Research Article

Global Attractivity Results on Complete Ordered Metric Spaces for Third-Order Difference Equations

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We establish fixed-point theorems for mixed monotone mappings in the setting of ordered metric spaces which satisfy a contractive condition for all points that are related by a given ordering. We also give a global attractivity result for all solutions of the difference equation $u_{n+1} = F(u_n, u_{n-1}, u_{n-2})$, $n = 2, 3, \dots$, where F satisfies certain monotonicity conditions with respect to the given ordering. As an application of our obtained results, we present some iterative algorithms to solve a class of matrix equations. A numerical example is also presented to test the validity of the algorithms.

1. Introduction

The following global attractivity result from [1] (see also [2]) is very useful in establishing convergence results in many situations.

Theorem 1 (see [1]). *Let $[a, b]$ be a closed and bounded interval of real numbers and let $F \in C([a, b]^3, [a, b])$ satisfy the following conditions.*

- (i) *The function $F(z_1, z_2, z_3)$ is monotonic in each of its arguments.*
- (ii) *For each $m, M \in [a, b]$ and for each $i \in \{1, 2, 3\}$, one defines*

$$M_i(m, M) = \begin{cases} M, & \text{if } F \text{ is increasing in } z_i, \\ m, & \text{if } F \text{ is decreasing in } z_i, \end{cases} \quad (1)$$

$$m_i(m, M) = M_i(M, m)$$

and assume that if (m, M) is a solution of the system

$$\begin{aligned} M &= F(M_1(m, M), M_2(m, M), M_3(m, M)), \\ m &= F(m_1(m, M), m_2(m, M), m_3(m, M)), \end{aligned} \quad (2)$$

then $M = m$.

Then there exists exactly one equilibrium $\bar{x}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}), \quad n = 0, 1, \dots \quad (3)$$

and every solution of the above equation converges to $\bar{x}$.

The above result in Theorem 1 attracted considerable attention of the leading specialists in difference equations and discrete dynamic systems and has been generalized and extended to the case of maps in $\mathbb{R}^n$, see [3], and maps in Banach space with cone, see [4–6].

Moreover, there has been recent interest in establishing fixed-point theorems in partially ordered complete metric spaces with a contractivity condition which holds for all points that are related by partial ordering, (see, e.g., [7–19]). These fixed-point results have been applied mainly to the existence of solutions of boundary value problems

for differential equations, namely [18], has been applied for solving a class of matrix equations.

In [20], Burgić et al. obtained the following global attractivity result for mixed monotone mappings in partially ordered complete metric spaces (see also Gnana Bhaskar and Lakshmikantham [10]).

Theorem 2 (see Burgić et al. [20]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \rightarrow X$ be a map such that $F(x, y)$ is nonincreasing in x for all $y \in X$ and nondecreasing in y for all $x \in X$. Suppose that the following conditions hold.*

(i) *There exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \quad \forall x \leq u, y \geq v. \quad (4)$$

(ii) *There exists $x_0, y_0 \in X$ such that the following condition holds:*

$$x_0 \leq F(x_0, y_0), \quad y_0 \geq F(y_0, x_0). \quad (5)$$

(iii) *If $\{x_n\} \subset X$ is a nondecreasing convergent sequence such that $\lim_{n \rightarrow \infty} x_n = x$, then $x_n \leq x$, for all $n \in \mathbb{N}$ and if $\{y_n\} \subset X$ is a nonincreasing convergent sequence such that $\lim_{n \rightarrow \infty} y_n = y$, then $y_n \geq y$, for all $n \in \mathbb{N}$; if $x_n \leq y_n$ for every n , then $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$.*

Then one has the following.

(a) *For every initial point $x_0, y_0 \in X$ such that condition (ii) holds, $F^n(x_0, y_0) \rightarrow x, F^n(y_0, x_0) \rightarrow y, n \rightarrow \infty$, where $x, y \in X$ satisfy*

$$x = F(x, y), \quad y = F(y, x). \quad (6)$$

(b) *If $x_0 \leq y_0$ in condition (ii), then $x = y$. If in addition $x = y$, then $\{x_n\}, \{y_n\}$ converge to the equilibrium of the equation*

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n), \quad n = 1, 2, \dots \quad (7)$$

(c) *In particular, every solution $\{z_n\}$ of*

$$z_{n+1} = F(z_n, z_{n-1}), \quad n = 2, 3, \dots \quad (8)$$

such that $x_0 \leq z_0, z_1 \leq y_0$ converges to the equilibrium of (7).

In this paper, motivated by the results and ideas in a recent work of Berinde and Borcut [9], we extend Theorem 2 to mappings $F : X \times X \times X \rightarrow X$. Such extension allows us to study the third-order difference equation

$$u_{n+1} = F(u_n, u_{n-1}, u_{n-2}), \quad n = 2, 3, \dots \quad (9)$$

The presented theorems also extend and generalize the work in [9]. We use our obtained results to build some iterative algorithms to solve a class of matrix equations. A numerical example is also presented to test the validity of the algorithms.

Now we introduce the following concepts.

Definition 3. Let X be a nonempty set and $F : X \times X \times X \rightarrow X$ a given mapping. One says that $(x, y, z) \in X \times X \times X$ is a fixed-point of the third order if

$$F(x, y, x) = x, \quad F(y, x, y) = y, \quad F(z, y, z) = z. \quad (10)$$

Definition 4. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \rightarrow X$ a given mapping. one says that F has the mixed monotone property if $F(x, y, z)$ is monotonously increasing in x and z and is monotonously decreasing in y ; that is, for any $x, y, z \in X$,

- (i) $x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z)$,
- (ii) $y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$, and
- (iii) $z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2)$.

Through this paper we will use the following notations.

Let $(X, \leq)$ be a partially ordered set.

(i) For $x, y, z \in X \times X \times X$ the notation $x \in [y, z]$ means that $x \geq y$ and $x \leq z$.

(ii) We endow $X \times X \times X$ with the partial order that we denote also by $\leq$, defined by

$$(x, y, z), (u, v, w) \in X \times X \times X, \quad (11)$$

$$(x, y, z) \leq (u, v, w) \iff x \leq u, y \geq v, z \leq w.$$

(iii) Let $F : X \times X \times X \rightarrow X$ be a given mapping. For all $x, y, z \in X$, we denote

$$F^2(x, y, z) = F(F(x, y, x), F(y, x, y), F(z, y, z)),$$

$$F^{p+1}(x, y, z) = F(F^p(x, y, x), F^p(y, x, y), F^p(z, y, z)),$$

$$\forall p \geq 2. \quad (12)$$

2. Main Result

Our first result is the following.

