Research Article

An Expansion Theorem Involving H -Function of Several Complex Variables

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The aim of this present paper is to obtain a general expansion theorem involving H -functions of several complex variables. This is done by making use of a Taylor-like expansion in terms of a rational function obtained by means of fractional derivatives given recently by the authors. Special cases are also computed.

1. Introduction

In 1971, Osler obtained with the use of Cauchy integral formula for the fractional derivatives the following generalization for the Taylor's series [1]:

$$\sum_{k \in \mathbb{K}} a^{-1} \omega^{-\gamma k} f(\theta^{-1}(\theta(z) \omega^k)) \\ = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{an+\gamma} [f(z) \theta'(z) [(z-z_0)/\theta(z)]^{an+\gamma+1}] \Big|_{z=z_0} \theta(z)^{an+\gamma}}{\Gamma(an+\gamma+1)}, \quad (1)$$

where a is a positive real number, $b \neq z_0$, $\omega = \exp(2\pi i/a)$, α and γ are arbitrary complex numbers, $f(z)$ is an analytic function in a simply connected region $\mathcal{R}$ and $\theta(z) = (z - z_0)q(z)$ with $q(z)$ is a regular and univalent function without zero in $\mathcal{R}$, and $K = 0, 1, \dots, [a]$, $[a]$ being the largest integer not greater than a . If $0 < a \leq 1$ and $\theta(z) = (z - z_0)$, then $K = 0$ and the formula (1) reduces to

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{[D_{z_0-b}^{an+\gamma} f(z_0)] (z - z_0)^{an+\gamma+1}}{\Gamma(an+\gamma+1)}. \quad (2)$$

This last formula is usually called the Taylor-Riemann formula and has been studied in several papers [2–6]. But none considered a more general expansion of $f(z)$ in terms of a power series of an arbitrary quadratic, cubic, or higher

degrees functions. Recently, the authors [7] obtained the power series of an analytic function $f(z)$ in terms of the rational expression $((z - z_1)/(z - z_2))$ where z_1 and z_2 are two arbitrary points inside the region of analyticity $\mathcal{R}$ of $f(z)$. In particular, we obtain the following expansion:

$$\frac{a^{-1} f(z) (z - z_1)^v (z - z_2)^u}{(z_1 - z_2)} \\ = \sum_{n=-\infty}^{\infty} \frac{e^{i\pi a(n+1)} \sin((\mu + an + \gamma)\pi)}{\sin((\mu - a + \gamma)\pi) \Gamma(1 - v + an + \gamma)} \\ \times D_{z-z_2}^{-v+an+\gamma} (z - z_2)^{\mu+an+\gamma-1} f(z) \Big|_{z=z_1} \left(\frac{z - z_1}{z - z_2} \right)^{an+\gamma}. \quad (3)$$

Several restrictions are imposed on the functions and parameters in (3). The following list is considered.

- (i) v , μ , and γ are arbitrary complex numbers.
- (ii) $0 < a \leq 1$ is a real and n is the integral index of summation.
- (iii) z_1, z_2 are fixed points in the z -plane and $\{z | |\theta(z)| = |\theta((z_1 + z_2)/2)|\}$, where $\theta(z) = (z - z_1)(z - z_2)$, defines a double-loop curve $C = C_1 \cup C_2$ on which the series (3) converges with $z_1 \neq z_2$.
- (iv) z is on the loop C_1 around the point z_1 but $z \neq (z_1 + z_2)/2$ as shown in Figure 1.

FIGURE 1: Multiloops contour.

