Research Article

Fractal Spherical Harmonics

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This paper tackles the construction of fractal maps on the unit sphere. The functions defined are a generalization of the classical spherical harmonics. The methodology used involves an iterated function system and a linear and bounded operator of functions on the sphere. For a suitable choice of the coefficients of the system, one obtains classical maps on the sphere. The different values of the system parameters provide Bessel sequences, frames, and Riesz fractal bases for the Lebesgue space of the square integrable functions on the sphere. The Laplace series expansion is generalized to a sum in terms of the new fractal mappings.

1. Introduction

The spherical patterns have an increasing importance in many recent scientific and technical fields like brain mapping, meteorology, oceanography, environment, etc. The problem of interpolation and approximation on the sphere tackles the reconstruction of an unknown function from a finite set of data. The targeted variable may be an electric brain potential, temperature, pressure, etc. If the accessible information is localized in a small zone of the surface, one can use two-dimensional Cartesian methods. However, if the data are disseminated all over the sphere, the global methods are essential.

The main objective of the present paper is the definition of new global functions on the sphere. The mappings are constructed here by means of a fractal methodology (an iterated function system and a linear and bounded operator), and they provide a perturbation of the spherical harmonics. For a suitable choice of the coefficients of the system, one obtains the classical functions. Different values of the parameters provide Bessel sequences, frames, and Riesz fractal bases for the Lebesgue space of square integrable functions on the sphere. The Laplace series expansion is generalized to a sum in terms of the new fractal maps.

A very different approach for the problem of approximation on the sphere is used in [1].

2. A Special Type of Fractal Functions

Let $t_0 < t_1 < \dots < t_N$ be real numbers and $I = [t_0, t_N] = [a, b]$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times \mathbb{R} : n = 0, 1, 2, \dots, N\}$ be given. Set $I_n = [t_{n-1}, t_n]$, and let $L_n: I \rightarrow I_n$, $n \in \{1, 2, \dots, N\}$ be contractive homeomorphisms such that

$$L_n(t_0) = t_{n-1}, \quad L_n(t_N) = t_n, \quad (1)$$

$$|L_n(c_1) - L_n(c_2)| \leq l |c_1 - c_2| \quad \forall c_1, c_2 \in I, \quad (2)$$

for some $0 \leq l < 1$.

Let $F = I \times \mathbb{R}$ and N continuous mappings, $F_n: F \rightarrow \mathbb{R}$, be given satisfying

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n = 1, 2, \dots, N \quad (3)$$

$$|F_n(t, x) - F_n(t, y)| \leq r |x - y|, \quad t \in I, \quad x, y \in \mathbb{R}, \quad 0 \leq r < 1. \quad (4)$$

Now, define functions $w_n(t, x) = (L_n(t), F_n(t, x))$, for all $n = 1, 2, \dots, N$. For any choices of $L_n(t)$ and $F_n(t, x)$ satisfying the conditions described and in particular (1), (2), (3), and (4), the next theorem holds.

Theorem 1 (Barnsley [2]). *The iterated function system (IFS) $\{F, w_n : n = 1, 2, \dots, N\}$ defined above admits a unique attractor G . G is the graph of a continuous function $g : I \rightarrow R$ which obeys $g(t_n) = x_n$ for $n = 0, 1, 2, \dots, N$.*

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^N$. The map g is unique satisfying the functional equation [2]:

$$\begin{aligned} g(t) &= F_n(L_n^{-1}(t), g \circ L_n^{-1}(t)), \\ n &= 1, 2, \dots, N, \quad t \in I_n = [t_{n-1}, t_n]. \end{aligned} \quad (5)$$

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{aligned} L_n(t) &= a_n t + b_n, \\ F_n(t, x) &= \alpha_n x + q_n(t), \end{aligned} \quad (6)$$

where $-1 < \alpha_n < 1$, for all $n = 1, 2, \dots, N$. α_n is called a vertical scaling factor of the transformation w_n , and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is the scale vector of the IFS. Following the equalities (1),

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0}, \quad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0}. \quad (7)$$

Let $f \in \mathcal{C}(I)$ be a continuous function. We consider the particular case

$$q_n(t) = f \circ L_n(t) - \alpha_n b(t), \quad (8)$$

where b is continuous and such that $b(t_0) = x_0$, and $b(t_N) = x_N$.