Theorem 5. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Suppose that the following conditions hold.*

(i) *There exists $\lambda \in [0, 1)$ with*

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) \\ & \leq \lambda \max \{d(x, u), d(y, v), d(z, w), d(F(x, y, x), x), \\ & d(F(x, y, x), u), d(F(y, x, y), y), \\ & d(F(y, x, y), v), d(F(z, y, z), z), \\ & d(F(z, y, z), w)\} \end{aligned} \quad (13)$$

for all $(x, y, z) \leq (u, v, w)$.

(ii) There exist $x_0, y_0, z_0 \in X$ such that

$$\begin{aligned} x_0 &\leq F(x_0, y_0, x_0), & y_0 &\geq F(y_0, x_0, y_0), \\ z_0 &\leq F(z_0, y_0, z_0). \end{aligned} \quad (14)$$

(iii) If $\{x_n\} \subset X$ is a nondecreasing convergent sequence such that $\lim_{n \rightarrow \infty} x_n = x$, then $x_n \leq x$ for all $n \in \mathbb{N}$, if $\{y_n\} \subset X$ is a nonincreasing convergent sequence such that $\lim_{n \rightarrow \infty} y_n = y$, then $y_n \geq y$ for every $n \in \mathbb{N}$, and if $x_n \leq y_n$ for every n , then $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$.

Then one has the following.

(a) For every initial point $(x_0, y_0, z_0) \in X \times X \times X$ such that condition (14) holds:

$$\begin{aligned} x_n &= F^n(x_0, y_0, x_0) \longrightarrow x, & y_n &= F^n(y_0, x_0, y_0) \longrightarrow y, \\ z_n &= F^n(z_0, y_0, z_0) \longrightarrow z \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (15)$$

where x, y , and z satisfy

$$x = F(x, y, x), \quad y = F(y, x, y), \quad z = F(z, y, z). \quad (16)$$

If $x_0 \leq y_0$ and $z_0 \leq y_0$ in condition (14), then $x = y = z$ and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to the equilibrium of the equation

$$\begin{aligned} x_{n+1} &= F(x_n, y_n, x_n), & y_{n+1} &= F(y_n, x_n, y_n), \\ z_{n+1} &= F(z_n, y_n, z_n), & n &= 0, 1, 2, \dots \end{aligned} \quad (17)$$

(b) In particular, every solution $\{u_n\}$ of

$$u_{n+1} = F(u_n, u_{n-1}, u_{n-2}), \quad n = 2, 3, \dots, \quad (18)$$

such that $u_0, u_1, u_2 \in [x_0, y_0]$ (or $[z_0, y_0]$) converges to the equilibrium of (18).

(c) The following estimates hold:

$$\begin{aligned} d(x, F^n(x_0, y_0, x_0)) &\leq \frac{\lambda^n}{1-\lambda} \max \{d(F(x_0, y_0, x_0), x_0), \\ &\quad d(F(y_0, x_0, y_0), y_0), \\ &\quad d(z_0, F(z_0, y_0, z_0))\}, \end{aligned}$$

$$\begin{aligned} d(y, F^n(y_0, x_0, y_0)) &\leq \frac{\lambda^n}{1-\lambda} \max \{d(F(x_0, y_0, x_0), x_0), \\ &\quad d(F(y_0, x_0, y_0), y_0), \\ &\quad d(z_0, F(z_0, y_0, z_0))\}, \\ d(z, F^n(z_0, y_0, z_0)) &\leq \frac{\lambda^n}{1-\lambda} \max \{d(F(x_0, y_0, x_0), x_0), \\ &\quad d(F(y_0, x_0, y_0), y_0), \\ &\quad d(F(z_0, y_0, z_0), z_0)\}. \end{aligned} \quad (19)$$

Proof. Let $x_0, y_0, z_0 \in X$ such that condition (14) is satisfied. Denote $x_1 = F(x_0, y_0, x_0)$, $y_1 = F(y_0, x_0, y_0)$, and $z_1 = F(z_0, y_0, z_0)$. Since $x_0 \leq F(x_0, y_0, x_0) = x_1$, $y_0 \geq F(y_0, x_0, y_0) = y_1$, and $z_0 \leq F(z_0, y_0, z_0) = z_1$, from the mixed monotone property of F , we get that

$$\begin{aligned} x_0 &\leq x_1 = F(x_0, y_0, x_0) \leq x_2 = F(x_1, y_1, x_1), \\ y_0 &\geq y_1 = F(y_0, x_0, y_0) \geq y_2 = F(y_1, x_1, y_1), \\ z_0 &\leq z_1 = F(z_0, y_0, z_0) \leq z_2 = F(z_1, y_1, z_1). \end{aligned} \quad (20)$$

Consider the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by (17). By induction and using the mixed monotone property of F , we obtain easily that

$$\begin{aligned} x_0 &\leq x_1 \leq \dots \leq x_n \leq \dots; & y_0 &\geq y_1 \geq \dots \geq y_n \geq \dots; \\ z_0 &\leq z_1 \leq \dots \leq z_n \leq \dots. \end{aligned} \quad (21)$$

For the sake of clarity, for all $n \geq 1$, denote

$$\begin{aligned} D_n^x &= d(x_{n-1}, x_n), & D_n^y &= d(y_{n-1}, y_n), \\ D_n^z &= d(z_{n-1}, z_n). \end{aligned} \quad (22)$$

We claim that, for all $n \geq 1$, we have

$$\begin{aligned} D_{n+1}^x &\leq \lambda^n \max \{D_1^x, D_1^y, D_1^z\}, \\ D_{n+1}^y &\leq \lambda^n \max \{D_1^x, D_1^y, D_1^z\}, \\ D_{n+1}^z &\leq \lambda^n \max \{D_1^x, D_1^y, D_1^z\}. \end{aligned} \quad (23)$$