The aim of this paper is to obtain a new expansion theorem involving the H -function of r complex variables $z_1, z_2, \dots, z_r$ defined by Srivastava and Panda [8–11]. We will define and represent it in the following form [12, page 251, equation (C.1)]:

$$\begin{aligned}
 & H[z_1, \dots, z_r] \\
 &= H_{P, Q; P', Q'; \dots; M^{(r)}, N^{(r)}}^{0, N; M', N'; \dots; M^{(r)}, N^{(r)}} \\
 &\quad \times \left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{c} (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1, P} : (c'_j, \gamma'_j)_{1, P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1, P^{(r)}} \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1, Q} : (d'_j, \delta'_j)_{1, Q'}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1, Q^{(r)}} \end{array} \right] \\
 &= \frac{1}{(2\pi i)^r} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) \\
 &\quad \times z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r,
 \end{aligned} \tag{4}$$

where $i = \sqrt{-1}$,

$$\begin{aligned}
 & \phi_k(\xi_k) \\
 &= \frac{\prod_{j=1}^{M^{(k)}} \Gamma(d_j^{(k)} - \delta_j^{(k)} \xi_k) \prod_{j=1}^{N^{(k)}} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} \xi_k)}{\prod_{j=M^{(k)}+1}^{Q^{(k)}} \Gamma(1 - d_j^{(k)} + \delta_j^{(k)} \xi_k) \prod_{j=N^{(k)}+1}^{P^{(k)}} \Gamma(c_j^{(k)} - \gamma_j^{(k)} \xi_k)},
 \end{aligned} \tag{5}$$

for all $k \in \{1, \dots, r\}$ and

$$\begin{aligned}
 & \psi(\xi_1, \dots, \xi_r) \\
 &= \frac{\prod_{j=1}^N \Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} \xi_k)}{\prod_{j=N+1}^P \Gamma(a_j - \sum_{k=1}^r \alpha_j^{(k)} \xi_k) \prod_{j=1}^Q \Gamma(1 - b_j + \sum_{k=1}^r \beta_j^{(k)} \xi_k)}.
 \end{aligned} \tag{6}$$

Here, for convenience, $(a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1, P}$ abbreviates the p -member array

$$\left((a_1; \alpha'_1, \dots, \alpha_1^{(r)}), \dots, (a_p; \alpha'_p, \dots, \alpha_p^{(r)}) \right), \tag{7}$$

while $(c_j^{(k)}, \gamma_j^{(k)})_{1, P^{(k)}}$ abbreviates the array of $P^{(k)}$ pairs of parameters:

$$(c_1^{(k)}, \gamma_1^{(k)}), \dots, (c_{P^{(k)}}^{(k)}, \gamma_{P^{(k)}}^{(k)}) \quad (k = 1, \dots, r), \tag{8}$$

and so on. Suppose, as usual, that the parameters:

$$\begin{aligned}
 & a_j, \quad j = 1, \dots, P; \quad c_j^{(k)}, \quad j = 1, \dots, P^{(k)} \\
 & b_j, \quad j = 1, \dots, Q; \quad d_j^{(k)}, \quad j = 1, \dots, Q^{(k)} \quad (\forall k \in \{1, \dots, r\})
 \end{aligned} \tag{9}$$

are complex numbers and the associated coefficients

$$\begin{aligned}
 & \alpha_j^{(k)}, \quad j = 1, \dots, P; \quad \gamma_j^{(k)}, \quad j = 1, \dots, P^{(k)} \\
 & \beta_j^{(k)}, \quad j = 1, \dots, Q; \quad \delta_j^{(k)}, \quad j = 1, \dots, Q^{(k)} \quad (\forall k \in \{1, \dots, r\})
 \end{aligned} \tag{10}$$

are positive real numbers such that

$$\Lambda_k := \sum_{j=1}^P \alpha_j^{(k)} - \sum_{j=1}^Q \beta_j^{(k)} + \sum_{j=1}^{P^{(k)}} \gamma_j^{(k)} - \sum_{j=1}^{Q^{(k)}} \delta_j^{(k)} \leq 0, \tag{11}$$