It is easy to check that condition (3) is fulfilled. The set of data points is here $\{(t_n, x_n = f(t_n)) \in I \times R : n = 0, 1, 2, \dots, N\}$. According to Theorem 1, the IFS provides in this case as an attractor the graph of a continuous function denoted here as $f^\alpha : I \rightarrow R$ such that

$$f^\alpha(t_n) = f(t_n), \quad (9)$$

for $n = 0, 1, \dots, N$.

The choice of IFS provides fractal analogues of any continuous function (see, e.g., [3–6]).

In particular, we consider in this paper the case

$$b = L(f), \quad (10)$$

where $L : \mathcal{C}[a, b] \rightarrow \mathcal{C}[a, b]$ is an operator linear, bounded with respect to the least square norm

$$\|f\|_2 = \left(\int_a^b |f|^2 dt \right)^{1/2} \quad (11)$$

and such that $Lf(t_0) = f(t_0)$, $Lf(t_N) = f(t_N)$, and $L \neq \text{Identity}$.

An example of operator L is defined by $Lf(t) = v(t)f(t)$, where $v(t)$ is continuous on I and $v(t_0) = v(t_N) = 1$ (see page 958 in [3]).

Definition 2. Let $\Delta : a = t_0 < t_1 < \dots < t_N = b$, where $N > 1$, be a partition of the interval $I = [a, b]$. A scale vector associated with Δ is a vector $\alpha \in (-1, 1)^N$.

Definition 3. Let f^α be the continuous function defined by the IFS (6), (7), (8), and (10). f^α is called α -fractal function associated with f with respect to L and the partition Δ .

Figure 1 represents the graph of the α -fractal function $(P_5^2)^\alpha(t)$ associated with the Ferrers polynomial (23) $P_5^2(t)$, with respect to the elements described in the legend. The oscillations of the graph display the self-similarity of the mapping on every subinterval $[t_{n-1}, t_n]$. Their large amplitudes are due to the greatest scale factors ($\alpha_3, \alpha_4, \alpha_6$, and α_{10} .)

According to (5), f^α satisfies the fixed point equation:

$$f^\alpha(t) = f(t) + \alpha_n (f^\alpha - Lf) \circ L_n^{-1}(t) \quad \forall t \in I_n. \quad (12)$$

f^α interpolates to f at t_n because of using (1), (12), and Barnsley's theorem:

$$\begin{aligned} f^\alpha(t_n) &= f(t_n) + \alpha_n (f^\alpha - Lf)(t_n) = f(t_n) \\ \forall n &= 0, 1, \dots, N. \end{aligned} \quad (13)$$

Let us call α -fractal operator $\mathcal{F}_L^\alpha = \mathcal{F}_{\Delta, L}^\alpha$ with respect to Δ and L the transformation which assigns f^α to the function f :

$$\mathcal{F}_L^\alpha(f) = f^\alpha. \quad (14)$$

The next results can be read in [4].

Theorem 4. (a) $\mathcal{F}_L^\alpha : \mathcal{C}[a, b] \rightarrow \mathcal{C}[a, b]$ is linear and bounded with respect to the least square norm.

(b) If $\alpha = 0$, $\mathcal{F}_L^\alpha = \text{Identity}$.

(c) The following inequalities hold:

$$\|\mathcal{F}_L^\alpha\| \leq 1 + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|, \quad (15)$$

$$\|I - \mathcal{F}_L^\alpha\| \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|,$$

where $|\alpha|_\infty = \max\{|\alpha_n| : n = 1, 2, \dots, N\}$ and $\|T\|$ is the operator norm defined as

$$\|T\| = \sup \{\|T(f)\|_2 : \|f\|_2 = 1, f \in \mathcal{C}[a, b]\}. \quad (16)$$

Consequence. According to Theorem 4(b), the collection of maps f^α constitutes a family of continuous functions containing f as a particular case (for $\alpha = 0$).

