Research Article

Optimum Operational Parameters for Yawed Wind Turbines

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A set of systematical optimum operational parameters for wind turbines under various wind directions is derived by using combined momentum-energy and blade-element-energy concepts. The derivations are solved numerically by fixing some parameters at practical values. Then, the interactions between the produced power and the influential factors of it are generated in the figures. It is shown that the maximum power produced is strongly affected by the wind direction, the tip speed, the pitch angle of the rotor, and the drag coefficient, which are specifically indicated by figures. It also turns out that the maximum power can take place at two different optimum tip speeds in some cases. The equations derived herein can also be used in the modeling of tethered wind turbines which can keep aloft and deliver energy.

1. Introduction

Wind turbines have been around for centuries, from the simple water pump in Arabia to the current widely used HAWT (horizontal axis wind turbine). With more and more public attention drawn to wind energy for its ease of development and environmental friendliness, wind farms that consist of hundreds of wind turbines at the same location are now being constructed rapidly all around the world [1]. Some of the wind farms are even built offshore by using large floating platforms due to the stronger and steadier wind at sea. However, the potential impact of wind farms on the biological, physical, and human environments is still under inspection, and the economics of offshore wind farms is still being demonstrated [2, 3]. Recently, much interest has arisen in new types of wind turbines like tethered wind turbines which can fly at high altitudes while delivering energy from wind [4–6]. Since the captured wind energy is proportional to the cube of the wind speed, a tethered wind turbine at high altitude may have 500 times as much available power as a wind turbine on the surface.

Although wind turbines vary in configuration, most of them incorporate the same mechanism of extracting wind energy by using rotor blades. To determine the optimum conditions when the maximum power is generated by the

individual rotor, CFD (computational fluid dynamics) is a widely used method due to its predictive accuracy. Other methods like FVM (free vortex models) have also been developed. However, these approaches become computationally intensive for the study on wind farms that involve hundreds of individual wind turbines [7], and they are of limited use in the modeling of new types of wind turbines such as tethered wind turbines. Therefore, BEM (blade element momentum) is still widely used because of its simplicity and overall accuracy.

BEM theory was formulated by Glauert and gives both the force in the axial direction and the tangential direction [7]. It analyzes the rotor blade in sections and adds up the forces on all sections to get the resultant force of the rotor [8]. However, wind direction is usually not considered in the BEM, and the wind is generally assumed to be perpendicular to the rotor. Therefore, it is still unknown what the true available power and the optimum operational condition would be for wind turbines in yawed flows. This gap has created obstacles in the consideration of wind turbines in yawed flows.

The purpose of the work described herein is to use momentum and blade-element considerations to determine the optimum operating conditions for a yawed wind turbine and to determine the maximum power that the turbine

FIGURE 1: Physical system model. W : wind speed and γ : wind direction angle which is between the wind and the rotor axis.

can generate under various wind directions. The work culminates in the design figures which can be used to design wind turbines. The derivation described herein also paves the way for our further study on tethered wind turbines.

2. Physical System

Figure 1 shows the physical system that will be analyzed in this paper. It should be noted that the yawed wind turbine herein is considered to be a rotor, the axis of which is fixed to some support. The wind is assumed to be horizontal, and it impinges on the rotor at a wind direction angle (between the wind direction and the axis of rotor disk), denoted as γ .

It is further assumed that the forces on the blades are determined solely by the lift and drag characteristics of the airfoil shape of the blades. Since momentum theory holds, the induced flow will be normal to the rotor disk. Swirl velocity is neglected in this study. No in-plane net loading is considered. Our desire is to determine the optimum operating conditions of the wind turbine under various wind directions in order to generate the most power possible. The results are shown in the figures at the end of the paper.

3. Case of a Horizontal Axis Wind Turbine

The first set of equations to be derived and solved are for the case of rotor with horizontal axis, $\gamma = 0$. This case provides a good baseline and insight for the more complicated yawed case that will be derived later.

By breaking a blade down into several small sections, we derive the forces on each of these small blade elements by analyzing the airfoil of the blade (Figure 2). These forces are then integrated along the entire blade. Thus, the resultant force on the rotor is obtained from the forces on all blades [8]. This is achieved by the following.

