad (29b)$$

$$\text{if } i_2 < 0, \text{ then } \bar{X} < \bar{X}_+(\bar{i}_y) < 0 \text{ or } 0 < \bar{X} < \bar{X}_-(\bar{i}_y). \quad (29c)$$

Here,

$$\bar{X}_\pm(\bar{i}_y) = \frac{-(i_1 + i_3 - \bar{i}_y) \pm \sqrt{(i_1 + i_3 - \bar{i}_y)^2 - 4i_2(i_0 - i_3 + \bar{i}_y)}}{2i_2} \quad (30)$$

are the values of $\bar{X}$ for which $g = 0$, whereas

$$\bar{J}_0 = i_1 + i_3 + 2i_2 + 2\sqrt{i_2(i_0 + i_1 + i_2)} \quad (31)$$

is the value of $\bar{i}_y$ for which the square root in (30) vanishes with $i_2 > 0$. Note that $\bar{J}_0 = \min_{\bar{X} > 1} \bar{J}(\bar{X})$, by (28).

4. An Example

The use of (27) and (29) can be elucidated on a practical example. Consider a fictitious long-endurance UAV. For this aircraft we adopt a standard drag-lift relation

$$C_D = C_{D0} + KC_L^2; \quad (32)$$

neglect $C_{D,u}$, $C_{L,u}$, and $C_{M,u}$; and set

$$C_{T,u} = -k_T(C_D + \gamma C_L), \quad (33)$$

where k_T is the scaling exponent of the propulsion system. It equals 2 for an ideal (constant thrust) jet and 3 for an ideal (constant power) propeller in [1, page 263]. Values of C_{D0} , K , μ , and the relevant stability derivatives are listed in Table 1.

Using this data, equation (4) was solved numerically with many combinations of lift coefficient, stability margin ($C_{M,\alpha}$), and reduced moment of inertia. Differentiating the stable and unstable combinations yielded the stability boundaries drawn by the thick lines in Figure 1. Approximate boundaries have been constructed as contours $\bar{J}(\bar{X}) = \bar{i}_y$ (see (27)) using (28), (8), and (19)–(25); they appear as thin lines on the same figure. The fit between the two seems fair.

With no u -derivatives, $x_s = x'_s = x_n$, by (9) and (16), and since $C_{M,q} < 0$, therefore $x_s - x_m > 0$, by (8). Thus, $\bar{X} > 1$, by (19) and (26a). Moreover,

$$i_1 = -\frac{C_{M,q}}{C_L} \left(\gamma - 2 \frac{C_{D,\alpha}}{C_{L,\alpha}} \right), \quad (34)$$

$$i_2 = -\frac{C_{M,q}}{C_L} \left(k_T \frac{C_D}{C_L} + (k_T - 3)\gamma \right), \quad (35)$$

by (21) and (22).

With $k_T = 2$ (an ideal jet), i_2 turns negative once γ exceeds $2C_D/C_L$. In this case (29c) applies, inferring that in a (sufficiently) steep climb, the aircraft turns unstable with $\bar{X} > \bar{X}_-(\bar{i}_y)$. This behavior is explicit in Figure 1(b). An exhaustive discussion of the stability of a climbing jet can be found on pages 329–346 in [1] and hence will not be repeated here.

With $k_T = 3$ (an ideal propeller), i_2 is positive unconditionally. In this case, (27) applies, inferring that an ideal-propeller-driven aircraft turns unstable for sufficiently large moment of inertia. This behavior is explicit in Figures 1(c) and 1(d). Since i_0 , i_2 , and i_3 are independent of γ by (20), (23), and (35) and i_1 increases with γ by (34), it follows from (28) that $\bar{J}(\bar{X})$ increases with γ as well. Consequently, the lowest (and hence the most constricting) limit on the longitudinal moment of inertia of a propeller-driven aircraft is obtained in a power-off glide, where $\gamma = -C_D/C_L$. Indeed, the curves of constant i_y in Figure 1(d) are lower than those in Figure 1(c).