Theorem 5. If $|\alpha|_\infty < 1/(1 + \|I - L\|)$, $\mathcal{F}_L^\alpha$ has a bounded inverse and

$$\|(\mathcal{F}_L^\alpha)^{-1}\| \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty \|L\|}. \quad (17)$$

FIGURE 1: Graph of the α -fractal function $(P_5^2)^\alpha(t)$ of the Ferrers polynomial $P_5^2(t)$ in the interval $I = [-1, 1]$, step $h = 0.1$, operator $Lf(t) = v(t)f(t)$, where $v(t) = \cos(2\pi t)$ and scale vector $\alpha = \{-0.3, 0.4, -0.6, 0.8, 0.2, -0.7, 0.4, 0.5, 0.2, -0.75\}$

3. Fractal Spherical Harmonics

A homogeneous polynomial V of degree n in the variables x , y , and z satisfying the Laplace equation $\Delta V = 0$ is called a Laplace or harmonic polynomial of degree n . If we consider spherical coordinates (ρ, θ, φ) for $P = (x, y, z) = \rho(\xi, \eta, \zeta) \in \mathbb{R}^3$, where

$$\xi = \sin(\varphi) \cos(\theta), \quad \eta = \sin(\varphi) \sin(\theta), \quad \zeta = \cos(\varphi) \quad (18)$$

(θ is the longitude and φ is the colatitude), then

$$V(x, y, z) = \rho^n V(\xi, \eta, \zeta). \quad (19)$$

In this case, the function

$$\begin{aligned} Y(\theta, \varphi) &= Y_n(\theta, \varphi) \\ &= V(\sin(\varphi) \cos(\theta), \sin(\varphi) \sin(\theta), \cos(\varphi)) \end{aligned} \quad (20)$$

is called the Laplace function or spherical harmonic of order n on the unit sphere.

Two spherical harmonics of different degree (or order) are orthogonal over the sphere:

$$\int_S Y_n(P) Y_m(P) dS = 0; \quad n \neq m, \quad (21)$$

where dS is the element of area of the sphere S . It is well known that the set of spherical harmonics of order n , $\mathcal{H}_n$, is a linear subspace of functions on the sphere with dimension $2n + 1$, and one of its orthogonal bases is

$$\begin{aligned} U_n^0(Q) &= P_n(\cos(\varphi)), \\ U_n^m(Q) &= P_n^m(\cos(\varphi)) \cos(m\theta), \\ V_n^m(Q) &= P_n^m(\cos(\varphi)) \sin(m\theta), \end{aligned} \quad (22)$$

if $Q = (\theta, \varphi)$, $m = 1, 2, \dots, n$. P_n is the n th Legendre polynomial, and P_n^m is the Ferrers or associated Legendre polynomial of degree n and order m defined as [7]

$$P_n^m(t) = (1 - t^2)^{m/2} P_n^{(m)}(t), \quad (23)$$

for $m = 1, 2, \dots, n$. The superindex m is only a notation on the left-hand side of (23). On the right, (m) represents the derivative of m th order.

A historical survey of this kind of functions can be found in [8].

The next equalities are satisfied by the basic spherical harmonics [7]:

$$\begin{aligned} \int_S (U_n^0(Q))^2 dS &= \frac{4\pi}{n+1}, \\ \int_S (U_n^m(Q))^2 dS &= \int_S (V_n^m(Q))^2 dS = \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!}. \end{aligned} \quad (24)$$

This basis is then unbounded. The family

$$\{U_n^0, U_n^m, V_n^m; \quad n = 0, 1, 2, \dots, \quad m = 1, 2, \dots, n\} \quad (25)$$

is an orthogonal and complete system of $\mathcal{L}^2(S)$. After normalization, one obtains an orthonormal basis (X_{nj}) of spherical harmonics, and every $f \in \mathcal{L}^2(S)$ can be expressed as

$$f = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj}, \quad (26)$$

in $\mathcal{L}^2$ -sense. The expansion of a function $f \in \mathcal{L}^2(S)$ in terms of this system is called Laplace series of f .

In the following, we extend the operator $\mathcal{F}_L^\alpha$ to the functions on the sphere $S \subset \mathbb{R}^3$.

For $f \in \mathcal{L}^2(S)$, let us define the least square norm on the sphere:

$$\|f\|_2 = \left(\int_S f^2(P) dS \right)^{1/2}. \quad (27)$$

Remark 6. The notation $\|\cdot\|_2$ represents the least square norm of a square integrable function defined either on an interval I or on the sphere S .

In [4], we defined a family of fractal functions close to the classical spherical harmonics and a finite sum corresponding to the new mappings. The fractal spherical harmonics are defined as in (22), substituting the functions P_n and P_n^m by their fractal analogues P_n^α and $(P_n^m)^\alpha$. We look for an expansion of any continuous function on the sphere in terms of these new maps.