By (13) and (21), we obtain

$$\begin{aligned}
D_2^x &= d(x_1, x_2) = d(F(x_0, y_0, x_0), F(x_1, y_1, x_1)) \\
&\leq \lambda \max \{d(x_0, x_1), d(y_0, y_1), d(x_0, x_1), \\
&\quad d(F(x_0, y_0, x_0), x_0), d(F(x_0, y_0, x_0), x_1), \\
&\quad d(F(y_0, x_0, y_0), y_0), d(F(y_0, x_0, y_0), y_1), \\
&\quad d(F(x_0, y_0, x_0), x_0), d(F(x_0, y_0, x_0), x_1)\} \\
&= \lambda \max \{d(x_0, x_1), d(y_0, y_1)\}, \\
D_2^y &= d(y_1, y_2) = d(F(y_0, x_0, y_0), F(y_1, x_1, y_1)) \\
&\leq \lambda \max \{d(y_0, y_1), d(x_0, x_1), d(y_0, y_1), \\
&\quad d(F(y_0, x_0, y_0), y_0), d(F(y_0, x_0, y_0), y_1), \\
&\quad d(F(x_0, y_0, x_0), x_0), d(F(x_0, y_0, x_0), x_1), \\
&\quad d(F(y_0, x_0, y_0), y_0), d(F(y_0, x_0, y_0), y_1)\} \\
&= \lambda \max \{d(y_0, y_1), d(x_0, x_1)\}, \\
D_2^z &= d(z_1, z_2) = d(F(z_0, y_0, z_0), F(z_1, y_1, z_1)) \\
&\leq \lambda \max \{d(z_0, z_1), d(y_0, y_1), d(z_0, z_1), \\
&\quad d(F(z_0, y_0, z_0), z_0), d(F(z_0, y_0, z_0), z_1), \\
&\quad d(F(y_0, z_0, y_0), y_0), d(F(y_0, z_0, y_0), y_1), \\
&\quad d(F(z_0, y_0, z_0), z_0), d(F(z_0, y_0, z_0), z_1)\} \\
&= \lambda \max \{d(z_0, z_1), d(y_0, y_1)\}.
\end{aligned} \tag{24}$$

Thus we get that

$$\begin{aligned}
D_2^x &\leq \lambda \max \{D_1^x, D_1^y, D_1^z\}, \quad D_2^y \leq \lambda \max \{D_1^x, D_1^y, D_1^z\}, \\
D_2^z &\leq \lambda \max \{D_1^x, D_1^y, D_1^z\}.
\end{aligned} \tag{25}$$

Then our claim holds for $n = 1$. Suppose now that (23) holds for some fixed $n \geq 1$. Similarly, by (13) and (21), we obtain

$$\begin{aligned}
D_{n+2}^x &= d(x_{n+1}, x_{n+2}) \\
&= d(F(x_n, y_n, x_n), F(x_{n+1}, y_{n+1}, x_{n+1})) \\
&\leq \lambda \max \{D_{n+1}^x, D_{n+1}^y\} \\
&\leq \lambda^{n+1} \max \{D_1^x, D_1^y, D_1^z\}.
\end{aligned} \tag{26}$$

Similarly, one can show that

$$\begin{aligned}
D_{n+2}^y &\leq \lambda^{n+1} \max \{D_1^x, D_1^y, D_1^z\}, \\
D_{n+2}^z &\leq \lambda^{n+1} \max \{D_1^x, D_1^y, D_1^z\}.
\end{aligned} \tag{27}$$

Then by the induction principle, (23) holds for all $n \geq 1$.

Now we will prove that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are Cauchy sequences in the metric space (X, d) . Using (23) and the triangular inequality, for $m > n$, we have

$$\begin{aligned}
d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) = \sum_{i=n}^{m-1} D_{i+1}^x \\
&\leq \sum_{i=n}^{m-1} \lambda^i \max \{D_1^x, D_1^y, D_1^z\} \\
&\leq \left(\sum_{i=n}^{m-1} \lambda^i \right) \max \{D_1^x, D_1^y, D_1^z\} \\
&\leq \frac{\lambda^n}{1-\lambda} \max \{D_1^x, D_1^y, D_1^z\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{28}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Similarly, one can prove that $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Since (X, d) is complete, there exist $x, y, z \in X$ such that

$$\begin{aligned}
x_n &\rightarrow x, \quad y_n \rightarrow y, \\
z_n &\rightarrow z \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{29}$$

From condition (iii) and (21), we get that

$$x_n \leq x, \quad y_n \geq y, \quad z_n \leq z, \quad \forall n. \tag{30}$$

We claim that $x = F(x, y, x)$. Indeed, we have

$$\begin{aligned}
&d(x, F(x, y, x)) \\
&\leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y, x)) \\
&= d(x, x_{n+1}) + d(F(x_n, y_n, x_n), F(x, y, x)) \\
&\leq d(x, x_{n+1}) \\
&\quad + \lambda \max \{d(x_n, x), d(y_n, y), \\
&\quad d(x_n, x), d(F(x_n, y_n, x_n), x_n), \\
&\quad d(F(x_n, y_n, x_n), x), d(F(y_n, x_n, y_n), y_n), \\
&\quad d(F(y_n, x_n, y_n), y), d(F(x_n, y_n, x_n), x_n), \\
&\quad d(F(x_n, y_n, x_n), x)\} \\
&= d(x, x_{n+1}) + \lambda \max \{d(x_n, x), d(y_n, y)\} \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{by (29)}.
\end{aligned} \tag{31}$$

Then, our claim holds; that is, $x = F(x, y, x)$. Similarly, one can show that $y = F(y, x, y)$ and $z = F(z, y, z)$. Thus we proved (16).

On the other hand, from (28), for $m > n$ with n being fixed, we have

$$\begin{aligned}
d(x_m, x_n) &= d(x_m, F^n(x_0, y_0, x_0)) \\
&\leq \frac{\lambda^n}{1-\lambda} \max \{D_1^x, D_1^y, D_1^z\}.
\end{aligned} \tag{32}$$

Letting $m \rightarrow \infty$ in the above inequality, we obtain

$$\begin{aligned} d(x, F^n(x_0, y_0, x_0)) \\ \leq \frac{\lambda^n}{1-\lambda} \max\{d(x_0, x_1), d(y_0, y_1), d(z_0, z_1)\}; \end{aligned} \quad (33)$$

that is,

$$\begin{aligned} d(x, F^n(x_0, y_0, x_0)) \\ \leq \frac{\lambda^n}{1-\lambda} \max\{d(x_0, F(x_0, y_0, x_0)), \\ d(y_0, F(y_0, x_0, y_0)), \\ d(z_0, F(z_0, y_0, z_0))\}. \end{aligned} \quad (34)$$

Similarly, one can show that

$$\begin{aligned} d(y, F^n(y_0, x_0, y_0)) \\ \leq \frac{\lambda^n}{1-\lambda} \max\{d(x_0, F(x_0, y_0, x_0)), \\ d(y_0, F(y_0, x_0, y_0)), \\ d(z_0, F(z_0, y_0, z_0))\}, \\ d(z, F^n(z_0, y_0, z_0)) \\ \leq \frac{\lambda^n}{1-\lambda} \max\{d(x_0, F(x_0, y_0, x_0)), \\ d(y_0, F(y_0, x_0, y_0)), \\ d(z_0, F(z_0, y_0, z_0))\}. \end{aligned} \quad (35)$$

Thus we proved (19).