All the forces can be normalized on $\rho\pi R^2 W^2/2$:

$$\begin{aligned} L &= \frac{1}{2}\rho\pi R^2 W^2 C_L, \\ D &= \frac{1}{2}\rho\pi R^2 W^2 C_D, \\ C_L &= a[\sin(\alpha)], \end{aligned} \quad (1)$$

where ρ is the air density, R is the radius of the rotor, W is the relative wind speed, a is the slope of the lift curve, α is the

FIGURE 2: Airfoil for one blade element. α : angle of attack; ϕ : angle between the relative wind and the plane of rotation; θ : pitch angle; L : lift force normal to the wind direction; D : profile drag parallel to the wind direction; dF_1 : thrust force on the blade element normal to the plane of rotation; dF_2 : drag force on the blade element tangential to the circle swept by the rotor; W : relative wind speed; W_p : component of W normal to the plane of rotation; W_T : component of W tangential to the circle swept by the rotor; c : chord length; x : radial location of a blade element.

angle of attack between the chord line and the relative wind, D is the profile drag parallel to the wind direction, C_D is the drag coefficient, L is the lift force normal to the wind, and C_L is the lift coefficient.

Since we are interested only in the force normal to and tangential to the plane of rotation, the lift and drag forces are projected in these directions

$$F_1 = L \cos \phi + D \sin \phi, \quad (2)$$

$$F_2 = L \sin \phi - D \cos \phi, \quad (3)$$

$$\sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi, \quad (4)$$

FIGURE 3: Physical system model in case of yawed wind turbine. W : initial wind speed; W_1 : component of W normal to the plane of rotation; W_2 : component of W parallel to the plane of rotation; v : induced flow; γ : wind direction angle between the wind and the rotor axis; ψ_i : angle between W_2 and the i th blade; Ω : angular velocity of rotor; dF_1^i : thrust on the element of the i th blade, which is normal to the plane of rotation; dF_2^i : drag on the element of the i th blade, which is tangential to the circle swept by the i th blade.

$$\cos \phi = \frac{W_T}{\sqrt{W_P^2 + W_T^2}}, \quad (5)$$

$$\sin \phi = \frac{W_P}{\sqrt{W_P^2 + W_T^2}}, \quad (6)$$

where θ is the pitch angle between the chord line and the plane of rotation, ϕ is the angle between the relative wind and the plane of rotation, F_1 is the thrust of each blade normal to the plane of rotation, F_2 is the drag of each blade tangential to the circle swept by the blade, W_P is the component of the relative wind speed normal to the plane of rotation, and W_T is the component of the relative wind speed tangential to the circle swept by the blade.

In this theoretical development, it is assumed that we are finding the optimum loading condition for a turbine in uniform air flow. For that reason, it is assumed that the airfoil sections (i.e., the chord and airfoils) have been chosen to give a reasonable life-to-drag ratio. Thus, this paper assumes that all airfoils are in the unstalled region such that the lift coefficient is a constant lift-curve slope multiplied by the angle of attack. Naturally, for a final wind turbine design, the optimum condition would have to be checked to make sure that the airfoils are not stalled and that the tip speeds are in a reasonable Mach range. From this and (1)–(6), it is possible to determine the thrust of each blade element normal to the plane of rotation, denoted as dF_1 , and the drag of each blade

(a)

(b)

(c)

FIGURE 4: (a) $\bar{C}_p$ versus J for $\theta = 0^\circ$, (b) $\bar{C}_p$ versus J for $\theta = 2^\circ$, and (c) $\bar{C}_p$ versus J for $\theta = 3^\circ$.

FIGURE 5: (a) $\bar{C}_p$ versus w for $\theta = 0^\circ$, (b) $\bar{C}_p$ versus w for $\theta = 2^\circ$, and (c) $\bar{C}_p$ versus w for $\theta = 3^\circ$.

FIGURE 6: $\text{Max } \bar{C}_p$ versus C_D .

FIGURE 7: Optimum J versus C_D .

element tangential to the circle swept by the blade, denoted as dF_2 , as follows (in the appendix):

$$\begin{aligned} dF_1 &= \frac{1}{2} \rho a c dx \left[-W_T^2 \theta + W_p W_T \right], \\ dF_2 &= \frac{1}{2} \rho a c dx \left[-W_T^2 \frac{C_D}{a} - W_p W_T \theta + W_p^2 \right], \end{aligned} \quad (7)$$

where c is the chord length and x is the radial location of a blade element.