$$\begin{aligned}
 \Omega_k := & - \sum_{j=N+1}^P \alpha_j^{(k)} - \sum_{j=1}^Q \beta_j^{(k)} + \sum_{j=1}^{N^{(k)}} \gamma_j^{(k)} - \sum_{j=N^{(k)}+1}^{P^{(k)}} \gamma_j^{(k)} \\
 & + \sum_{j=1}^{M^{(k)}} \delta_j^{(k)} - \sum_{j=M^{(k)}+1}^{Q^{(k)}} \delta_j^{(k)} > 0, \quad (\forall k \in \{1, \dots, r\}),
 \end{aligned} \tag{12}$$

where the integers $N, P, Q, M^{(k)}, N^{(k)}, P^{(k)}$, and $Q^{(k)}$ are constrained by the inequalities $0 \leq N \leq P, Q \geq 0, 1 \leq M^{(k)} \leq Q^{(k)}$, and $0 \leq N^{(k)} \leq P^{(k)}$ (for all $k \in \{1, \dots, r\}$) and the equality in (11) holds true for suitably restricted values of the complex variables $z_1, \dots, z_r$.

The multiple Mellin-Barnes contour integral [12, page 251, equation (C.1)] representing the multivariable H -function (4) converges absolutely, under the conditions (12), when

$$|\arg(z_k)| < \frac{1}{2} \Omega_k \pi, \quad (\forall k \in \{1, \dots, r\}), \tag{13}$$

the points $z_k = 0$ ($k = 1, \dots, r$) and various exceptional parameter values being tacitly excluded. Furthermore, we have (cf. [9, page 131, equation (1.9)]):

$$H[z_1, \dots, z_r] = \begin{cases} O(|z_1|^{\xi_1} \dots |z_r|^{\xi_r}), \\ \quad (\max\{|z_1|, \dots, |z_r|\} \rightarrow 0), \\ O(|z_1|^{\eta_1} \dots |z_r|^{\eta_r}), \\ \quad (N = 0; \min\{|z_1|, \dots, |z_r|\} \rightarrow \infty), \end{cases} \tag{14}$$

where $(k = 1, \dots, r)$

$$\xi_k = \min \left\{ \frac{\operatorname{Re}(d_j^{(k)})}{\delta_j^{(k)}} \right\}, \quad (j = 1, \dots, M^{(k)}), \quad (15)$$

$$\eta_k = \max \left\{ \frac{\operatorname{Re}(c_j^{(k)} - 1)}{\gamma_j^{(k)}} \right\}, \quad (j = 1, \dots, N^{(k)}), \quad (16)$$

provided that each of the inequalities in (11)–(13) holds true.

Note that throughout this work, we will assume that the convergence and existence conditions corresponding appropriately to the ones detailed above are satisfied by each of the various H -functions involved.

2. Pochhammer Contour Integral Representation for Fractional Derivative

The use of contour of integration in the complex plane provides a very powerful tool in both classical and fractional calculus. The most familiar representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [13–15], that is,

$$D_z^\alpha z^p f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi) \xi^p (\xi - z)^{-\alpha-1} d\xi, \quad (17)$$

which is valid for $\operatorname{Re}(\alpha) < 0$, $\operatorname{Re}(p) > 1$ and where the integration is done along a straight line from 0 to z in the ξ -plane. By integrating by part m times, we obtain

$$D_z^\alpha z^p f(z) = \frac{d^m}{dz^m} D_z^{\alpha-m} z^p f(z). \quad (18)$$

This allows to modify the restriction $\operatorname{Re}(\alpha) < 0$ to $\operatorname{Re}(\alpha) < m$ [15]. Another used representation for the fractional derivative is the one based on the Cauchy integral formula widely used by Osler [4, 16–18]. These two representations have been used in many interesting research papers. It appears that the less restrictive representation of fractional derivative according to parameters is the Pochhammer's contour definition introduced in [19, 20].