Now, if $x_0 \leq y_0$ and $z_0 \leq y_0$, we claim that, for all $n \in \mathbb{N}$, $x_n \leq y_n$ and $z_n \leq y_n$. Indeed, by the mixed monotone property of F ,

$$\begin{aligned} x_1 &= F(x_0, y_0, x_0) \leq F(y_0, x_0, y_0) = y_1, \\ z_1 &= F(z_0, y_0, z_0) \leq F(y_0, x_0, y_0) = y_1. \end{aligned} \quad (36)$$

Assume that $x_n \leq y_n$ and $z_n \leq y_n$ for some n . Then,

$$x_{n+1} = F(x_n, y_n, x_n) \leq F(y_n, x_n, y_n) = y_{n+1}, \quad (37)$$

and similarly for y_n and z_n . Thus we proved that

$$x_n \leq y_n, \quad z_n \leq y_n, \quad \forall n \in \mathbb{N}. \quad (38)$$

Next, from (38), we have

$$\begin{aligned} d(x, y) &\leq d(x, x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y_{n+1}, y) \\ &\leq d(x, x_{n+1}) + d(F(x_n, y_n, x_n), F(y_n, x_n, y_n)) \\ &\quad + d(y_{n+1}, y) \\ &\leq d(x, x_{n+1}) + \lambda d(x_n, z_n) + d(y_{n+1}, y) \\ &\vdots \\ &\leq d(x, x_{n+1}) + \lambda^n d(x_0, y_0) + d(y_{n+1}, y) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (39)$$

which implies that $x = y$. Similarly, we obtain that $d(y, z) = 0$, that is, $y = z$. Then $x = y = z$.

Now, assume that $x_0 \leq y_0$. Then, in view of the monotonicity of F , we have

$$\begin{aligned} x_1 &= F(x_0, y_0, x_0) \leq x_2 = F(x_1, y_1, x_1) \\ &\leq F(y_1, x_1, y_1) = y_2 \leq F(y_0, x_0, y_0) = y_1. \end{aligned} \quad (40)$$

Continuing this process, one can show that

$$x_n \leq x_{n+1} \leq y_{n+1} \leq y_n, \quad \forall n. \quad (41)$$

If we assume that $u_0, u_1, u_2 \in [x_0, y_0]$, in view of the monotonicity of F , we have

$$\begin{aligned} x_1 &= F(x_0, y_0, x_0) \leq F(u_2, u_1, u_0) \\ &= u_3 \leq F(y_0, x_0, y_0) = y_1, \\ x_1 &= F(x_0, y_0, x_0) \leq F(u_3, u_2, u_1) \\ &= u_4 \leq F(y_0, x_0, y_0) = y_1, \\ x_1 &= F(x_0, y_0, x_0) \leq F(u_4, u_3, u_2) \\ &= u_5 \leq F(y_0, x_0, y_0) = y_1. \end{aligned} \quad (42)$$

Continuing in a similar way, we can prove that

$$x_k \leq u_{3k}, u_{3k+1}, u_{3k+2} \leq y_k, \quad \forall k \geq 1. \quad (43)$$

Letting $k \rightarrow \infty$ and using (iii) and (29), we get that $u_n \rightarrow x$ as $n \rightarrow \infty$, where x is the equilibrium of (17). Similarly, if $z_0 \leq y_0$ and $u_0, u_1, u_2 \in [z_0, y_0]$, we obtain also the same result. $\square$

Remark 6. If we replace the condition, if $\{x_n\} \subset X$ is a nondecreasing convergent sequence such that $\lim_{n \rightarrow \infty} x_n = x$, then $x_n \leq x$ for all $n \in \mathbb{N}$ and if $\{y_n\} \subset X$ is a nonincreasing convergent sequence such that $\lim_{n \rightarrow \infty} y_n = y$, then $y_n \geq y$ for every $n \in \mathbb{N}$, by the continuity of F , we can check easily that the result of Theorem 5 holds also in this case.

Theorem 7. In addition to the hypotheses of Theorem 5, suppose that for every $(x, y, z), (x', y', z') \in X \times X \times X$, there

exists (u, v, w) such that $(x, y, z) \leq (u, v, w)$ and $(x', y', z') \leq (u, v, w)$. Then one obtains the uniqueness of the fixed point of the third order.

Proof. From (a) of Theorem 5, we know that F admits a fixed point of the third order $(x, y, z) \in X \times X \times X$; that is,

$$x = F(x, y, x), \quad y = F(y, x, y), \quad z = F(z, y, z), \quad (44)$$

where

$$\begin{aligned} x_n = F^n(x_0, y_0, x_0) &\longrightarrow x, & y_n = F^n(y_0, x_0, y_0) &\longrightarrow y, \\ z_n = F^n(z_0, y_0, z_0) &\longrightarrow z \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (45)$$

Suppose that $(x^*, y^*, z^*) \in X \times X \times X$ is another fixed point of the third order of F ; that is,

$$\begin{aligned} x^* &= F(x^*, y^*, x^*), & y^* &= F(y^*, x^*, y^*), \\ z^* &= F(z^*, y^*, z^*). \end{aligned} \quad (46)$$

We will prove that

$$\eta((x, y, z), (x^*, y^*, z^*)) = 0, \quad (47)$$

where

$$\begin{aligned} \eta((x, y, z), (x^*, y^*, z^*)) \\ = \max\{d(x, x^*), d(y, y^*), d(z, z^*)\}. \end{aligned} \quad (48)$$

From the hypothesis of Theorem 7, there exists (u, v, w) such that

$$(x, y, z) \leq (u, v, w), \quad (x^*, y^*, z^*) \leq (u, v, w). \quad (49)$$

Since F is a mixed monotone operator, we have

$$\begin{aligned} (F^n(x, y, x), F^n(y, x, y), F^n(z, y, z)) \\ = (x, y, z) \leq (F^n(u, v, w), F^n(v, u, v), F^n(w, v, w)), \\ (F^n(x^*, y^*, x^*), F^n(y^*, x^*, y^*), F^n(z^*, y^*, z^*)) \\ = (x^*, y^*, z^*) \leq (F^n(u, v, w), F^n(v, u, v), \\ F^n(w, v, w)). \end{aligned} \quad (50)$$