FIGURE 8: Optimum w versus C_D .

From the flow geometry,

$$\begin{aligned} W_P &= W - v, \\ W_T &= \Omega x, \end{aligned} \quad (8)$$

where v is the induced flow and Ω is the angular velocity of the blade.

Thus, the thrust force on the entire rotor, denoted as T , can be obtained from integrated forces on all blades and the extracted power, denoted as P , can be obtained from the work done by the drag force:

$$\begin{aligned} T &= b \int_0^R dF_1, \\ P &= b \int_0^R dF_2 \Omega x, \end{aligned} \quad (9)$$

where b is the number of blades.

Substituting (7) and (8) into (9), we obtain

$$T = \rho abc \left[-\frac{1}{6} \Omega^2 R^3 \theta + \frac{1}{4} \Omega R^2 (W - v) \right], \quad (10)$$

$$\begin{aligned} P &= \rho abc \left[-\frac{1}{8} \Omega^3 R^4 C_D \frac{1}{a} - \frac{1}{6} \Omega^2 R^3 (W - v) \theta \right. \\ &\quad \left. + \frac{1}{4} \Omega R^2 (W - v)^2 \right]. \end{aligned} \quad (11)$$

The above two equations are derived from blade-element theory. It is obvious that the thrust and the power are both relative to the blade number and the angular velocity of the rotor.

We can also obtain the thrust T from momentum theory:

$$T = \rho \pi R^2 (W - v) 2v. \quad (12)$$

Combining (10) and (12) and normalizing all velocities on W , we obtain:

$$w = \frac{1}{2} + \frac{1}{16} a \sigma J - \frac{1}{2} \sqrt{\frac{1}{64} a^2 \sigma^2 J^2 + 1 - \frac{1}{4} a \sigma J + \frac{1}{3} a \theta \sigma J^2}, \quad (13)$$

$$\sigma = \frac{bc}{\pi R}, \quad (14)$$

where σ is the rotor property parameter, w is the induced flow coefficient, which is v/W and should be less than 0.5, J is the normalized tip speed, which is $\Omega R/W$.

For (11), we normalize power on $\rho \pi R^2 W^3/2$ and all velocities on W :

$$\bar{C}_P = a \sigma \left[-C_D \frac{1}{4a} J^3 - \frac{\theta}{3} (1 - w) J^2 + \frac{1}{2} (1 - w)^2 J \right], \quad (15)$$

where $\bar{C}_P$ is the normalized power coefficient extracted from wind.

Equations (13) and (15) are used to determine optimum operational conditions for the horizontal axis rotor. It is useful to firstly assign practical values to a and σ , both of which are only determined by the shape of blades:

$$\bar{C}_P = -0.1528 \theta (1 - w) J^2 + 0.2292 (1 - w)^2 J - 0.02 C_D J^3, \quad (16)$$

$$\begin{aligned} w &= 0.5 + 0.0287 J \\ &\quad - 0.5 \sqrt{0.0033 J^2 + 0.1528 \theta J^2 - 0.1146 J + 1}, \end{aligned} \quad (17)$$

where $a = 5.73$ and $\sigma = 0.08$.

Equations (16) and (17) consist of five variables: $\bar{C}_P$, w , J , θ and C_D . It is difficult to find the direct relationship among five variables with only two equations. To find the optimum operating conditions, we fix one of the two variables (θ or J) at a certain value. By varying C_D from 0.01 to 0.09 discretely, we use (16) and (17) to find the relationships among the three remaining variables. Thus, the discussion can be divided into two parts: θ fixed and J fixed.

3.1. Numerical Analysis of θ -Fixed Case. Figures 4(a), 4(b), 4(c), 5(a), 5(b), and 5(c) show how the produced power

FIGURE 9: (a) $\bar{C}_p$ versus θ for $J = 1$, (b) $\bar{C}_p$ versus θ for $J = 3$ (c) $\bar{C}_p$ versus θ for $J = 5$, and (d) $\bar{C}_p$ versus θ for $J = 7$.

interacts with the tip speed and the induced flow when the pitch angle is fixed at practical values. For each value of θ and C_D , we use (17) to find each corresponding J with w varied from 0 to 0.5. Then, (16) is used to find each corresponding $\bar{C}_p$. It is common in these figures that each curve has one peak which indicates optimum condition, except for the one case with $C_D = 0$ in Figure 4(b), which has two clear peaks. That special curve will be specifically discussed later in our yawed wind turbine. As θ becomes relatively larger, the phenomenon of the curve twisting is seen in Figures 5(b) and 5(c), which indicate the possibility of one induced flow value corresponding to two different powers produced.