Definition 1. Let $f(z)$ be analytic in a simply connected region $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$ and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$ then if α is not a negative integer, p is not an integer, and z is in $\mathcal{R} - \{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$\begin{aligned} D_{g(z)}^\alpha g(z)^p f(z) &= \frac{e^{-i\pi p} \Gamma(1+\alpha)}{4\pi \sin(\pi p)} \\ &\times \int_{C(z+, g^{-1}(0)+, z-, g^{-1}(0)-; F(a), F(a))} \frac{f(\xi) g(\xi)^p g'(\xi)}{(g(\xi) - g(z))^{\alpha+1}} d\xi. \end{aligned} \quad (19)$$

For noninteger α and p , the functions $g(\xi)^p$ and $(g(\xi) - g(z))^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{-1}(0)$, and both pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 2. $F(a)$ denotes the principal value of the integrand in (19) at the beginning and ending point of the Pochhammer contour $P(a)$ which is closed on Riemann surface of the multiple-valued function $F(\xi)$.

Remark 2. In Definition 1, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However it is interesting to note here that we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, and (19) would still be valid.

Remark 3. The Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (19), then we know that the integral is analytic for all p and for all α and for z in $\mathcal{R} - \{g^{-1}(0)\}$. Indeed, the only possible singularities of $D_{g(z)}^\alpha g(z)^p f(z)$ are $\alpha = -1, -2, \dots$ and $p = 0, \pm 1, \pm 2, \dots$ which can directly be identified from the coefficient of the integral (19). However, integrating by parts N times the integral in (19) by two different ways, we can show that $\alpha = -1, -2, \dots$, and $p = 0, 1, 2, \dots$ are removable singularities (see [19]).

In 1985, Srivastava and Goyal [21, page 644, equation (17)] obtained the following fractional derivative formula, by using the well-known Riemann-Liouville's definition of the fractional derivative, for the multivariable H -function (4):

$$\begin{aligned} D_x^\alpha \{ &x^\beta (x + \xi)^\lambda H[z_1 x^{\rho_1} (x + \xi)^{\sigma_1}, \dots, z_r x^{\rho_r} (x + \xi)^{\sigma_r}] \} \\ &= \xi^\lambda x^{\beta-\alpha} \sum_{m=0}^{\infty} \frac{(x/\xi)^m}{m!} \times H_{P+2, Q+2; P', Q'; \dots; P^{(r)}, Q^{(r)}}^{0, N+2; M', N'; \dots; M^{(r)}, N^{(r)}} \\ &\times \left[\begin{array}{c} z_1 x^{\rho_1} \xi^{\sigma_1} \\ \vdots \\ z_r x^{\rho_r} \xi^{\sigma_r} \end{array} \middle| \begin{array}{c} (-\lambda; \sigma_1, \dots, \sigma_r), (-\beta - m; \rho_1, \dots, \rho_r), \\ (m - \lambda; \sigma_1, \dots, \sigma_r), (\alpha - \beta - m; \rho_1, \dots, \rho_r), \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,P} : (c'_j, \gamma'_j)_{1,P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,Q} : (d'_j, \delta'_j)_{1,Q'}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} \end{array} \right] \end{aligned} \quad (20)$$

provided (in addition to the usual convergence and existence conditions) that $\min\{\rho_i, \sigma_i\} > 0$, $i = 1, \dots, r$, $|\arg(x/\xi)| < \pi$, $\operatorname{Re}(\mu) < 0$ and

$$\operatorname{Re}(\beta) + \sum_{i=1}^r \rho_i \xi_i > -1, \quad (21)$$

where $\xi_1, \dots, \xi_r$ are given by (15).

FIGURE 2: Pochhammer's contour.

Remark 4. Adopting the representation based on the Pochhammer contour for fractional derivative and in view of Remark 3, the restrictions $\text{Re}(\beta) + \sum_{i=1}^r \rho_i \xi_i > -1$ can be modified to $\text{Re}(\beta) + \sum_{i=1}^r \rho_i \xi_i$ not a negative integer and $\text{Re}(\mu) < 0$ to μ not a negative integer.