The next result is proved in [4, Proposition 3.2].

Proposition 7. *There exists an operator $\mathcal{S}_n^\alpha : \mathcal{H}_n \rightarrow \mathcal{L}^2(S)$, where $\mathcal{H}_n$ is the space of spherical harmonics of order n , linear, and injective and such that*

$$\|\mathcal{S}_n^\alpha\| \leq \|\mathcal{F}_L^\alpha\|, \quad (28)$$

where $\|\cdot\|$ is the operator norm with respect to the least square norm and $\mathcal{F}_L^\alpha$ is the operator of $\mathcal{C}[-1, 1]$ studied in the previous section.

The image of the elements of the basis is defined as

$$\begin{aligned}\mathcal{S}_n^\alpha(U_n^0)(\theta, \varphi) &= (U_n^0)^\alpha(\theta, \varphi) = P_n^\alpha(\cos(\varphi)), \\ \mathcal{S}_n^\alpha(U_n^m)(\theta, \varphi) &= (U_n^m)^\alpha(\theta, \varphi) = (P_n^m)^\alpha(\cos(\varphi)) \cos(m\theta), \\ \mathcal{S}_n^\alpha(V_n^m)(\theta, \varphi) &= (V_n^m)^\alpha(\theta, \varphi) = (P_n^m)^\alpha(\cos(\varphi)) \sin(m\theta),\end{aligned}\quad (29)$$

where $(P_n^m)^\alpha(\cos(\varphi)) = \mathcal{F}_L^\alpha(P_n^m)(\cos(\varphi))$ and $\mathcal{F}_L^\alpha$ is the operator defined in Section 2.

We consider in this case the interval $I = [-1, 1]$ and a uniform partition Δ of I in order to define the fractal analogues. By linearity, $\mathcal{S}_n^\alpha$ is extended to the rest of $\mathcal{H}_n$. Let us denote

$$\mathcal{H}_n^\alpha = \mathcal{S}_n^\alpha(\mathcal{H}_n). \quad (30)$$

$\mathcal{H}_n^\alpha$ is spanned by $\{(U_n^0)^\alpha, (U_n^m)^\alpha, (V_n^m)^\alpha; m = 1, 2, \dots, n\}$. These fractal elements are mutually orthogonal as well. For instance,

$$\begin{aligned}((U_n^m)^\alpha, (U_n^j)^\alpha)_2 &= \int_0^{2\pi} \int_0^\pi (P_n^m)^\alpha(\cos(\varphi)) (P_n^j)^\alpha \\ &\quad \times (\cos(\varphi)) \cos(m\theta) \cos(j\theta) \sin \varphi \, d\varphi \, d\theta \\ &= 0\end{aligned}\quad (31)$$

if $m \neq j$, due to the orthogonality of $\cos(m\theta)$, $\cos(j\theta)$.

Proposition 8. For all $n = 0, 1, \dots$ and for all $m = 1, 2, \dots, n$,

$$\begin{aligned}\|(U_n^0)^\alpha\|_2 &= \sqrt{2\pi} \|P_n^\alpha\|_2 \leq \sqrt{2\pi} \|\mathcal{F}_L^\alpha\| \|P_n\|_2, \\ \|(U_n^m)^\alpha\|_2 &= \|(V_n^m)^\alpha\|_2 = \sqrt{\pi} \|(P_n^m)^\alpha\|_2 \\ &\leq \sqrt{\pi} \|\mathcal{F}_L^\alpha\| \|P_n^m\|_2.\end{aligned}\quad (32)$$

Proof. For instance,

$$\begin{aligned}\|(U_n^m)^\alpha\|_2^2 &= \int_0^{2\pi} \int_0^\pi |(P_n^m)^\alpha(\cos(\varphi))|^2 \cos^2(m\theta) \sin(\varphi) \, d\varphi \, d\theta.\end{aligned}\quad (33)$$

