Research Article

Common Fixed Point Theorems Using the E.A. and CLR Properties in 2-Menger Spaces

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First, we prove a common fixed point theorem using weakly compatible maps in 2-Menger space with t -norm of Hadzic type. Second, we prove a common fixed point theorem using the E.A. property along with weakly compatible maps. Further, we obtained a common fixed point theorem using the CLR property along with weakly compatible maps. At the end, we provide an application of our main theorem for four finite families of mappings.

1. Introduction

The theory of probabilistic metric spaces is an important part of stochastic analysis, and so it is of interest to develop the fixed point theory in such spaces. The first result from the fixed point theory in probabilistic metric spaces was obtained by Sehgal and Bharucha-Reid [1]. Since then many fixed points theorems for single valued and multivalued mappings in probabilistic metric spaces have been proved in [2–8]. The study of 2-metric spaces was initiated by Gähler [9] and some fixed point theorems in 2-metric spaces were proved in [2, 8, 10–13]. In 1987, Zeng [14] gave the generalization of 2-metric to probabilistic 2-metric as follows.

A probabilistic 2-metric space is an ordered pair (X, F) , where X is an arbitrary set and F is a mapping from X^3 into the set of distribution functions. The distribution function $F_{x,y,z}(t)$ will denote the value of $F_{x,y,z}$ at the positive real number t . The function $F_{x,y,z}$ is assumed to satisfy the following conditions:

- (i) $F_{x,y,z}(0) = 0$ for all $x, y, z \in X$;
- (ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of the three points x, y, z are equal;
- (iii) for distinct points $x, y \in X$, there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$;

- (iv) $F_{x,y,z}(t) = F_{y,z,x}(t) = F_{z,x,y}(t) = \dots$ for all $x, y, z \in X$ and $t > 0$;
- (v) if $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$, and $F_{w,y,z}(t_3) = 1$, then $F_{x,y,z}(t_1 + t_2 + t_3) = 1$ for all $x, y, z, w \in X$ and $t_1, t_2, t_3 > 0$.

In 2003, Shi et al. [15] gave the notion of n th order t -norm as follows.

Definition 1. A mapping $\Delta : \prod_{i=1}^n [0, 1] \rightarrow [0, 1]$ is called an n th order t -norm if the following conditions are satisfied:

- (i) $\Delta(0, 0, \dots, 0) = 0$, $\Delta(a, 1, 1, \dots, 1) = a$ for all $a \in [0, 1]$;
- (ii) $\Delta(a_1, a_2, a_3, \dots, a_n) = \Delta(a_2, a_1, a_3, \dots, a_n) = \Delta(a_2, a_3, a_1, \dots, a_n) = \dots = \Delta(a_2, a_3, a_4, \dots, a_n, a_1)$;
- (iii) $a_i \geq b_i, i = 1, 2, 3, \dots, n$, implies $\Delta(a_1, a_2, a_3, \dots, a_n) \geq \Delta(b_1, b_2, b_3, \dots, b_n)$;
- (iv) $\Delta(\Delta(a_1, a_2, a_3, \dots, a_n), b_2, b_3, \dots, b_n) = \Delta(a_1, \Delta(a_2, a_3, \dots, a_n, b_2), b_3, \dots, b_n) = \Delta(a_1, a_2, \Delta(a_3, a_4, \dots, a_n, b_2, b_3), b_4, \dots, b_n) = \dots = \Delta(a_1, a_2, a_3, \dots, a_{n-1}, \Delta(a_n, b_2, b_3, \dots, b_n))$.

For $n = 2$, we have a binary t -norm, which is commonly known as t -norm.

Basic examples of t -norm are the Lukasiewicz t -norm $\Delta_L, \Delta_L(a, b) = \max(a + b - 1, 0)$, t -norm $\Delta_P, \Delta_P(a, b) = ab$, and t -norm $\Delta_M, \Delta_M(a, b) = \min\{a, b\}$.

Definition 2 (see [6]). A special class of t -norms (called a Hadzic-type t -norm) is introduced as follows.

Let Δ be a t -norm and let $\Delta_n : [0, 1] \rightarrow [0, 1]$ ($n \in N$) be defined in the following way:

$$\Delta_1(x) = \Delta(x, x), \quad \Delta_{n+1}(x) = \Delta(\Delta_n(x), x) \quad (1)$$

$$(n \in N, x \in [0, 1]).$$

We say that the t -norm Δ is of H type if Δ is continuous and the family $\{\Delta_n(x), n \in N\}$ is equicontinuous at $x = 1$.