Letting $\sigma_i \downarrow 0$, for all $i \in \{1, \dots, r\}$ and $\lambda = 0$ in (20) yields the following useful particular case:

$$\begin{aligned}
 & D_x^\alpha \{x^\beta H[z_1 x^{\rho_1}, \dots, z_r x^{\rho_r}]\} \\
 &= x^{\beta-\alpha} H_{P+1, Q+1: P', Q'; \dots; P^{(r)}, Q^{(r)}}^{0, N+1: M', N'; \dots; M^{(r)}, N^{(r)}} \\
 & \times \left[\begin{array}{l} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{array} \middle| \begin{array}{l} (-\beta; \rho_1, \dots, \rho_r), (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,P} : \\ (\alpha - \beta; \rho_1, \dots, \rho_r), (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,Q} : \\ (c'_j, \gamma'_j)_{1,P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ (d'_j, \delta'_j)_{1,Q'}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} \end{array} \right]. \quad (22)
 \end{aligned}$$

3. Expansion Theorem and Special Cases

In this section, we establish the new expansion theorem for the H -functions of several complex variables. Two special cases (presumably new) are also computed to demonstrate the importance of this new expansion theorem to the theory of special functions of mathematical physics.

Theorem 5. Assume that the convergence and existence conditions corresponding appropriately to the ones detailed in Section 1 are satisfied for each H -functions involved. Assume

also that the list of restrictions of (3) is satisfied, then the following expansion

$$\begin{aligned}
 & H_{P, Q: P', Q'; \dots; P^{(r)}, Q^{(r)}}^{0, N: M', N'; \dots; M^{(r)}, N^{(r)}} \\
 & \times \left[\begin{array}{l} y_1 x^{\rho_1} \\ \vdots \\ y_r x^{\rho_r} \end{array} \middle| \begin{array}{l} (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,P} : \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,Q} : \\ (c'_j, \gamma'_j)_{1,P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ (d'_j, \delta'_j)_{1,Q'}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} \end{array} \right] \\
 &= \frac{(az_1)}{x^\mu (x - z_1)^\nu} \\
 & \times \sum_{n=-\infty}^{\infty} \frac{e^{i\pi a(n+1)} \sin((\mu + an + \gamma)\pi)}{\sin((\mu - a + \gamma)\pi) \Gamma(1 - \nu + an + \gamma)} \\
 & \times \left(\frac{x - z_1}{x} \right)^{an + \gamma} \times z_1^{\mu + \nu - 1} H_{P+1, Q+1: P', Q'; \dots; P^{(r)}, Q^{(r)}}^{0, N+1: M', N'; \dots; M^{(r)}, N^{(r)}} \\
 & \times \left[\begin{array}{l} y_1 z_1^{\rho_1} \\ \vdots \\ y_r z_1^{\rho_r} \end{array} \middle| \begin{array}{l} (-\mu - an - \gamma + 1; \rho_1, \dots, \rho_r), \\ (1 - \mu - \nu; \rho_1, \dots, \rho_r), \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,P} : (c'_j, \gamma'_j)_{1,P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,Q} : (d'_j, \delta'_j)_{1,Q'}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} \end{array} \right] \quad (23)
 \end{aligned}$$

holds true.