We have

$$\begin{aligned} \eta((x, y, z), (x^*, y^*, z^*)) \\ = \eta((F^n(x, y, x), F^n(y, x, y), F^n(z, y, z)), \\ (F^n(x^*, y^*, x^*), F^n(y^*, x^*, y^*), \\ F^n(z^*, y^*, z^*))) \\ \leq \eta((F^n(x, y, x), F^n(y, x, y), F^n(z, y, z)), \\ (F^n(u, v, u), F^n(v, u, v), F^n(w, v, w))) \\ + \eta((F^n(u, v, u), F^n(v, u, v), F^n(w, v, w)), \\ (F^n(x^*, y^*, x^*), F^n(y^*, x^*, y^*), \\ F^n(z^*, y^*, z^*))) \\ \leq \lambda^n \max\{d(x, u), d(y, v), d(z, w)\} \\ + \lambda^n \max\{d(u, x^*), d(v, y^*), d(w, z^*)\} \\ \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \quad (51)$$

which implies that

$$\eta((x, y, z), (x^*, y^*, z^*)) = 0; \quad (52)$$

that is,

$$x = x^*, \quad y = y^*, \quad z = z^*. \quad (53)$$

Example 8. Let $X = [0, 1]$ be an ordered set with the natural ordering of real numbers and d a usual metric on X . Let $F : X \times X \times X \rightarrow X$ be defined by

$$F(x, y, z) = \frac{1 + x - y + 2z}{8} \quad \forall x, y, z \in X. \quad (54)$$

It is easy to check that F has the mixed monotone property on X . For $x, y, z, u, v, w \in X$ with $(x, y, z) \leq (u, v, w)$, we have

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \\ = \frac{1}{8} |x - u - (y - v) + 2(z - w)| \\ \leq \frac{1}{8} [|x - u| + |y - v| + 2|z - w|] \\ \leq \frac{3}{4} \max\{|x - u|, |y - v|, |z - w|\} \\ = \frac{3}{4} \max\{d(x, u), d(y, v), d(z, w)\} \\ \leq \frac{3}{4} \max\{d(x, u), d(y, v), d(z, w), \\ d(F(x, y, x), x), d(F(x, y, x), u), \\ d(F(y, x, y), y), d(F(y, x, y), v), \\ d(F(z, y, z), z), d(F(z, y, z), w)\}. \end{aligned} \quad (55)$$

Thus (13) is satisfied for $\lambda = 3/4 \in [0, 1]$. Thus all the conditions of Theorems 5 and 7 are satisfied. Moreover, there exists a unique $(x, y, z) = (1/6, 1/6, 1/6)$ in $X \times X \times X$ such that

$$F(x, y, x) = x, \quad F(y, x, y) = y, \quad F(z, y, z) = z. \quad (56)$$

□

Corollary 9. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Suppose that the following conditions hold.

- (1) There exists $a_i \geq 0$ for $i = 1, 2, \dots, 9$ and $\sum_{i=1}^9 a_i < 1$ with

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \\ \leq a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w) \\ + a_4 d(F(x, y, x), x) + a_5 d(F(x, y, x), u) \\ + a_6 d(F(y, x, y), y) + a_7 d(F(y, x, y), v) \\ + a_8 d(F(z, y, z), z) + a_9 d(F(z, y, z), w) \end{aligned} \quad (57)$$

for all $(x, y, z) \leq (u, v, w)$.

- (2) There exist $x_0, y_0, z_0 \in X$ such that

$$\begin{aligned} x_0 \leq F(x_0, y_0, x_0), \quad y_0 \geq F(y_0, x_0, y_0), \\ z_0 \leq F(z_0, y_0, z_0). \end{aligned} \quad (58)$$

- (3) If $\{x_n\} \subset X$ is a nondecreasing convergent sequence such that $\lim_{n \rightarrow \infty} x_n = x$, then $x_n \leq x$ for all $n \in \mathbb{N}$, if $\{y_n\} \subset X$ is a nonincreasing convergent sequence such that $\lim_{n \rightarrow \infty} y_n = y$, then $y_n \geq y$ for every $n \in \mathbb{N}$, and if $x_n \leq y_n$ for every $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$.

Then one has the following.

- (a) For every initial point $(x_0, y_0, z_0) \in X \times X \times X$ such that condition (14) holds,

$$\begin{aligned} x_n = F^n(x_0, y_0, x_0) \longrightarrow x, \quad y_n = F^n(y_0, x_0, y_0) \longrightarrow y, \\ z_n = F^n(z_0, y_0, z_0) \longrightarrow z \quad \text{as } n \longrightarrow \infty, \end{aligned} \quad (59)$$

where x, y , and z satisfy

$$x = F(x, y, x), \quad y = F(y, x, y), \quad z = F(z, y, z). \quad (60)$$

If $x_0 \leq y_0$ and $z_0 \leq y_0$ in condition (14), then $x = y = z$ and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to the equilibrium of the equation

$$\begin{aligned} x_{n+1} = F(x_n, y_n, x_n), \quad y_{n+1} = F(y_n, x_n, y_n), \\ z_{n+1} = F(z_n, y_n, z_n), \quad n = 0, 1, 2, \dots \end{aligned} \quad (61)$$

- (b) In particular, every solution $\{u_n\}$ of

$$u_{n+1} = F(u_n, u_{n-1}, u_{n-2}), \quad n = 2, 3, \dots \quad (62)$$

such that $u_0, u_1, u_2 \in [x_0, y_0]$ (or $[z_0, y_0]$) converges to the equilibrium of (18).

Proof. Note that (14) implies that

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \\ \leq h \max \{d(x, u), d(y, v), d(z, w), \\ d(F(x, y, x), x), d(F(x, y, x), u), \\ d(F(y, x, y), y), d(F(y, x, y), v), \\ d(F(z, y, z), z), d(F(z, y, z), w)\} \end{aligned} \quad (63)$$

for all $(x, y, z) \leq (u, v, w)$, where $h = \sum_{i=1}^9 a_i < 1$ and the result follows from Theorem 5. □

Corollary 10. In addition to the hypotheses of Corollary 9, suppose that for every $(x, y, z), (x', y', z') \in X \times X \times X$, there exists (u, v, w) such that $(x, y, z) \leq (u, v, w)$ and $(x', y', z') \leq (u, v, w)$. Then one obtains the uniqueness of the fixed point of the third order.

Corollary 11. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Suppose that the following conditions hold.

- (1) There exists $\lambda \in [0, 1]$ with

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \\ \leq \lambda \max \{d(x, u), d(y, v), d(z, w)\} \\ \forall (x, y, z) \leq (u, v, w). \end{aligned} \quad (64)$$

- (2) There exist $x_0, y_0, z_0 \in X$ such that

$$\begin{aligned} x_0 \leq F(x_0, y_0, x_0), \quad y_0 \geq F(y_0, x_0, y_0), \\ z_0 \leq F(z_0, y_0, z_0). \end{aligned} \quad (65)$$

- (3) If $\{x_n\} \subset X$ is a nondecreasing convergent sequence such that $\lim_{n \rightarrow \infty} x_n = x$, then $x_n \leq x$ for all $n \in \mathbb{N}$; if $\{y_n\} \subset X$ is a nonincreasing convergent sequence such that $\lim_{n \rightarrow \infty} y_n = y$, then $y_n \geq y$ for every $n \in \mathbb{N}$, and if $x_n \leq y_n$ for every n , then $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$.