In a glide,

$$\begin{aligned} \bar{J}(\bar{X}) = & -KC_{M,q} \left(\left(3 + \frac{3C_{D0}}{KC_L^2} \right) (\bar{X} - 1) \right. \\ & + \left(1 + \frac{5C_{D0}}{KC_L^2} + \frac{2k_q}{KC_{L,\alpha}} \frac{C_{L,q}}{2\mu} \right) \\ & \left. + \left(\frac{2k_q}{KC_{L,\alpha}} - 2 + \frac{2C_{D0}}{KC_L^2} \right) \frac{1}{\bar{X} - 1} \right), \end{aligned} \quad (36)$$

by (28), (32)–(35), (20), and (23). Since $\bar{X} > 1$ (see above), $\bar{J}(\bar{X})$ decreases with increasing lift coefficient. Indeed, the lines of constant i_y in Figure 1(d) are higher for lower values

TABLE 1: Dimensionless stability derivatives of the model aircraft.

Quantity	Value	Quantity	Value
C_L	variable	$C_{M,u}$	0
$C_{L,\alpha}$	6.3	C_{D0}	0.03
$C_{L,\dot{\alpha}}$	1	K	0.03
$C_{L,q}$	4.5	$C_{D,\alpha}$	$0.38C_L$
$C_{L,u}$	0	$C_{D,u}$	0
$C_{M,\alpha}$	variable	$C_{T,u}$	variable
$C_{M,\dot{\alpha}}$	-4.0	μ	150
$C_{M,q}$	-20.0	i_y	variable

of i_y . Accordingly, the most constricting limit on the moment of inertia in a glide is obtained when the aircraft flies at the highest possible lift coefficient.

5. Maximal Moment of Inertia

Let $\bar{J}_1 = \min_{(\bar{X} > 1, C_L > 0)} \bar{J}(\bar{X})$ be the minimum of $\bar{J}(\bar{X})$ in a glide with respect to both $\bar{X}$ and C_L . Under the same assumptions as those of the preceding section (the parabolic drag polar and the neglect of the u -derivatives) it is given by

$$\bar{J}_1 = -2C_{M,q}P, \quad (37)$$

where

$$P = K \left(\frac{1}{2} + \frac{k_q}{KC_{L,\alpha}} \frac{C_{L,q}}{2\mu} + \sqrt{\frac{6k_q}{KC_{L,\alpha}} - 6} \right). \quad (38)$$

This minimum corresponds to $C_L \rightarrow \infty$ (which is, of course, beyond reach) and

$$\bar{X} = 1 + \sqrt{\frac{2k_q/(KC_{L,\alpha}) - 2 + 2C_{D0}/(KC_L^2)}{3 + 3C_{D0}/(KC_L^2)}}. \quad (39)$$

Clearly, an aircraft which is designed with sufficiently small moment of inertia to satisfy $\bar{i}_y < \bar{J}_1$ and with sufficiently forward center-of-gravity to satisfy (26c) (and (26a)) will be stable at all flight conditions.

It is shown in Appendix B that P depends mainly on the aspect ratio of the wing. Since the aspect ratio is dictated by performance (rather than stability) requirements, $C_{M,q}$ remains the main design parameter affecting the maximal moment of inertia for which the aircraft is stable.

For a conventional subsonic aircraft, $C_{M,q}$ is associated mainly with the contribution of its horizontal tail; the contribution of the other parts is small and has the same (negative) sign as the tail. Hence, the contribution of the tail,

$$C_{M,q} = -2a_t(x_n - x_t)^2 \left(\frac{S_t}{S_w} \right), \quad (40)$$

provides a lower bound on $|C_{M,q}|$. Here, S_t/S_w is the tail to wing areas ratio, $x_n - x_t$ is the distance between the aircraft neutral point and the aerodynamic center of the tail, and a_t is the lift slope coefficient of the tail [1, page 269].