With the change of variable $t = \cos(\varphi)$ and the integration on the longitude θ , the following inequality holds:

$$\begin{aligned}\|(U_n^m)^\alpha\|_2^2 &= \pi \int_{-1}^1 |(P_n^m)^\alpha(t)|^2 dt = \pi \|(P_n^m)^\alpha\|_2^2 \\ &\leq \pi \|\mathcal{F}_L^\alpha\|^2 \|P_n^m\|_2^2,\end{aligned}\quad (34)$$

and the result is obtained. $\square$

Let us denote by (X_{nj}) the orthonormal basis of spherical harmonics and by (X_{nj}^α) the transformed elements by $\mathcal{S}_n^\alpha$. That is to say:

$$\mathcal{S}_n^\alpha(X_{nj}) = X_{nj}^\alpha. \quad (35)$$

Theorem 9 (linear and bounded operator Theorem [9]). *If an operator $S : X \rightarrow Y$ is linear and bounded, Y is Banach, and X is dense in X' , then S can be extended to X' preserving the norm of S .*

Theorem 10. For any $f \in \mathcal{L}^2(S)$, let us consider the Laplace series (in terms of spherical harmonics)

$$f = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj}. \quad (36)$$

The operator $\kappa^\alpha : \mathcal{L}^2(S) \rightarrow \mathcal{L}^2(S)$ such that

$$\kappa^\alpha(f) = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj}^\alpha, \quad (37)$$

where $X_{nj}^\alpha = \mathcal{S}_n^\alpha(X_{nj})$ is well defined, linear, and continuous.

Proof. Let us consider the linear mapping $\mathcal{G}^\alpha : \text{span}(X_{mj}) \subset \mathcal{L}^2(S) \rightarrow \mathcal{L}^2(S)$, defined by

$$\mathcal{G}^\alpha(f) = \mathcal{G}^\alpha\left(\sum_{m=m_0}^M \sum_{j=j_0}^q c_{mj} X_{mj}\right) = \sum_{m=m_0}^M \sum_{j=j_0}^q c_{mj} X_{mj}^\alpha. \quad (38)$$

According to this definition, the operator $\mathcal{G}^\alpha$ is linear.

If one considers that for any $f \in \text{span}(X_{mj})$, $f \in \mathcal{H}_n$ for n sufficiently large, then

$$\mathcal{G}^\alpha(f) = \mathcal{S}_n^\alpha(f), \quad (39)$$

and thus (28)

$$\|\mathcal{G}^\alpha(f)\|_2 = \|\mathcal{S}_n^\alpha(f)\|_2 \leq \|\mathcal{S}_n^\alpha\| \|f\|_2 \leq \|\mathcal{F}_L^\alpha\| \|f\|_2. \quad (40)$$

Consequently,

$$\|\mathcal{G}^\alpha\| \leq \|\mathcal{F}_L^\alpha\|, \quad (41)$$

and $\mathcal{G}^\alpha$ is bounded and linear. Since

$$\overline{\text{span}}(X_{nj}) = \mathcal{L}^2(S), \quad (42)$$

then by the bounded and linear operator theorem (Theorem 9), $\mathcal{G}^\alpha$ can be extended to $\mathcal{L}^2(S)$ preserving the norm. Let us denote this extension as

$$\kappa^\alpha : \mathcal{L}^2(S) \longrightarrow \mathcal{L}^2(S). \quad (43)$$

The linearity and continuity of κ^α imply that if

$$f = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj}, \quad (44)$$

then

$$\kappa^\alpha(f) = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj}^\alpha. \quad (45)$$

Moreover,

$$\|\kappa^\alpha\| = \|\mathcal{G}^\alpha\| \leq \|\mathcal{F}_L^\alpha\|. \quad (46)$$

□

4. Fractal Spanning Systems of $\mathcal{L}^2(S)$

Let us consider now the process of expanding the operator $L : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$, linear and bounded with respect to the least square norm, quoted in Section 2, to the space $\mathcal{L}^2(S)$. The extension is similar to the method employed for $\mathcal{F}_L^\alpha$. Let L_H be defined as

$$L_H(U_n^m)(\theta, \varphi) = L(P_n^m)(\cos(\varphi)) \cos(m\theta), \quad (47)$$

$$L_H(V_n^m)(\theta, \varphi) = L(P_n^m)(\cos(\varphi)) \sin(m\theta)$$

and extended by linearity to the space of spherical harmonics of order n :

$$L_H : \mathcal{H}_n \longrightarrow \mathcal{L}^2(S), \quad (48)$$

such that

$$\|L_H\| \leq \|L\|. \quad (49)$$

The operator L_H admits an extension to (see Theorem 10)

$$L_S : \mathcal{L}^2(S) \longrightarrow \mathcal{L}^2(S), \quad (50)$$

such that

$$\|L_S\| = \|L_H\| \leq \|L\|. \quad (51)$$

The proof of the next result can be read in [4] (expression (17) of Theorem 2.4).