Then one has the following.

- (a) For every initial point $(x_0, y_0, z_0) \in X \times X \times X$ such that condition (14) holds,

$$\begin{aligned} x_n = F^n(x_0, y_0, x_0) \longrightarrow x, \quad y_n = F^n(y_0, x_0, y_0) \longrightarrow y, \\ z_n = F^n(z_0, y_0, z_0) \longrightarrow z \quad \text{as } n \longrightarrow \infty, \end{aligned} \quad (66)$$

where x, y , and z satisfy

$$x = F(x, y, x), \quad y = F(y, x, y), \quad z = F(z, y, z). \quad (67)$$

If $x_0 \leq y_0$ and $z_0 \leq y_0$ in condition (14), then $x = y = z$ and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to the equilibrium of the equation

$$\begin{aligned} x_{n+1} &= F(x_n, y_n, x_n), & y_{n+1} &= F(y_n, x_n, y_n), \\ z_{n+1} &= F(z_n, y_n, z_n), & n &= 0, 1, 2, \dots \end{aligned} \quad (68)$$

(b) In particular, every solution $\{u_n\}$ of

$$u_{n+1} = F(u_n, u_{n-1}, u_{n-2}), \quad n = 2, 3, \dots \quad (69)$$

such that $u_0, u_1, u_2 \in [x_0, y_0]$ (or $[z_0, y_0]$) converges to the equilibrium of (18).

(c) The following estimates hold:

$$\begin{aligned} & d(x, F^n(x_0, y_0, x_0)) \\ & \leq \frac{\lambda^n}{1-\lambda} \max \{d(F(x_0, y_0, x_0), x_0), \\ & \quad d(F(y_0, x_0, y_0), y_0), \\ & \quad d(z_0, F(z_0, y_0, z_0))\}, \\ & d(y, F^n(y_0, x_0, y_0)) \\ & \leq \frac{\lambda^n}{1-\lambda} \max \{d(F(x_0, y_0, x_0), x_0), \\ & \quad d(F(y_0, x_0, y_0), y_0), \\ & \quad d(z_0, F(z_0, y_0, z_0))\}, \\ & d(z, F^n(z_0, y_0, z_0)) \\ & \leq \frac{\lambda^n}{1-\lambda} \max \{d(F(x_0, y_0, x_0), x_0), \\ & \quad d(F(y_0, x_0, y_0), y_0), \\ & \quad d(F(z_0, y_0, z_0), z_0)\}. \end{aligned} \quad (70)$$

Corollary 12. In addition to the hypotheses of Corollary 11, suppose that for every $(x, y, z), (x', y', z') \in X \times X \times X$, there exists (u, v, w) such that $(x, y, z) \leq (u, v, w)$ and $(x', y', z') \leq (u, v, w)$. Then one obtains the uniqueness of the fixed point of the third order.

3. Application: Solving a Class of Third-Order Difference Matrix Equations

In this section, we apply our main results to the study of a class of third-order difference matrix equations. At first, we start by fixing some notations and recalling some preliminaries.

We will use the symbol $H(N)$ for the set of all $N \times N$ Hermitian matrices. We denote by $P(N)$ the set of all $N \times N$ Hermitian positive definite matrices. Instead of $X \in P(N)$ we will also write $X > 0$. Furthermore, $X \geq 0$ means that X is positive semidefinite. As a different notation for $X - Y \geq 0, X - Z \geq 0$ and $X \leq Z \leq Y$ we will use, respectively, $X \geq Y, X \geq Z$, and $Z \in [X, Y]$. The symbol $\|\cdot\|$ denotes the spectral norm, that is, $\|A\| = \lambda_{\max}^{1/2}(A^*A)$, the largest eigenvalue of A^*A . We denote by $\|\cdot\|_1$ the Ky Fan norm defined by $\|A\|_1 = \sum_{j=1}^N s_j(A)$, where $s_j(A), j = 1, \dots, N$ are the singular values of A . For a given $Q \in P(N)$, we define the modified norm $\|\cdot\|_{1,Q}$ given by $\|A\|_{1,Q} = \|Q^{1/2}AQ^{1/2}\|_1$. The set $H(N)$ endowed with this norm is a complete metric space for any positive definite matrix Q . For any $N \times N$ matrix A , we denote by $\text{tr}(A)$ the trace of the matrix A .

The following lemmas will be useful later.

Lemma 13 (see [18]). Let $A \geq 0$ and $B \geq 0$ be $N \times N$ matrices, then

$$0 \leq \text{tr}(AB) \leq \|A\| \text{tr}(B). \quad (71)$$

Lemma 14 (See [21]). Let $A \in H(N)$ satisfy $-I < A < I$, then $\|A\| < 1$.

Finally, we recall the well-known Schauder fixed-point theorem.

Theorem 15 (The Schauder fixed-point theorem). Let S be a nonempty, compact, and convex subset of a normed vector space. Every continuous function $f : S \rightarrow S$ mapping S into itself has a fixed point.

Now, we consider the class of third-order difference matrix equations:

$$\begin{aligned} U_{n+1} &= Q + \sum_{i=1}^m A_i U_n A_i - \sum_{i=1}^m B_i U_{n-1} B_i \\ &+ \sum_{i=1}^m C_i U_{n-2} C_i, \quad n = 2, 3, \dots \end{aligned} \quad (72)$$

for given $U_0, U_1, U_2 > 0$, where $Q > 0$ and A_i, B_i , and C_i are $N \times N$ Hermitian matrices. This type of difference equations often arises from many areas such as ladder networks [22, 23], dynamic programming [24, 25], and control theory [26, 27].

3.1. A Convergence Result

Theorem 16. Suppose that

$$\begin{aligned} \sum_{i=1}^m A_i Q A_i &< \frac{Q}{4}, & \sum_{i=1}^m B_i Q B_i &< \frac{Q}{2}, \\ \sum_{i=1}^m C_i Q C_i &< \frac{Q}{4}. \end{aligned} \quad (73)$$

Then, one has the following.

(i) Equation (72) has one and only one equilibrium point $\widehat{X} > 0$.