Figures 6, 7, and 8 show the optimum operating conditions when the greatest power is produced. Each of these plots is produced by starting with a value of C_D , stepping through values of J , and getting each corresponding w and

$\bar{C}_p$. Then, it is possible to determine the maximum $\bar{C}_p$, the corresponding optimum J and w . It is obvious that the maximum power, the optimum tip speed, and the optimum induced flow decrease as the drag coefficient gets larger. This can be verified by comparing peak positions of each curve in Figures 4(a), 4(b), 4(c), 5(a), 5(b), and 5(c). Although the optimum induced flow coefficient is $1/3$ for an ideal rotor in axial flows, the optimum values in Figure 8 are not always $1/3$.

3.2. Numerical Analysis of J -Fixed Case. Figures 9(a), 9(b), 9(c), 9(d), 10(a), 10(b), 10(c), and 10(d) show how the power interacts with the pitch angle and the induced flow when the tip speed is fixed at practical values. The plotting method is the same numerical process as that for Figures 4(a), 4(b), 4(c), 5(a), 5(b), and 5(c). As J increases from Figures 9(a)

FIGURE 10: (a) $\bar{C}_p$ versus w for $J = 1$, (b) $\bar{C}_p$ versus w for $J = 3$, (c) $\bar{C}_p$ versus w for $J = 5$, and (d) $\bar{C}_p$ versus w for $J = 7$.

to 9(d), the curves in each figure move apart from each other and the peak of each curve gradually comes out and moves rightward. However, the movement of the peak is unclear in Figures 10(a), 10(b), 10(c), and 10(d). These optimum conditions can be seen in Figure 13. It should be noticed that the two curves with C_D being 0.08 and 0.09 in Figure 10(d) totally lie below zero and the turbine is unable to produce power in these cases.

Figures 11, 12, and 13 show the optimum operating conditions when the greatest power is produced. The plotting method is the same as that of Figures 6, 7, and 8. It is clear from Figure 11 that the maximum power decreases linearly as the drag coefficient gets larger with the tip speed drastically affecting the velocity of decreasing. In Figures 12 and 13, however, the drag coefficient does not affect the optimum

pitch angle and the induced flow, both of which are only determined by the tip speed.

4. Case of a Yawed Wind Turbine

When the wind direction angle γ is an arbitrary value (Figure 3), we can divide the initial wind speed into components normal and parallel to the rotor axis and obtain the thrust and the power based on momentum and blade-element derivations. We generalize the momentum equations in the following:

$$W_1 = W \cos \gamma, \quad (18)$$

$$W_2 = W \sin \gamma, \quad (19)$$

FIGURE 11: Max $\bar{C}_p$ versus C_D . FIGURE 13: Optimum w versus C_D . FIGURE 12: Optimum θ versus C_D .

$$\dot{m} = \rho \pi R^2 \sqrt{(W_1 - v)^2 + W_2^2}, \quad (20)$$

$$T = 2\dot{m}v, \quad (21)$$

where W is the initial wind speed, W_1 is the component of W normal to plane of rotation, W_2 is the component of W parallel to the plane of rotation, v is the induced flow, $\dot{m}$ is the mass flow rate at the rotor, and T is the thrust on the rotor.

Substituting (18)–(20) into (21), we obtain

$$T = 2\rho\pi R^2 v \sqrt{W^2 + v^2 - 2vW \cos \gamma}. \quad (22)$$

We then use the blade-element derivations to obtain the thrust and the drag on the element of the i th blade:

$$W_T^i = \Omega x + W_2 \sin \psi_i, \quad (23)$$

$$W_P^i = W_1 - v, \quad (24)$$

$$dF_1^i = \frac{1}{2} \rho a c dx \left[-\left(W_T^i\right)^2 \theta + W_P^i W_T^i \right], \quad (25)$$

$$dF_2^i = \frac{1}{2} \rho a c dx \left[-\left(W_T^i\right)^2 \frac{C_D}{a} - W_P^i W_T^i \theta + \left(W_P^i\right)^2 \right], \quad (26)$$

where ψ_i is the angle between W_2 and the i th blade, W_T^i is the component of the relative wind speed which is tangential to the circle swept by the i th blade, W_P^i is the component of the relative wind speed which is normal to the plane of rotation, dF_1^i is the thrust on the element of the i th blade which is normal to the plane of rotation, and dF_2^i is the drag on the element of the i th blade which is tangential to the circle swept by the i th blade.