Proof. By setting $f(x) = H[y_1 x^{\rho_1}, \dots, y_r x^{\rho_r}]$, $z_2 = 0$ in (3) and with the help of (22), we find first that

$$\begin{aligned}
 & D_x^{-\nu + an + \gamma} x^{\mu + an + \gamma - 1} H[y_1 x^{\rho_1}, \dots, y_r x^{\rho_r}] \Big|_{x=z_1} \\
 &= z_1^{\mu + \nu - 1} H_{P+1, Q+1: P', Q'; \dots; P^{(r)}, Q^{(r)}}^{0, N+1: M', N'; \dots; M^{(r)}, N^{(r)}} \\
 & \times \left[\begin{array}{l} y_1 z_1^{\rho_1} \\ \vdots \\ y_r z_1^{\rho_r} \end{array} \middle| \begin{array}{l} (-\mu - an - \gamma + 1; \rho_1, \dots, \rho_r), \\ (1 - \mu - \nu; \rho_1, \dots, \rho_r), \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,P} : (c'_j, \gamma'_j)_{1,P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,Q} : (d'_j, \delta'_j)_{1,Q'}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} \end{array} \right]. \quad (24)
 \end{aligned}$$

Substituting the last result into (3) yields the desired result. $\square$

Example 6. For $N = P = Q = 0$ and $\rho_1 = \rho_2 = \dots = \rho_r = 1$ the multivariable H -function in the L.H.S. of (23) breaks up

into product of r H -functions and there holds the following result [22]:

$$H_{0,0:P',Q';\dots;P^{(r)},Q^{(r)}}^{0,0:M',N';\dots;M^{(r)},N^{(r)}} \times \left[\begin{array}{c} y_1 x \\ \vdots \\ y_r x \end{array} \middle| \begin{array}{c} - : (c'_j, \gamma'_j)_{1,P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ - : (d'_j, \delta'_j)_{1,Q'}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} \end{array} \right] \quad (25)$$

$$= \prod_{i=1}^r H_{P^{(i)},Q^{(i)}}^{M^{(i)},N^{(i)}} \left[\begin{array}{c} y_i x \\ \vdots \end{array} \middle| \begin{array}{c} (c_j^{(i)}, \gamma_j^{(i)})_{1,P^{(i)}} \\ (d_j^{(i)}, \delta_j^{(i)})_{1,Q^{(i)}} \end{array} \right].$$

Now setting $P^{(i)} = Q^{(i)} = M^{(i)} = N^{(i)} = \gamma_j^{(i)} = \delta_j^{(i)} = 1$, $d_j^{(i)} = 0$, $i = 1, \dots, r$, substituting $c_j^{(i)}$ by $1 - c_j^{(i)}$, $i = 1, \dots, r$ into (25), replacing x by $-x$, and making use of the following special case [23]

$$H_{1,1}^{1,1} \left[\begin{array}{c} (1 - \nu, 1) \\ z \end{array} \middle| \begin{array}{c} (0, 1) \end{array} \right] = \Gamma(\nu) (1 + z)^{-\nu}, \quad |z| < 1, \quad (26)$$

gives

$$H_{0,0:1,1;\dots;1,1}^{0,0:1,1;\dots;1,1} \times \left[\begin{array}{c} -y_1 x \\ \vdots \\ -y_r x \end{array} \middle| \begin{array}{c} - : (1 - c'_j, 1); \dots; (1 - c_1^{(r)}, 1) \\ - : (0, 1); \dots; (0, 1) \end{array} \right] \quad (27)$$

$$= \prod_{i=1}^r \Gamma(c_1^{(i)}) (1 - y_i x)^{-c_1^{(i)}}.$$

Moreover, we also know from [21, page 649, equation (3.6)] that

$$D_x^{\lambda-\mu} x^{\lambda-1} \prod_{i=1}^r (1 - xz_i)^{-\alpha_i} \quad (28)$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} F_D^{(r)} [\lambda, \alpha_1, \dots, \alpha_r; \mu; xz_1, \dots, xz_r],$$

where $F_D^{(r)}$ denotes the Lauricella's hypergeometric function of r variables [24, page 33, equation (4)]. Thus, we have

$$D_x^{-\nu+an+\gamma} x^{\mu+an+\gamma-1} \prod_{i=1}^r (1 - y_i x)^{-c_1^{(i)}} \bigg|_{x=z_1} \quad (29)$$