Lemma 11. For any $f \in \mathcal{C}(I)$,

$$\|f^\alpha - f\|_2 \leq |\alpha|_\infty \|f^\alpha - L(f)\|_2. \quad (52)$$

Lemma 12. Let κ^α be the operator defined in Theorem 10, then

$$\begin{aligned} \|\kappa^\alpha(U_n^m) - U_n^m\|_2 &\leq |\alpha|_\infty \|\kappa^\alpha(U_n^m) - L_S(U_n^m)\|_2, \\ \|\kappa^\alpha(V_n^m) - V_n^m\|_2 &\leq |\alpha|_\infty \|\kappa^\alpha(V_n^m) - L_S(V_n^m)\|_2. \end{aligned} \quad (53)$$

Proof. κ^α is the extension of $\mathcal{S}_n^\alpha$. According to its definition,

$$\begin{aligned} &\|\kappa^\alpha(U_n^m) - U_n^m\|_2^2 \\ &= \int_0^{2\pi} \int_0^\pi |(P_n^m)^\alpha(\cos(\varphi)) - P_n^m(\cos(\varphi))|^2 \\ &\quad \times \cos^2(m\theta) \sin(\varphi) d\varphi d\theta \\ &= \pi \int_{-1}^1 |(P_n^m)^\alpha(t) - P_n^m(t)|^2 dt \\ &= \pi \|(P_n^m)^\alpha - P_n^m\|_2^2 \\ &\leq \pi |\alpha|_\infty^2 \|(P_n^m)^\alpha - L P_n^m\|_2^2. \end{aligned} \quad (54)$$

The last inequality agrees with (52) for $f = P_n^m$. Furthermore,

$$\begin{aligned} &\|\kappa^\alpha(U_n^m) - L_S(U_n^m)\|_2^2 \\ &= \int_0^{2\pi} \int_0^\pi |((P_n^m)^\alpha - L(P_n^m))(\cos(\varphi))|^2 \\ &\quad \times \cos^2(m\theta) \sin(\varphi) d\varphi d\theta \\ &= \pi \int_{-1}^1 |(P_n^m)^\alpha(t) - L P_n^m(t)|^2 dt \\ &= \pi \|(P_n^m)^\alpha - L P_n^m\|_2^2. \end{aligned} \quad (55)$$

As a consequence,

$$\|\kappa^\alpha(U_n^m) - U_n^m\|_2 \leq |\alpha|_\infty \|\kappa^\alpha(U_n^m) - L_S(U_n^m)\|_2. \quad (56)$$

□

Proposition 13. For any $f \in \mathcal{L}^2(S)$,

$$\|\kappa^\alpha(f) - f\|_2 \leq |\alpha|_\infty \|\kappa^\alpha(f) - L_S f\|_2, \quad (57)$$

$$\|\kappa^\alpha(f) - f\|_2 \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L_S\| \|f\|_2. \quad (58)$$

Proof. Using arguments of density [3, Lemma 4.7], the inequalities of Lemma 12 can be extended to any $f \in \mathcal{L}^2(S)$:

$$\|\kappa^\alpha(f) - f\|_2 \leq |\alpha|_\infty \|\kappa^\alpha(f) - L_S f\|_2. \quad (59)$$

Consequently,

$$\|\kappa^\alpha(f) - f\|_2 \leq |\alpha|_\infty (\|\kappa^\alpha(f) - f\|_2 + \|f - L_S f\|_2), \quad (60)$$

and thus

$$\|\kappa^\alpha(f) - f\|_2 \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L_S\| \|f\|_2. \quad (61)$$

□

Proposition 14. If $|\alpha|_\infty < \|L\|^{-1}$, κ^α is injective and its range is closed.