Substituting (23) and (24) into (25) and (26), we find

$$\begin{aligned} dF_1^i &= \frac{1}{2} \rho a c dx \\ &\times \left[-\theta \left(\Omega^2 x^2 + W^2 \sin^2 \gamma \sin^2 \psi_i + 2\Omega x W \sin \gamma \sin \psi_i \right) \right. \\ &\quad \left. + (\Omega x W \cos \gamma - \Omega x v + W^2 \sin \gamma \cos \gamma \sin \psi_i \right. \\ &\quad \left. - v W \sin \gamma \sin \psi_i) \right], \end{aligned} \quad (27)$$

FIGURE 14: (a) $\bar{C}_p$ versus J for $\theta = 0^\circ$ and $\gamma = 15^\circ$ and (b) $\bar{C}_p$ versus J for $\theta = 0^\circ$ and $\gamma = 60^\circ$.

$$\begin{aligned}
 dF_2^i &= \frac{1}{2} \rho a c d x \\
 &\times \left[-\frac{C_D}{a} (\Omega^2 x^2 + W^2 \sin^2 \gamma \sin^2 \psi_i \right. \\
 &\quad \left. + 2\Omega x W \sin \gamma \sin \psi_i) \right. \\
 &\quad \left. - \theta (\Omega x W \cos \gamma - \Omega x \nu + W^2 \sin \gamma \cos \gamma \sin \psi_i \right. \\
 &\quad \left. - \nu W \sin \gamma \sin \psi_i) \right. \\
 &\quad \left. + (W^2 \cos^2 \gamma + \nu^2 - 2\nu W \cos \gamma) \right].
 \end{aligned} \tag{28}$$

Again, the thrust force on the entire rotor T can be obtained from integrated forces on all blades, and the

extracted power P can be obtained from the work done by the drag forces:

$$T = \sum_{i=1}^b \int_0^R dF_1^i, \tag{29}$$

$$P = \sum_{i=1}^b \int_0^R dF_2^i \Omega x,$$

where b is the number of blades.

Substituting (28) into (29), we obtain

$$\begin{aligned}
 T &= \frac{1}{2} \rho a c \left[-\theta \left(\frac{1}{3} b \Omega^2 R^3 + W^2 R \sin^2 \gamma \sum_{i=1}^b \sin^2 \psi_i + \Omega R^2 W \sin \gamma \sum_{i=1}^b \sin \psi_i \right) \right. \\
 &\quad \left. + \left(\frac{1}{2} b \Omega R^2 W \cos \gamma - \frac{1}{2} b \Omega R^2 \nu + W^2 R \sin \gamma \cos \gamma \sum_{i=1}^b \sin \psi_i - \nu W R \sin \gamma \sum_{i=1}^b \sin \psi_i \right) \right], \\
 P &= \frac{1}{2} \rho a c \left[-\frac{C_D}{a} \left(\frac{1}{4} b \Omega^3 R^4 + \frac{1}{2} \Omega R^2 W^2 \sin^2 \gamma \sum_{i=1}^b \sin^2 \psi_i + \frac{2}{3} \Omega^2 R^3 W \sin \gamma \sum_{i=1}^b \sin \psi_i \right) \right. \\
 &\quad \left. - \theta \left(\frac{1}{3} b \Omega^2 R^3 W \cos \gamma - \frac{1}{3} b \Omega^2 R^3 \nu + \frac{1}{2} \Omega R^2 W^2 \sin \gamma \cos \gamma \sum_{i=1}^b \sin \psi_i - \frac{1}{2} \Omega R^2 \nu W \sin \gamma \sum_{i=1}^b \sin \psi_i \right) \right. \\
 &\quad \left. + \left(\frac{1}{2} b \Omega R^2 W^2 \cos^2 \gamma + \frac{1}{2} b \Omega R^2 \nu^2 - b \Omega R^2 \nu W \cos \gamma \right) \right].
 \end{aligned} \tag{30}$$