$$= \frac{\Gamma(\mu + an + \gamma)}{\Gamma(\mu + \nu)} z_1^{\mu+\nu-1} F_D^{(r)}$$

$$\times [\mu + an + \gamma, c_1^{(1)}, \dots, c_1^{(r)}; \mu + \nu; y_1 z_1, \dots, y_r z_1]$$

with $\max\{|y_1|, \dots, |y_r|\} < 1$. Combining (27) and (29) with Theorem 5 yields the following new expansion involving the Lauricella's hypergeometric function of r variables

$$\prod_{i=1}^r (1 - y_i x)^{-c_1^{(i)}} = \frac{a}{x^\mu (x - z_1)^\nu}$$

$$\times \sum_{n=-\infty}^{\infty} \frac{e^{i\pi a(n+1)} \sin((\mu + an + \gamma)\pi) \Gamma(\mu + an + \gamma)}{\sin((\mu - a + \gamma)\pi) \Gamma(1 - \nu + an + \gamma)}$$

$$\times \frac{z_1^{\mu+\nu}}{\Gamma(\mu + \nu)} F_D^{(r)} [\mu + an + \gamma, c_1^{(1)}, \dots, c_1^{(r)}; \mu + \nu; y_1 z_1, \dots, y_r z_1] \left(\frac{x - z_1}{x} \right)^{an+\gamma} \quad (30)$$

with $\max\{|y_1 z_1|, \dots, |y_r z_1|\} < 1$.

Example 7. Setting $\rho_1 = \dots = \rho_r = 1$ and $\alpha_j^{(i)} = \beta_j^{(i)} = \gamma_j^{(i)} = \delta_j^{(i)} = 1$, $i = 1, \dots, r$ in (23) and with the help of the result given by Srivastava and Panda [10] connecting the generalized Lauricella function and the multivariable H -function:

$$H_{P,Q:P',Q'+1;\dots;P^{(r)},Q^{(r)+1}}^{0,P:1,P';\dots;1,P^{(r)}} \times \left[\begin{array}{c} xy_1 \\ \vdots \\ xy_r \end{array} \middle| \begin{array}{c} (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,P} : \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,Q} : \\ (c'_j, \gamma'_j)_{1,P'}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ (0, 1), (d'_j, \delta'_j)_{1,Q'}; \dots; (0, 1), (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} \end{array} \right]$$

$$= \frac{[\prod_{j=1}^P \Gamma(1 - a_j)] [\prod_{j=1}^{P'} \Gamma(1 - c'_j)] \dots [\prod_{j=1}^{P^{(r)}} \Gamma(1 - c_j^{(r)})]}{[\prod_{j=1}^Q \Gamma(1 - b_j)] [\prod_{j=1}^{Q'} \Gamma(1 - d'_j)] \dots [\prod_{j=1}^{Q^{(r)}} \Gamma(1 - d_j^{(r)})]}$$

$$\times F_{Q:Q';\dots;Q^{(r)}}^{P:P';\dots;P^{(r)}} \times \left[\begin{array}{c} (1 - a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,P} : (1 - c'_j, \gamma'_j)_{1,P'}; \dots; \\ (1 - b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,Q} : (1 - d'_j, \delta'_j)_{1,Q'}; \dots; \\ (1 - c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}} \\ (1 - d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}} - x y_1, \dots, -x y_r \end{array} \right], \quad (31)$$

we obtain the following expansion involving an extension of the Kampé de Fériet series [24, page 38, equation (24)]