Proof. If $\kappa^\alpha(f) = 0$, according to (57),

$$\|f\|_2 \leq |\alpha|_\infty \|L\| \|f\|_2. \quad (62)$$

Since $|\alpha|_\infty \|L\| < 1$, one obtains that $f = 0$. The inequality (57)

$$\|\kappa^\alpha(f) - f\|_2 \leq |\alpha|_\infty \|\kappa^\alpha(f) - L_S f\|_2, \quad (63)$$

implies that

$$\begin{aligned} \|f\|_2 - \|\kappa^\alpha(f)\|_2 &\leq |\alpha|_\infty (\|\kappa^\alpha(f)\|_2 + \|L\| \|f\|_2), \\ \|f\|_2 &\leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty \|L\|} \|\kappa^\alpha(f)\|_2. \end{aligned} \quad (64)$$

By linearity,

$$\|f_n - f_m\|_2 \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty \|L\|} \|\kappa^\alpha(f_n) - \kappa^\alpha(f_m)\|_2. \quad (65)$$

The rest of the proof is similar to Theorem 3.5 of [3]. □

Definition 15. A sequence $(x_m) \subseteq H$, where H is a Hilbert space, is a Bessel sequence if there exists a constant $B > 0$ such that for all $x \in H$

$$\sum_{m=0}^{\infty} |\langle x, x_m \rangle|^2 \leq B \|x\|^2. \quad (66)$$

Let us consider the basis of spherical harmonics (X_{nj}) and its image by κ^α , (X_{nj}^α) .

Proposition 16. For any scale vector such that $|\alpha|_\infty < 1$, (X_{nj}^α) is a Bessel sequence.

Proof. For any $f \in \mathcal{L}^2(S)$, the definition of adjoint operator $(\kappa^\alpha)^*$ of κ^α implies that

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |\langle f, X_{nj}^\alpha \rangle|^2 \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |\langle f, \kappa^\alpha(X_{nj}) \rangle|^2 \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |\langle (\kappa^\alpha)^*(f), X_{nj} \rangle|^2 = \|(\kappa^\alpha)^*(f)\|_2^2 \\ &\leq \|(\kappa^\alpha)^*\|^2 \|f\|_2^2 = B \|f\|_2^2, \end{aligned} \quad (67)$$

where $B = \|\kappa^\alpha\|^2$. The penultimate equality is due to Parseval's formula applied to the basis (X_{nj}) and the element $(\kappa^\alpha)^*(f)$. The norm of the adjoint operator $(\kappa^\alpha)^*$ agrees with the norm of κ^α (see, e.g., Lemma 5.5.1 of [9]). $\square$

Definition 17. A sequence $(x_m) \subseteq H$, where H is a Hilbert space, is a frame if there exist constants $A, B > 0$ such that for all $x \in H$

$$A \|x\|^2 \leq \sum_{m=0}^{\infty} |\langle x, x_m \rangle|^2 \leq B \|x\|^2. \quad (68)$$

Definition 18. A sequence $(x_m) \subseteq H$, where H is a Hilbert space, is a frame sequence if it is a frame for its closed span $[x_m] = \overline{\text{span}}(x_m)$.

Proposition 19. If $|\alpha|_\infty < \|L\|^{-1}$, (X_{nj}^α) is a frame sequence.

Proof. The right-hand side of (68) is proved in Proposition 16. For the left one, let us consider that κ^α is injective with closed range (Proposition 14). A closed space of a Hilbert space is Hilbert as well, and $(\kappa^\alpha)^{-1}$ is well defined, linear, and bounded as κ^α is (see, e.g., Theorem 3.5.3 of [10]).

Therefore, $\kappa^\alpha \circ (\kappa^\alpha)^{-1}$ is the identity on the range of κ^α (denoted by $rg(\kappa^\alpha)$).

If $g \in [X_{nj}^\alpha]$, then $g \in rg(\kappa^\alpha)$ since

$$\text{span}(X_{nj}^\alpha) \subseteq rg(\kappa^\alpha), \quad (69)$$

and thus

$$[X_{nj}^\alpha] = \overline{\text{span}}(X_{nj}^\alpha) \subseteq rg(\kappa^\alpha), \quad (70)$$

as $rg(\kappa^\alpha)$ is closed. For any $g \in [X_{nj}^\alpha]$,

$$g = ((\kappa^\alpha)^{-1})^* \circ (\kappa^\alpha)^*(g), \quad (71)$$

and thus

$$\|g\|_2^2 \leq \|(\kappa^\alpha)^{-1}\|^2 \|(\kappa^\alpha)^*(g)\|_2^2, \quad (72)$$

since

$$\|(\kappa^\alpha)^{-1}\| = \|((\kappa^\alpha)^{-1})^*\|. \quad (73)$$

The definition of adjoint operator implies that (see the proof of Proposition 16)

$$\|(\kappa^\alpha)^*(g)\|_2^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |\langle g, X_{nj}^\alpha \rangle|^2. \quad (74)$$