To simplify the equation, we consider the most widely used rotor which has three blades:

$$\begin{aligned}
 b &= 3, \\
 \sum_{i=1}^3 \sin \psi_i &= 0, \\
 \sum_{i=1}^3 \sin^2 \psi_i &= \frac{3}{2}.
 \end{aligned} \tag{31}$$

Thus,

$$\begin{aligned}
 T &= \frac{1}{2} \rho a c \left[-\theta \left(\Omega^2 R^3 + \frac{3}{2} W^2 R \sin^2 \gamma \right) \right. \\
 &\quad \left. + \left(\frac{3}{2} \Omega R^2 W \cos \gamma - \frac{3}{2} \Omega R^2 \gamma \right) \right], \\
 P &= \frac{1}{2} \rho a c \\
 &\quad \times \left[-\frac{C_D}{a} \left(\frac{3}{4} \Omega^3 R^4 + \frac{3}{4} \Omega R^2 W^2 \sin^2 \gamma \right) \right. \\
 &\quad \left. - \theta (\Omega^2 R^3 W \cos \gamma - \Omega^2 R^3 \gamma) \right. \\
 &\quad \left. + \left(\frac{3}{2} \Omega R^2 W^2 \cos^2 \gamma + \frac{3}{2} \Omega R^2 \gamma^2 - 3 \Omega R^2 \gamma W \cos \gamma \right) \right].
 \end{aligned} \tag{32}$$

Then, we combine (22) and (32) and normalize all power on $\rho \pi R^2 W^3/2$, and all velocities on W :

$$\begin{aligned}
 &\frac{4}{a\sigma} w \sqrt{1 + w^2 - 2w \cos \gamma} + \frac{1}{2} J w + \theta \left(\frac{1}{3} J^2 + \frac{1}{2} \sin^2 \gamma \right) \\
 &\quad - \frac{1}{2} J \cos \gamma = 0, \\
 \overline{C}_P &= a\sigma \left[-\frac{C_D}{4a} (J^3 + J \sin^2 \gamma) \right. \\
 &\quad \left. - \frac{\theta}{3} (\cos \gamma - w) J^2 + \frac{1}{2} (\cos \gamma - w)^2 J \right],
 \end{aligned} \tag{34}$$

where w is the induced flow coefficient, J is the normalized tip speed, and σ is the rotor property parameter, which is $bc/(\pi R)$.

TABLE 1: Optimum condition at $C_D = 0$.

θ	γ	Maximum $\overline{C}_P$	Optimum J
1.6°	0°	0.59	28.50
	15°	0.58	25.15
	30°	0.52	14.56
1.7°	0°	0.59	26.23
	15°	0.58	22.86
	30°	0.52	14.01
1.8°	0°	0.59	7.67; 24.16
	15°	0.58	9.25; 20.71
	30°	0.51	13.98
1.9°	0°	0.59	7.88; 22.27
	15°	0.58	9.73; 18.65
	30°	0.51	12.91
2.0°	0°	0.59	8.13; 20.52
	15°	0.58	10.43; 16.54
	30°	0.51	12.42
2.1°	0°	0.59	8.42; 18.86
	15°	0.58	13.64
	30°	0.50	12.51
2.2°	0°	0.59	17.28
	15°	0.58	12.56
	30°	0.50	11.61
2.3°	0°	0.59	15.66
	15°	0.58	11.85
	30°	0.49	11.65

Again, we assign a and σ the practical values:

$$\begin{aligned}
 &w \sqrt{1 + w^2 - 2w \cos \gamma} + 0.0573 J w \\
 &\quad + 0.1146 \theta \left(\frac{1}{3} J^2 + \frac{1}{2} \sin^2 \gamma \right) - 0.0573 J \cos \gamma = 0, \\
 \overline{C}_P &= -0.1528 \theta (\cos \gamma - w) J^2 + 0.2292 (\cos \gamma - w)^2 J \\
 &\quad - 0.02 C_D (J^3 + J \sin^2 \gamma),
 \end{aligned} \tag{35}$$

where $a = 5.73$ and $\sigma = 0.08$.