FIGURE 1: Stability boundaries for the model aircraft. Combinations of lift coefficient and stability margin are unstable if situated to the right (b) or above (a, c, d) the respective line. Thick lines mark the exact boundaries obtained from multiple solutions of (4); thin lines are solutions of $\bar{J}(\bar{X}) = \bar{i}_y$.

$\bar{i}_y$ can be related with longitudinal (reduced) radius of gyration, r_y . In fact, $\bar{i}_y = 4r_y^2$ (see the Nomenclature). Hence, one may set $\bar{J}_1 = 4R_1^2$ and use R_1 as a conservative upper bound of r_y ;

$$R_1 = (x_n - x_t) \sqrt{\frac{a_t P S_t}{S_w}}, \quad (41)$$

by (37), (38), and (40). With $a_t = 5$, $S_t/S_w = 6$, and the data of our model aircraft, R_1 equals, approximately, $0.4(x_n - x_t)$ —almost half the distance between the tail and the center-of-gravity (it is implied here that $(x_{cg} - x_n) \ll (x_n - x_t)$). This result marks out a single configuration which may be at risk of becoming unstable because of the excessive inertia. This

is the configuration with an aft engine, counterbalanced by the equipment in the nose. All conventional configurations of propeller-driven aircraft should be stable as long as their center-of-gravity is properly placed.

6. Concluding Remarks

The key element underlying the preceding derivations is that, under assumptions (10) and (17), the coefficients $a, \dots, e$ of the characteristic polynomial can be approximated by a few dominant terms only. These dominant terms were used in (11)–(14) and (18) to approximate the respective members in the Routh sequence, leading to stability conditions (26b) and (26c). The former sets an upper limit on the longitudinal

moment of inertia; the latter sets an aft limit on the center-of-gravity position.

Perhaps the most conspicuous result of this study is that most configurations of propeller-driven aircraft are practically unlimited by inertia and hence are stable as long as their center-of-gravity is placed forward of the neutral-speed-stability (and maneuver) points—condition (26c). At the same time, aircraft with high-aspect ratio wing and an aft engine are at risk of becoming unstable because of an excessive inertia—condition (26b). The critical moment of inertia (28) depends on the center-of-gravity position and the flight conditions, but $\bar{J}_1$ from (37) may provide an early warning when longitudinal inertia becomes an issue.

Appendices

A. Coefficients of the Characteristic Polynomial

The coefficients in (4) are

$$a = 2i_y\mu(2\mu + C_{L,\dot{\alpha}}), \quad (\text{A.1})$$

$$b = 2\mu \left(-2\mu (C_{m,q} + C_{m,\dot{\alpha}}) + i_y (C_{L,\alpha} + C_D) \right. \\ \left. + (C_{m,\dot{\alpha}}C_{L,q} - C_{L,\dot{\alpha}}C_{m,q}) \right) \\ - i_y (2\mu + C_{L,\dot{\alpha}}) (C_{T,u} - C_{D,u} + 2\gamma C_L), \quad (\text{A.2})$$

$$c = 2\mu \left(- (2\mu - C_{L,q}) C_{m,\alpha} - C_{m,q} (C_{L,\alpha} + C_D + \gamma C_L) \right. \\ \left. + (C_{T,u} - C_{D,u} + 3\gamma C_L) (C_{m,q} + C_{m,\dot{\alpha}}) \right) \\ + i_y \left((2C_L + C_{L,u}) (C_L - C_{D,\alpha}) \right. \\ \left. - (C_{L,\alpha} + C_D) (C_{T,u} - C_{D,u} + 2\gamma C_L) \right) \\ - (C_{T,u} - C_{D,u} + 2\gamma C_L) (C_{m,\dot{\alpha}}C_{L,q} - C_{m,q}C_{L,\dot{\alpha}}), \quad (\text{A.3})$$