The family $\{\Delta_n(x), n \in N\}$ is equicontinuous at $x = 1$ if for every $\lambda \in (0, 1)$ there exists $\delta(\lambda) \in (0, 1)$ such that the following implication holds:

$$x > 1 - \delta(\lambda) \quad \text{implies} \quad \Delta_n(x) > 1 - \lambda \quad \forall n \in N. \quad (2)$$

A trivial example of t -norm of H type is $\Delta = \Delta_M$.

Remark 3. Every t -norm Δ_M is of Hadzic type but the converse need not be true; see [7].

There is a nice characterization of continuous t -norm.

- (i) If there exists a strictly increasing sequence $\{b_n\}_{n \in N} \in [0, 1]$ such that $\lim_{n \rightarrow \infty} b_n = 1$ and $\Delta(b_n, b_n) = b_n$ for all $n \in N$, then Δ is of Hadzic type.
- (ii) If Δ is continuous and Δ is of Hadzic type, then there exists a sequence $\{b_n\}_{n \in N}$ as in (i).

Definition 4 (see [7]). If Δ is a t -norm and $(x_1, x_2, x_3, \dots, x_n) \in [0, 1]^n$ ($n \in N$), then $\Delta_{i=1}^n x_i$ is defined recurrently by 1, if $n = 1$ and $\Delta_{i=1}^n x_i = \Delta(\Delta_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. If $\{b_n\}_{n \in N}$ is a sequence of numbers from $[0, 1]$, then $\Delta_{i=1}^\infty x_i$ is defined as $\lim_{n \rightarrow \infty} \Delta_{i=1}^n x_i$ (this limit always exists) and $\Delta_{i=1}^\infty x_i$ as $\Delta_{i=1}^\infty x_{n+i}$.

Definition 5. Let X be any nonempty set and D the set of all left-continuous distribution functions. A triplet (X, F, Δ) is said to be a 2-Menger space if the probabilistic 2-metric space (X, F) satisfies the following condition:

$$F_{x,y,z}(t) \geq \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3)), \text{ where } t_1, t_2, t_3 > 0, t_1 + t_2 + t_3 = t; x, y, z, w \in X; \text{ and } \Delta \text{ is the 3rd order } t\text{-norm.}$$

Definition 6. A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be

- (i) converge with limit x if $\lim_{n \rightarrow \infty} F_{x_n, x, a}(t) = 1$ for all $t > 0$ and for every $a \in X$,
- (ii) Cauchy sequence in X , if given $\epsilon > 0, \lambda > 0$, there exists a positive integer $N_{\epsilon, \lambda}$ such that

$$F_{x_n, x_m, a}(\epsilon) > 1 - \lambda \quad \forall m, n > N_{\epsilon, \lambda}, \text{ for every } a \in X, \quad (3)$$

- (iii) complete if every Cauchy sequence in X is convergent in X .

In 1996, Jungck's [16] introduced the notion of weakly compatible as follows.

Definition 7. Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

In 2002, Aamri and Moutawakil [17] generalized the notion of noncompatible mapping to the E.A. property. It was pointed out in [17] that the property E.A. buys containment of ranges without any continuity requirements besides minimizing the commutativity conditions of the maps to the commutativity at their points of coincidence. Moreover, the E.A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range. Recently, some common fixed point theorems in probabilistic metric spaces/fuzzy metric spaces by the E.A. property under weak compatibility have been obtained in [18–20].

Definition 8 (see [17]). Let f and g be two self-maps of a metric (X, d) . The maps f and g are said to satisfy the E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = u \quad \text{for some } u \in X. \quad (4)$$

Now in a similar mode, we can state the E.A. property in 2-Menger space as follows.

Definition 9. A pair of self-mappings (f, g) of 2-Menger spaces (X, F, Δ) is said to hold the E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} F_{f x_n, g x_n, p}(t) = 1 \quad \forall t > 0, p \in X. \quad (5)$$

Example 10. Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \rightarrow X$ by $fx = x/4$ and $gx = 3x/4$ for all $x \in X$. Consider the sequence $\{x_n\} = 1/n$. Since $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = 0$, then f and g satisfy the E.A. property.

Although E.A property is generalization of the concept of noncompatible maps, yet it requires either completeness of the whole space or any of the range spaces or continuity of maps. But on the contrary, the new notion of the CLR property (common limit range property) recently given by Sintunavarat and Kumam [21] does not impose such conditions. The importance of the CLR property ensures that one does not require the closeness of range subspaces.