(ii) $\widehat{X} \in [F(0, Q, 0), F(2Q, 0, 2Q)]$, where

$$F(0, 2Q, 0) = Q - 2 \sum_{i=1}^m B_i Q B_i, \quad (74)$$

$$F(2Q, 0, 2Q) = Q + 2 \sum_{i=1}^m A_i Q A_i + 2 \sum_{i=1}^m C_i Q C_i.$$

(iii) The sequences (X_k) and (Y_k) defined by $X_0 = 0 = Z_0$, $Y_0 = 2Q$, and

$$\begin{aligned} X_{k+1} &= Q + \sum_{i=1}^m A_i X_k A_i - \sum_{i=1}^m B_i Y_k B_i + \sum_{i=1}^m A_i X_k A_i, \\ Y_{k+1} &= Q + \sum_{i=1}^m A_i Y_k A_i - \sum_{i=1}^m B_i X_k B_i + \sum_{i=1}^m A_i Y_k A_i, \\ Z_{k+1} &= Q + \sum_{i=1}^m A_i Z_k A_i - \sum_{i=1}^m B_i Y_k B_i + \sum_{i=1}^m A_i Z_k A_i, \\ k &= 0, 1, 2, \dots \end{aligned} \quad (75)$$

converge to $\widehat{X}$, and the error estimation is given by

$$\begin{aligned} &\max \left\{ \|X_{k+1} - \widehat{X}\|_{1,Q}, \|Y_{k+1} - \widehat{X}\|_{1,Q}, \|Z_{k+1} - \widehat{Z}\|_{1,Q} \right\} \\ &\leq \frac{\lambda^k}{(1-k)} \max \left\{ \|X_1 - X_0\|_{1,Q}, \right. \\ &\quad \left. \|Y_1 - Y_0\|_{1,Q}, \|Z_1 - Z_0\|_{1,Q} \right\}, \end{aligned} \quad (76)$$

for all $k = 0, 1, \dots$, where λ is a certain constant in $[0, 1)$.

(iv) For every $U_0, U_1, U_2 \in [0, 2Q]$, every solution $\{U_n\}$ of (72) converges to $\widehat{X}$.

Proof. In order to make the proof easy, we divide it into several steps.

Step 1. We claim that there exists a unique $(X, Y, Z) \in H(N) \times H(N) \times H(N)$ solution to the system

$$\begin{aligned} X &= Q + \sum_{i=1}^m A_i X A_i - \sum_{i=1}^m B_i Y B_i + \sum_{i=1}^m C_i X C_i, \\ Y &= Q + \sum_{i=1}^m A_i Y A_i - \sum_{i=1}^m B_i X B_i + \sum_{i=1}^m C_i Y C_i, \\ Z &= Q + \sum_{i=1}^m A_i Z A_i - \sum_{i=1}^m B_i Y B_i + \sum_{i=1}^m C_i Z C_i. \end{aligned} \quad (77)$$

Consider the mapping $F : H(N) \times H(N) \times H(N) \rightarrow H(N)$ defined by

$$F(X, Y, Z) = Q + \sum_{i=1}^m A_i X A_i - \sum_{i=1}^m B_i Y B_i + \sum_{i=1}^m C_i Z C_i, \quad (78)$$

for all $X, Y, Z \in H(N)$. It is clear that F is a mapping having the mixed monotone property with respect to the partial order $\leq$. Let $X, Y, Z, U, V, W \in H(N)$ such that $X \geq U, Y \leq V$, and $Z \geq W$. Using Lemma 13, we have

$$\begin{aligned} &\|F(X, Y, Z) - F(U, V, W)\|_{1,Q} \\ &= \text{tr} \left(Q^{1/2} (F(X, Y, Z) - F(U, V, W)) Q^{1/2} \right) \\ &= \text{tr} \left(Q^{1/2} \left(\sum_{i=1}^m A_i X A_i - \sum_{i=1}^m B_i Y B_i + \sum_{i=1}^m C_i Z C_i \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^m A_i U A_i + \sum_{i=1}^m B_i V B_i \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m C_i W C_i \right) Q^{1/2} \right) \\ &= \sum_{i=1}^m \text{tr} \left(Q^{1/2} (A_i (X - U) A_i + B_i (V - Y) B_i \right. \\ &\quad \left. + C_i (Z - W) C_i) Q^{1/2} \right) \\ &= \sum_{i=1}^m \text{tr} \left(Q^{1/2} (A_i (X - U) A_i) Q^{1/2} \right) \\ &\quad + \text{tr} \left(Q^{1/2} (B_i (V - Y) B_i) Q^{1/2} \right) \\ &\quad + \text{tr} \left(Q^{1/2} (C_i (Z - W) C_i) Q^{1/2} \right) \\ &= \sum_{i=1}^m \text{tr} (A_i Q A_i (X - U)) + \text{tr} (B_i Q B_i (V - Y)) \\ &\quad + \text{tr} (C_i Q C_i (Z - W)) \\ &= \sum_{i=1}^m \text{tr} (A_i Q A_i Q^{-1/2} Q^{1/2} (X - U) Q^{1/2} Q^{-1/2}) \\ &\quad + \text{tr} (B_i Q B_i Q^{-1/2} Q^{1/2} (V - Y) Q^{1/2} Q^{-1/2}) \\ &\quad + \text{tr} (C_i Q C_i Q^{-1/2} Q^{1/2} (Z - W) Q^{1/2} Q^{-1/2}) \\ &= \sum_{i=1}^m \text{tr} (Q^{-1/2} A_i Q A_i Q^{-1/2} Q^{1/2} (X - U) Q^{1/2}) \\ &\quad + \text{tr} (Q^{-1/2} B_i Q B_i Q^{-1/2} Q^{1/2} (V - Y) Q^{1/2}) \\ &\quad + \text{tr} (Q^{-1/2} C_i Q C_i Q^{-1/2} Q^{1/2} (Z - W) Q^{1/2}) \end{aligned}$$

$$\begin{aligned}
&= \text{tr} \left(\sum_{i=1}^m Q^{-1/2} A_i Q A_i Q^{-1/2} Q^{1/2} (X - U) Q^{1/2} \right) \\
&\quad + \text{tr} \left(\sum_{i=1}^m Q^{-1/2} B_i Q B_i Q^{-1/2} Q^{1/2} (V - Y) Q^{1/2} \right) \\
&\quad + \text{tr} \left(\sum_{i=1}^m Q^{-1/2} C_i Q C_i Q^{-1/2} Q^{1/2} (Z - W) Q^{1/2} \right) \\
&\leq \left\| \sum_{i=1}^m Q^{-1/2} A_i Q A_i Q^{-1/2} \right\| \|X - U\|_{1,Q} \\
&\quad + \left\| \sum_{i=1}^m Q^{-1/2} B_i Q B_i Q^{-1/2} \right\| \|V - Y\|_{1,Q} \\
&\quad + \left\| \sum_{i=1}^m Q^{-1/2} C_i Q C_i Q^{-1/2} \right\| \|Z - W\|_{1,Q} \\
&\leq \delta \max \{ \|X - U\|_{1,Q}, \|V - Y\|_{1,Q}, \|Z - W\|_{1,Q} \}, \tag{79}
\end{aligned}$$

where

$$\delta = \max \left\{ \left\| \sum_{i=1}^m Q^{-1/2} A_i Q A_i Q^{-1/2} \right\|, \left\| \sum_{i=1}^m Q^{-1/2} B_i Q B_i Q^{-1/2} \right\|, \left\| \sum_{i=1}^m Q^{-1/2} C_i Q C_i Q^{-1/2} \right\| \right\}. \tag{80}$$

From (73) and Lemma 14, we have $\delta < 1$. Thus, the contractive condition of Theorem 5 is satisfied for all $X, Y, Z, U, V, W \in H(N)$ with $X \geq U$, $Y \leq V$, and $Z \geq W$. Moreover, from (73), we have $X_0 = Z_0 = 0 < F(0, 2Q, 0)$ and $F(2Q, 0, 2Q) < 2Q = Y_0$.