Let us denote that

$$A = \|(\kappa^\alpha)^{-1}\|^{-2}. \quad (75)$$

The inequality (72) implies that,

$$A \|g\|_2^2 \leq \|(\kappa^\alpha)^*(g)\|_2^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |\langle g, X_{nj}^\alpha \rangle|^2, \quad (76)$$

and the proof is complete. $\square$

Definition 20. A sequence $(x_m) \subseteq H$, where H is a Hilbert space, is a Riesz sequence if there exist $k_1, k_2 > 0$ such that for any $(c_m) \in l^2$

$$k_1 \sum_{m=0}^{\infty} |c_m|^2 \leq \left\| \sum_{m=0}^{\infty} c_m x_m \right\|^2 \leq k_2 \sum_{m=0}^{\infty} |c_m|^2. \quad (77)$$

Proposition 21. If $|\alpha|_\infty < \|L\|^{-1}$, (X_{nj}^α) is a Riesz sequence.

Proof. If $(c_{nj}) \in l^2$,

$$f = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj} \in \mathcal{L}^2(S),$$

$$\begin{aligned} \|\kappa^\alpha(f)\|_2^2 &= \left\| \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj}^\alpha \right\|_2^2 \\ &\leq \|\kappa^\alpha\|^2 \|f\|_2^2 = k_2 \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |c_{nj}|^2, \end{aligned} \quad (78)$$

where

$$k_2 = \|\kappa^\alpha\|^2. \quad (79)$$

Let us consider now the left inequality of (77). With the hypothesis on α , κ^α has closed range and is injective. $(\kappa^\alpha)^{-1}$ is

defined on the range $(rg(\kappa^\alpha))$, being continuous as well. Let l_1 be defined by the expression

$$l_1 = \frac{1 - |\alpha|_\infty \|L\|}{1 + |\alpha|_\infty}. \quad (80)$$

Due to Parseval's equality,

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |c_{nj}|^2. \quad (81)$$

Therefore, by (64),

$$l_1^2 \|f\|_2^2 = l_1^2 \sum_{n=0}^{\infty} \sum_{j=0}^{2n} |c_{nj}|^2 \leq \|\kappa^\alpha f\|_2^2 = \left\| \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj}^\alpha \right\|_2^2, \quad (82)$$

and (X_{nj}^α) is a Riesz sequence. $\square$

Definition 22. A sequence $(x_m) \subseteq H$, where H is a Hilbert space, is a Riesz basis if it is equivalent to an orthonormal basis of H . That is to say, there exists a topological isomorphism T on H and an orthonormal basis (e_m) , such that $T(e_m) = (x_m)$.

Lemma 23 (see [9]). *If L is a bounded and linear operator from a Banach space into itself such that $\|I - L\| < 1$, then L^{-1} exists and is bounded.*

Theorem 24. *If $|\alpha|_\infty < (1 + \|I - L\|)^{-1}$, (X_{nj}^α) is a Riesz basis.*

Proof. The extension of the operator $I - L$ to $\mathcal{L}^2(S)$ is $I - L_S$ [3, Lemma 4.5], and thus

$$\|I - L\| = \|I - L_S\|. \quad (83)$$

If $|\alpha|_\infty < (1 + \|I - L_S\|)^{-1}$, then

$$\begin{aligned} \frac{1}{1 - |\alpha|_\infty} &< \frac{1 + \|I - L_S\|}{\|I - L_S\|}, \\ \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L_S\| &< 1, \\ \|I - \kappa^\alpha\| &< 1, \end{aligned} \quad (84)$$

due to (58). The operator κ^α is an isomorphism (Lemma 23), and (X_{nj}^α) is a Riesz basis. $\square$

Using the identity,

$$f = \kappa^\alpha \circ (\kappa^\alpha)^{-1}(f), \quad (85)$$

one obtains the Laplace series of $f \in \mathcal{L}^2(S)$, with respect to the fractal spherical harmonics:

$$\begin{aligned} f &= \kappa^\alpha \left((\kappa^\alpha)^{-1}(f) \right) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} \left((\kappa^\alpha)^{-1}(f) \right) X_{nj}^\alpha. \end{aligned} \quad (86)$$

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