Figures 14(a), 14(b), 15(a), 15(b), 16(a), 16(b), 17(a), 17(b), 18(a), 18(b), 19(a), and 19(b) show how the produced power interacts with the tip speed and the induced flow when the pitch angle and the wind direction angle are fixed at practical values. All plots are produced by starting with a value of C_D , stepping through values of J , and getting each corresponding w and $\overline{C}_P$ based on (35). Most figures in the yawed rotor are similar to the figures in the horizontal axis rotor: as the tip speed or the induced flow gets larger,

FIGURE 15: (a) $\overline{C}_P$ versus J for $\theta = 2^\circ$ and $\gamma = 15^\circ$ and (b) $\overline{C}_P$ versus J for $\theta = 2^\circ$ and $\gamma = 60^\circ$.

FIGURE 16: (a) $\overline{C}_P$ versus J for $\theta = 3^\circ$ and $\gamma = 15^\circ$ and (b) $\overline{C}_P$ versus J for $\theta = 3^\circ$ and $\gamma = 60^\circ$.

the power increases to a global maximum followed by monotonous decreases; the phenomenon of twisting comes out in $\overline{C}_P$ versus w at the relatively large pitch angle and wind direction angle.

However, the curve with $C_D = 0$ in Figure 15(a) has two global maxima, which is the same as those in Figure 4(b). For θ being 2° and γ being 0° , the maximum $\overline{C}_P$ is 0.59 when J is 8.13 and 20.52; for θ being 2° and γ being 15° , the maximum $\overline{C}_P$ is 0.58 when J is 10.43 and 16.54. It is

somewhat surprising to get two optimum values of J for only one maximum $\overline{C}_P$. To study this further, we also fixed θ at values around 2° and computed the maximum $\overline{C}_P$ and its corresponding J when C_D is 0. The numerical results are shown in Table 1.

It can be concluded from the table that, at some pitch angle value around 1.8° , 1.9° , 2.0° , and 2.1° , two optimum tip speeds may coexist provided that the drag coefficient equals 0 and the wind direction angle is small.

FIGURE 17: (a) $\bar{C}_P$ versus w for $\theta = 0^\circ$ and $\gamma = 15^\circ$ and (b) $\bar{C}_P$ versus w for $\theta = 0^\circ$ and $\gamma = 60^\circ$.

FIGURE 18: (a) $\bar{C}_P$ versus w for $\theta = 2^\circ$ and $\gamma = 15^\circ$ and (b) $\bar{C}_P$ versus w for $\theta = 2^\circ$ and $\gamma = 60^\circ$.

Figures 20(a), 20(b), 20(c), 21(a), 21(b), 21(c), 22(a), 22(b), and 22(c) show the optimum operating conditions when the greatest power is produced. All plots are generated by starting at a value of γ , stepping through values of J , and getting each corresponding w and $\bar{C}_P$. Then, it is possible to determine the maximum $\bar{C}_P$, the corresponding optimum J and w . It is clear that the maximum power goes down monotonously as the wind direction angle gets larger. For

the optimum tip speed and the optimum induced flow, most of them also decrease monotonously as the wind direction angle increases except for the ones with $C_D = 0$. It should be noted that the $C_D = 0$ cases are numerically sensitive and so behave differently from the $C_D \neq 0$ cases. As a verification of the coexisting optimum tip speeds mentioned above, the optimum J jumps upside and down when $C_D = 0$ in Figure 21(b).

FIGURE 19: (a) $\bar{C}_p$ versus w for $\theta = 3^\circ$ and $\gamma = 15^\circ$ and (b) $\bar{C}_p$ versus w for $\theta = 3^\circ$ and $\gamma = 60^\circ$.

5. Conclusion

Momentum-energy and blade-element-energy equations have been derived to find optimum conditions for both yawed wind turbines and new types of wind turbines using rotor blades. They are solved numerically by fixing some parameters at practical values. Then, the interactions between the produced power and the influential factors of it are generated in the figures.

It is a general principle that, under various wind direction angles, the maximum possible power produced is strongly affected by the tip speed, the pitch angle of the rotor, and the drag coefficient from the wind, which are specifically indicated by figures. For each of optimum operating conditions, most maximum powers correspond to unique optimum tip speed except for some special cases in which two optimum tip speeds coexist. As the wind direction angle increases to right angle, the maximum power decreases monotonously to zero.