$$d = - (2C_L + C_{L,u}) \left(C_L (C_{m,q} + C_{m,\dot{\alpha}}) - C_{m,q}C_{D,\alpha} \right) \\ + (C_{T,u} - C_{D,u} + 2\gamma C_L) \\ \times \left(-\gamma C_L C_{m,\dot{\alpha}} + (C_D + C_{L,\alpha}) C_{m,q} + (2\mu - C_{L,q}) C_{m,\alpha} \right) \\ + C_{m,u} \left(C_L (C_{L,q} + C_{L,\dot{\alpha}}) + (2\mu - C_{L,q}) C_{D,\alpha} \right) \\ + 2\mu C_L \gamma C_{m,\alpha}, \quad (\text{A.4})$$

$$e = -C_{m,\alpha} C_L (2C_L + C_{L,u} + (C_{T,u} - C_{D,u} + 2C_L \gamma) \gamma) \\ + C_{m,u} C_L (C_{L,\alpha} + C_D + (C_L - C_{D,\alpha}) \gamma). \quad (\text{A.5})$$

Using (6), (7), and (9), the last expression can be recast as

$$e = 2C_L^2 C_{L,\alpha} (x_{cg} - x'_s) \\ \times \left(1 - \frac{C_D C_{L,u}}{2C_L C_{L,\alpha}} + \gamma \left(\gamma - \frac{C_{L,u} (C_L - C_{D,\alpha})}{2C_L C_{L,\alpha}} + \frac{C_{T,u} - C_{D,u}}{2C_L} \right) \right), \quad (\text{A.6})$$

where

$$x'_s = x_n - \frac{C_{M,u}}{2C_L} \frac{C_1}{C_2}, \quad (\text{A.7})$$

and in which

$$C_1 = 1 - \frac{C_D (C_L + C_{L,u})}{C_L C_{L,\alpha}} \\ + 2\gamma \left(\gamma - \frac{(C_L - C_{D,\alpha}) (C_L + C_{L,u})}{2C_L C_{L,\alpha}} + \frac{(C_{T,u} - C_{D,u})}{2C_L} \right), \quad (\text{A.8})$$

$$C_2 = 1 - \frac{C_D C_{L,u}}{2C_L C_{L,\alpha}} \\ + \gamma \left(\gamma - \frac{(C_L - C_{D,\alpha}) C_{L,u}}{2C_L C_{L,\alpha}} + \frac{(C_{T,u} - C_{D,u})}{2C_L} \right) \quad (\text{A.9})$$

B. Parameter P

In this appendix we seek the leading-order approximation for P , which was defined in (38). It depends on four parameters: k_q , $C_{L,\alpha}$, K , and the ratio $C_{L,q}/\mu$. The former is given by (25). Neglecting the respective contributions of the horizontal tail, and assuming that the wing loading is elliptic, the simplest approximations for the next two are

$$C_{L,\alpha} = \frac{2\pi A_w}{A_w + 2}, \quad (\text{B.1})$$

$$K = \frac{1}{\pi A_w}, \quad (\text{B.2})$$

where A_w is the aspect ratio of the wing [4, page 130]. With these,

$$P = \frac{C_{L,q}}{4\pi\mu} \left(1 + \frac{C_{M,\dot{\alpha}}}{C_{M,q}} \right) \\ + \frac{1}{\pi} \sqrt{\frac{3}{A_w} + \frac{3}{A_w} \frac{C_{M,\dot{\alpha}}}{C_{M,q}} + \frac{6}{A_w^2} \frac{C_{M,\dot{\alpha}}}{C_{M,q}}} \\ + \frac{1}{2\pi A_w} \left(1 + \frac{C_{L,q}}{\mu} + \frac{C_{L,q}}{\mu} \frac{C_{M,\dot{\alpha}}}{C_{M,q}} \right). \quad (\text{B.3})$$

The ratio $C_{M,\dot{\alpha}}/C_{M,q}$ is, approximately, the downwash derivative at the tail [1, pp. 269, 284], about 0.3. The ratio $C_{L,q}/\mu$ for a conventional subsonic aircraft (ultralights excluded) is of the order of 10^{-1} —see (10) and (10); the aspect ratio A_w is of the order of 10^1 . Hence, the leading-order (10^{-1}) term in (B.3) is $\pi^{-1} \sqrt{3A_w^{-1}}$.