$$\begin{aligned}
 & F_{Q:P'; \dots; Q^{(r)}}^{P:P'; \dots; P^{(r)}} \left[\begin{matrix} (1-a_P) : (1-c_{P'}^{(r)}); \dots; (1-c_{P^{(r)}}^{(r)}); \\ (1-b_Q) : (1-d_{Q'}^{(r)}); \dots; (1-d_{Q^{(r)}}^{(r)}); \end{matrix} \right. \left. x y_1, \dots, x y_r \right] \\
 &= \frac{a z_1^{\mu+\nu}}{x^\mu (x-z_1)^\nu} \sum_{n=-\infty}^{\infty} \frac{e^{i\pi a(n+1)} \sin((\mu+an+\gamma)\pi) \Gamma(\mu+an+\gamma)}{\sin((\mu-a+\gamma)\pi) \Gamma(1-\nu+an+\gamma) \Gamma(\mu+\nu)} \\
 &\quad \times \left(\frac{x-z_1}{x} \right)^{an+\gamma} \\
 & F_{Q+1:P'; \dots; Q^{(r)}}^{P+1:P'; \dots; P^{(r)}} \left[\begin{matrix} \mu+an+\gamma, (1-a_P) : (1-c_{P'}^{(r)}); \dots; \\ \mu+\nu, (1-b_Q) : (1-d_{Q'}^{(r)}); \dots; \\ (1-c_{P^{(r)}}^{(r)}); \\ (1-d_{Q^{(r)}}^{(r)}); \end{matrix} \right. \left. z_1 y_1, \dots, z_1 y_r \right]. \quad (32)
 \end{aligned}$$

If we put $r = 1$, $P = 1$, $Q = 0$, and $z_1 = 1$ in (32), we have

$$\begin{aligned}
 & {}_1F_0 \left[\begin{matrix} 1-a_1; \\ -; \end{matrix} \right. \left. x y_1 \right] \\
 &= \frac{a}{x^\mu (x-1)^\nu} \\
 &\quad \times \sum_{n=-\infty}^{\infty} \frac{e^{i\pi a(n+1)} \sin((\mu+an+\gamma)\pi) \Gamma(\mu+an+\gamma)}{\sin((\mu-a+\gamma)\pi) \Gamma(1-\nu+an+\gamma) \Gamma(\mu+\nu)} \\
 &\quad \times \left(\frac{x-1}{x} \right)^{an+\gamma} \times {}_2F_1 \left[\begin{matrix} \mu+an+\gamma, 1-a_1; \\ \mu+\nu; \end{matrix} \right. \left. y_1 \right]. \quad (33)
 \end{aligned}$$

Making the following changes

$$y_1 \mapsto \frac{1-y_1}{2}, \quad a_1 \mapsto 1+\gamma \quad (34)$$

and using the hypergeometric representation for the Jacobi functions [25, page 35]

$$P_\mu^{(\theta, \gamma)}(z) = \frac{\Gamma(1+\theta+\mu)}{\Gamma(1+\mu)\Gamma(1+\theta)} {}_2F_1 \left[\begin{matrix} \mu+\gamma+\theta+1, -\mu; \\ 1+\theta; \end{matrix} \right. \left. \frac{1-z}{2} \right] \quad (35)$$

yields

$$\begin{aligned}
 & {}_1F_0 \left[\begin{matrix} -\gamma; \\ -; \end{matrix} \right. \left. \frac{x(1-y_1)}{2} \right] \\
 &= \left[1 - \left(\frac{x(1-y_1)}{2} \right) \right]^\gamma = \frac{a}{x^\mu (x-1)^\nu} \\
 &\quad \times \sum_{n=-\infty}^{\infty} \frac{e^{i\pi a(n+1)} \sin((\mu+an+\gamma)\pi) \Gamma(\mu+an+\gamma)}{\sin((\mu-a+\gamma)\pi) \Gamma(1-\nu+an+\gamma) \Gamma(\mu+\nu)} \\
 &\quad \times \left(\frac{x-1}{x} \right)^{an+\gamma} \times \frac{\Gamma(1+\gamma) \Gamma(\mu+\nu)}{\Gamma(\mu+\nu+\gamma)} P_\gamma^{(\mu+\nu-1, an-\nu)}(y_1). \quad (36)
 \end{aligned}$$

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