Definition 11 (see [21]). Two maps f and g on 2-Menger spaces X satisfy the common limit in the range of g (CLRg) property if $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = gx$ for some $x \in X$.

Example 12. Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \rightarrow X$ by $fx = x + 1$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{x_n\} = 1 + (1/n)$. Since $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = 2 = g1$, f and g satisfy the CLRg property.

Now we state a lemma which is useful in our study.

Lemma 13 (see [22]). Let (X, F, Δ) be a 2-Menger space. If there exists $q \in (0, 1)$ such that $F(x, y, z, qt) \geq F(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq x, z \neq y$, and $t > 0$, then $x = y$.

2. Weakly Compatible Maps

Theorem 14. Let (X, F, Δ) be a complete 2-Menger space with continuous t -norm Δ of H type. Let S and T be self-mappings of X . Then S and T have a unique common fixed point in X if and only if there exist two self-mappings A, B of X satisfying the following:

- (2.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;
- (2.2) the pairs (A, S) and (B, T) are weakly compatible;
- (2.3) there exists $q \in (0, 1)$ such that for every $x, y, p \in X$ and $t > 0$, $F(Ax, By, p, qt) \geq \min\{F(Sx, Ty, p, t), F(Ax, Sx, p, t), F(By, Ty, p, t), F(Ax, Ty, p, t)\}$;
- (2.4) one of the subsets $A(X), B(X), S(X)$, or $T(X)$ is a closed subset of X .

Indeed, A, B, S , and T have a unique common fixed point in X .

Proof. Suppose that S and T have a unique common fixed point, say $z \in X$.

Define $A : X \rightarrow X$ by $Ax = z$ for all $x \in X$ and $B : X \rightarrow X$ by $Bx = z$ for all $x \in X$.

Then one can see that (2.1)–(2.4) are satisfied.

Conversely, assume that there exist two self-mappings A, B of X satisfying conditions (2.1)–(2.4). From condition (2.1) we can construct two sequences $\{x_n\}$ and $\{y_n\}$ of X such that

$$\begin{aligned} y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}, \quad y_{2n} = Sx_{2n} = Bx_{2n-1} \\ \text{for } n = 1, 2, \dots \end{aligned} \quad (6)$$

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (2.3), we have that for all $p \in X$ and $t > 0$

$$\begin{aligned} F(y_{2n+1}, y_{2n+2}, p, qt) \\ = F(Ax_{2n}, Bx_{2n+1}, p, qt) \\ \geq \min\{F(Sx_{2n}, Tx_{2n+1}, p, t), \\ F(Ax_{2n}, Sx_{2n}, p, t), \\ F(Bx_{2n+1}, Tx_{2n+1}, p, t), \\ F(Ax_{2n}, Tx_{2n+1}, p, t)\} \\ = \min\{F(y_{2n}, y_{2n+1}, p, qt), \\ F(y_{2n+1}, y_{2n+2}, p, qt)\} \end{aligned} \quad (7)$$

implies $F(y_{2n+1}, y_{2n+2}, p, qt) \geq F(y_{2n}, y_{2n+1}, p, t)$, because F is nondecreasing.

Also, letting $x = x_{2n+2}$ and $y = x_{2n+1}$ in (2.3), we have that

$$\begin{aligned} F(y_{2n+2}, y_{2n+3}, p, qt) \geq F(y_{2n+1}, y_{2n+2}, p, t) \\ \forall p \in X, t > 0. \end{aligned} \quad (8)$$

In general, we have

$$F(y_n, y_{n+1}, p, qt) \geq F(y_{n-1}, y_n, p, t). \quad (9)$$

Thus for all $p \in X$, $t > 0$ and $n = 1, 2, 3, \dots$

$$F(y_n, y_{n+1}, p, t) \geq F\left(y_0, y_1, p, \frac{t}{q^n}\right). \quad (10)$$

We now show that $\{y_n\}$ is a Cauchy sequence in X .

Let $\epsilon \in (0, 1)$ be given. Since the t -norm Δ is of H type, there exists $\lambda \in (0, 1)$ such that for all $m, n \in N$ with $m > n$

$$\Delta^{2^{m-n}}(1 - \lambda) > 1 - \epsilon. \quad (11)$$

Since $\lim_{n \rightarrow \infty} F(y_0, y_1, p, t/q^n) = 1$, there exists $n_0 \in N$ such that for all $p \in X$ and $t > 0$, $F(y_0, y_1, p, t/q^n) > 1 - \lambda$ for all $n \geq n_0$.