Nomenclature

a_t :	Lift slope coefficient of the horizontal tail
$a, \dots, e$:	Coefficients in the characteristic equation
A_w :	Aspect ratio of the wing
c_w :	Mean aerodynamic chord of the wing
C_D, C_L, C_T :	Drag, lift, and thrust coefficients at trim; the unit is $(1/2)\rho V^2 S_w$
C_{D0} :	Parasite (zero lift) drag coefficient
C_m, C_M :	Pitching moment coefficients; the former is referred to the center of gravity, the latter is referred to the neutral point; the unit is $(1/2)\rho V^2 S_w c_w$
f, g :	The third and the fourth terms in the Routh sequence
$i_0, \dots, i_3$:	Combinations of stability derivatives
$i_y, \bar{i}_y$:	Reduced moment of inertia; the units are $(1/8)\rho S_w c_w^3$ and $(1/8)\rho S_w c_w^3 \mu$
$\bar{J}(\bar{X})$:	Critical value of $\bar{i}_y$ for a given $\bar{X}$
$\bar{J}_0, \bar{J}_1$:	$\min_{\bar{X}>1} \bar{J}(\bar{X})$ and $\min_{(\bar{X}>1, C_L>0)} \bar{J}(\bar{X})$, respectively
k_q :	Combination of the pitch damping derivatives defined in (25)
k_T :	Scaling exponent of the thrust coefficient variation with airspeed at constant throttle
K :	Induced drag coefficient
P :	Parameter defined in (38)
q :	Reduced pitch rate, $\dot{\theta}$
r_y :	Reduced radius of gyration, $(1/2)\sqrt{\bar{i}_y}$; the unit is c_w
R_1 :	An upper stability limit for r_y
S_w, S_t :	Wing and tail areas
t :	Reduced time; the unit is $c_w/(2V)$
u :	Reduced airspeed perturbation from trim; the unit is V
V :	Airspeed at trim
x_{cg} :	Reduced longitudinal center-of-gravity position; the unit is c_w
x_m, x_n, x_s :	Maneuver, neutral, and neutral-speed-stability points; the unit is c_w
$\bar{X}$:	Reduced maneuver margin; the unit is $x_s - x_m$
$\bar{X}_{\pm}(\bar{i}_y)$:	Critical values of $\bar{X}$ for a given $\bar{i}_y$
α :	Angle of attack, measured from trim
γ :	Flight path angle at trim
θ :	Pitch angle, measured from trim
μ :	Reduced mass; the unit is $(1/2)\rho S_w c_w$
ρ :	Air density.

Overheads

- $\dot{\cdot}$: (overdot) a derivative with respect to the reduced time.
 $\bar{\cdot}$: (overbar) a nonstandard reduced quantity.

Subscripts

$\dots, \bullet$: Partial derivative with respect to $\bullet$, computed at trim.

References

- [1] B. Etkin, *Dynamics of Atmospheric Flight*, John Wiley & Sons, New York, NY, USA, 1972.
- [2] R. D. Milne and G. D. Padfield, "Strongly controlled aircraft," *Aeron Quart*, vol. 22, no. 2, pp. 146–168, 1971.
- [3] K. Ogata, *Modern Control Engineering*, Prentice Hall, Upper Saddle River, NJ, USA, 1970.
- [4] B. Thwaites, *Incompressible Aerodynamics*, Oxford University Press, Oxford, UK, 1960.