Appendix

Derivation for the Differential Thrust and Drag

Substituting (1), (4), (5), and (6) into differential forms of (2) and (3), we obtain

$$dF_1 = \frac{1}{2} a \rho c dx [\cos \theta W_p W_T - \sin \theta W_T^2] + \frac{1}{2} \rho c dx [W_p \sqrt{W_T^2 + W_p^2}] C_D,$$

$$dF_2 = \frac{1}{2} a \rho c dx [\cos \theta W_p^2 - \sin \theta W_p W_T]$$

$$- \frac{1}{2} \rho c dx \left(W_T \sqrt{W_T^2 + W_p^2} \right) C_D.$$

(A.1)

Since W_p is far less than W_T , the pitch angle θ is very little, and the drag coefficient C_D is far less than the lift curve slope a in practical situation, we find approximations

$$\sqrt{W_T^2 + W_p^2} = W_T,$$

$$\sin \theta = \theta,$$

$$\cos \theta = 1,$$

$$\frac{C_D}{a} = 0.$$

(A.2)

Thus,

$$dF_1 = \frac{1}{2} \rho a c dx [-W_T^2 \theta + W_p W_T],$$

(A.3)

$$dF_2 = \frac{1}{2} \rho a c dx \left[-W_T^2 \frac{C_D}{a} - W_p W_T \theta + W_p^2 \right].$$

FIGURE 20: (a) $\text{Max } \bar{C}_p$ versus γ for $\theta = 0^\circ$, (b) $\text{Max } \bar{C}_p$ versus γ for $\theta = 2^\circ$, and (c) $\text{Max } \bar{C}_p$ versus γ for $\theta = 3^\circ$.

FIGURE 21: (a) Optimum J versus γ for $\theta = 0^\circ$, (b) Optimum J versus γ for $\theta = 2^\circ$, and (c) Optimum J versus γ for $\theta = 3^\circ$.

FIGURE 22: (a) Optimum w versus γ for $\theta = 0^\circ$, (b) Optimum w versus γ for $\theta = 2^\circ$, and (c) Optimum w versus γ for $\theta = 3^\circ$.

Nomenclature

- a : Slope of lift curve
 b : Number of blades
 c : Chord length (m)
 C_D : Drag coefficient ($D/(\rho\pi R^2 W^2/2)$)
 C_L : Lift coefficient ($L/(\rho\pi R^2 W^2/2)$)
 $\bar{C}_P$: Normalized power coefficient ($P/(\rho\pi R^2 W^3/2)$)
 dF_1 : Thrust on blade element, which is normal to the plane of rotation (N)
 dF_2 : Drag on blade element, which is tangential to the circle swept by rotor (N)
 dF_1^i : Thrust on element of the i th blade, which is normal to the plane of rotation (N)
 dF_2^i : Drag on element of the i th blade, which is tangential to the circle swept by the i th blade (N)
 D : Profile drag parallel to the wind direction (N)
 F_1 : Thrust on blade, which is normal to the plane of rotation (N)
 F_2 : Drag on blade, which is tangential to the circle swept by the rotor (N)
 J : Normalized tip speed ($\Omega R/W$)
 L : Lift force on the rotor (N)
 $\dot{m}$: Mass flow rate (kg/s)
 P : Power extracted from wind (Watts)
 R : Radius of blade (m)
 T : Thrust on the rotor (N)
 v : Induced flow (m/s)
 w : Induced flow coefficient (v/W)
 W : Wind speed (m/s)
 W_1 : Component of the initial wind speed which is normal to the plane of rotation (m/s)
 W_2 : Component of the initial wind speed which is parallel to the plane of rotation (m/s)
 W_P : Component of the relative wind speed which is normal to the plane of rotation (m/s)
 W_T : component of the relative wind speed which is tangential to the circle swept by the rotor (m/s)
 W_P^i : Component of the relative wind speed which is normal to the plane of rotation (m/s)
 W_T^i : Component of the relative wind speed which is tangential to the circle swept by the i th blade (m/s)
 x : Radial location of a blade element (m)
 ρ : Density of air (kg/m³)
 α : Angle of attack (rad)
 γ : Wind direction angle, which is between the wind direction and the rotor axis (rad)
 θ : Pitch angle (rad)
 ϕ : Angle between the relative wind and the plane of rotation (rad)
 Ω : Blade angular velocity (rad/s)
 ψ_i : Angle between W_2 and the i th blade (rad)
 σ : Rotor property parameter ($bc/(\pi R)$).

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