Now, all the hypotheses of Theorem 5 are satisfied. Consequently, there exists $(X, Y, Z) \in H(N) \times H(N) \times H(N)$ solution to (77). Since for every $X, Y, Z \in H(N)$ there is a greatest lower bound and a least upper bound with respect to the partial order $\leq$, we deduce from Theorem 7 the uniqueness of the solution to (77). Then, our claim holds.

Step 2. We claim that $X = Y = Z = \widehat{X}$.

Since $X_0 = Z_0 = 0 < Y_0 = 2Q$ (note that Q is a positive definite matrix), applying Theorem 5, we obtain the equality $X = Y = Z$. This proves our claim.

From Steps 1 and 2, we know that (72) has a unique equilibrium point $\widehat{X} \in H(N)$. Now, we need to prove that $\widehat{X} \in P(N)$. This is the goal of the next step.

Step 3 (proof of (i)–(iii)). Define the mapping $G : [F(0, 2Q, 0), F(2Q, 0, 2Q)] \rightarrow H(N)$ by

$$GX = Q + \sum_{i=1}^m A_i X A_i - \sum_{i=1}^m B_i X B_i + \sum_{i=1}^m C_i X C_i, \tag{81}$$

for all $X \in [F(0, 2Q, 0), F(2Q, 0, 2Q)]$.

From (73), one can show that $G([F(0, 2Q, 0), F(2Q, 0, 2Q)]) \subseteq [F(0, 2Q, 0), F(2Q, 0, 2Q)]$. Now, applying the Schauder fixed-point theorem (see Theorem 15), we deduce the existence of a fixed point of G in $[F(0, 2Q, 0), F(2Q, 0, 2Q)]$. But a fixed point of G is an equilibrium point of (72), and from Step 2, we know that (72) has a unique equilibrium point in $H(N)$. Consequently, $0 < F(0, 2Q, 0) \leq \widehat{X} \leq F(2Q, 0, 2Q)$. This proves (i) and (ii). For the proof of (iii), we have only to apply the inequalities (19).

Step 4. Let $U_0 = 0$, $U_1 = Q$, and $U_2 = 2Q$. Then, we have $U_0, U_1, U_2 \in [0, 2Q]$, thus by (b) Theorem 5, every solution $\{U_n\}$ to (72) converges to $\widehat{X}$. $\square$

3.2. Numerical Example. Consider a third-order difference matrix equation

$$U_{n+1} = Q + AU_nA - BU_{n-1}B + CU_{n-2}C, \tag{82}$$

where Q, A, B, C, U_0, U_1 , and U_2 are given by

$$\begin{aligned}
Q &= \begin{pmatrix} 1.5 & 0.3 & 0.3 \\ 0.3 & 1.5 & 0.3 \\ 0.3 & 0.3 & 1.5 \end{pmatrix}, & A &= \begin{pmatrix} 0.3 & 0.05 & 0.05 \\ 0.05 & 0.3 & 0.05 \\ 0.05 & 0.05 & 0.3 \end{pmatrix}, \\
B &= \begin{pmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.1 \end{pmatrix}, & C &= \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}, \tag{83}
\end{aligned}$$

$$U_0 = 0, \quad U_1 = Q, \quad U_2 = 2Q,$$

We are interested to approximate the unique positive equilibrium to (82).

We use Algorithm (iii) of Theorem 16 with $X_0 = 0 = Z_0$ and $Y_0 = 2Q$.

After 35 iterations, we get the unique solution $\widehat{X}$ given by

$$\begin{aligned}
\widehat{X} = X_{35} = Y_{35} = Z_{35} &= \begin{pmatrix} 1.615643487571 & 0.31874620339355 & 0.33564348757095 \\ 0.31874620339355 & 1.5374221323462 & 0.31874620339355 \\ 0.33564348757095 & 0.31874620339355 & 1.615643487571 \end{pmatrix}, \\
U_{35} &= \begin{pmatrix} 1.6156434875817 & 0.31874620340583 & 0.33564348758171 \\ 0.31874620340583 & 1.5374221323604 & 0.31874620340583 \\ 0.33564348758171 & 0.31874620340583 & 1.6156434875817 \end{pmatrix}. \tag{84}
\end{aligned}$$

FIGURE 1: Convergence history for (82).

The residual errors are

$$\begin{aligned}
 R(U)_{35} &= \|U_{35} - (Q + A_1 U_{35} A_1 + A_2 U_{35} A_2 \\
 &\quad - B_1 U_{35} B_1 - B_2 U_{35} B_2)\| \\
 &= 3.352 \times 10^{-11}. \\
 R(X)_{35} &= \|X_{35} - (Q + A_1 X_{35} A_1 + A_2 X_{35} A_2 \\
 &\quad - B_1 X_{35} B_1 - B_2 X_{35} B_2)\| \\
 &= 3.792 \times 10^{-16}, \\
 R(Y)_{35} &= \|Y_{35} - (Q + A_1 Y_{35} A_1 + A_2 Y_{35} A_2 \\
 &\quad - B_1 Y_{35} B_1 - B_2 Y_{35} B_2)\| \\
 &= 1.796 \times 10^{-16}, \\
 R(Z)_{35} &= \|Z_{35} - (Q + A_1 Z_{35} A_1 + A_2 Z_{35} A_2 \\
 &\quad - B_1 Z_{35} B_1 - B_2 Z_{35} B_2)\| \\
 &= 2.351 \times 10^{-16},
 \end{aligned} \tag{85}$$

The convergence history is given by Figure 1, where (a) corresponds to $\|X_k - F(X_k, X_k, X_k)\|$, (b) corresponds to $\|Y_k - F(Y_k, Y_k, Y_k)\|$, (c) corresponds to $\|Z_k - F(Z_k, Z_k, Z_k)\|$, and (d) corresponds to $\|U_k - F(U_k, U_k, U_k)\|$.

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