From (10) we have for all $p \in X$ and $t > 0$, $F(y_n, y_{n+1}, p, t) > 1 - \lambda$ for all $n \geq n_0$.

Let $m > n \geq n_0$. Then for all $p \in X$ and $t > 0$, we have

$$\begin{aligned} F(y_m, y_n, p, t) \\ = F\left(y_m, y_n, p, \frac{t}{3} + \frac{t}{3} + \frac{t}{3}\right) \\ \geq \Delta\left(F\left(y_m, y_n, y_{n+1}, \frac{t}{3}\right), \right. \\ \left. F\left(y_m, y_{n+1}, p, \frac{t}{3}\right), F\left(y_{n+1}, y_n, p, \frac{t}{3}\right)\right) \\ \geq \Delta\left(F\left(y_{n+1}, y_n, p, \frac{t}{3}\right), \right. \\ \left. \Delta\left(F\left(y_{n+1}, y_n, y_m, \frac{t}{3}\right), F\left(y_{n+1}, y_m, p, \frac{t}{3}\right)\right)\right) \\ \geq \Delta\left(\Delta^2\left((1 - \lambda), F\left(y_{n+1}, y_m, p, \frac{t}{3}\right)\right)\right). \end{aligned} \quad (12)$$

Since

$$\begin{aligned} F\left(y_{n+1}, y_m, p, \frac{t}{3}\right) \geq \Delta\left(F\left(y_{n+2}, y_{n+1}, p, \frac{t}{3}\right), \right. \\ \left. \Delta\left(F\left(y_{n+2}, y_{n+1}, y_m, \frac{t}{3^2}\right), \right. \right. \\ \left. \left. F\left(y_{n+2}, y_m, p, \frac{t}{3^2}\right)\right)\right). \end{aligned} \quad (13)$$

From (13), we get

$$F(y_m, y_n, p, t) \geq \Delta\left(\Delta^{2^2}(1 - \lambda), F\left(y_{n+2}, y_m, p, \frac{t}{3^2}\right)\right). \quad (14)$$

Inductively, we obtain

$$\begin{aligned} F(y_m, y_n, p, t) \geq \Delta\left(\Delta^{2^{m-n}}(1 - \lambda), F\left(y_m, y_m, p, \frac{t}{3^{m-n}}\right)\right) \\ = \Delta^{2^{m-n}}(1 - \lambda). \end{aligned} \quad (15)$$

From (11) and (13) we get, for all $p \in X$ and $t > 0$, $F(y_m, y_n, p, t) > 1 - \epsilon$ for all $m > n \geq n_0$. Thus $\{y_n\}$ is

a Cauchy sequence in X . Since X is complete, there exists a point z in X such that $\lim_{n \rightarrow \infty} y_n = z$ and this gives

$$\begin{aligned} \lim_{n \rightarrow \infty} Sx_{2n} &= \lim_{n \rightarrow \infty} Tx_{2n-1} = \lim_{n \rightarrow \infty} Ax_{2n-2} \\ &= \lim_{n \rightarrow \infty} Bx_{2n-1} = z \quad \forall n \in \mathbb{N}. \end{aligned} \quad (16)$$

Without loss of generality, we assume that $S(X)$ is a complete subspace of X ; therefore, $z = Su$ for some $u \in X$.

Subsequently, we have

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z = Su. \quad (17)$$

Next, we claim that $Au = Su$.

For this purpose, we put $x = u$ and $y = x_n$ in (2.3); we have

$$\begin{aligned} F(Au, Bx_n, p, qt) \\ \geq \min \{F(Su, Tx_n, p, t), F(Au, Su, p, t), \\ F(Bx_n, Tx_n, p, t), F(Au, Tx_n, p, t)\}. \end{aligned} \quad (18)$$

Taking limit as $n \rightarrow \infty$

$$\begin{aligned} F(Au, z, p, qt) &\geq \min \{F(z, z, p, t), F(Au, z, p, t), \\ &F(z, z, p, t), F(Au, z, p, t)\} \\ &\geq F(Au, z, p, t). \end{aligned} \quad (19)$$

By Lemma 13, we have $Au = z$.

Hence $Au = Su = z$.

Since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $Au = z = Tv$.

Next we claim that $Tv = Bv$.

Putting $x = u$ and $y = v$ in (2.3), we have

$$\begin{aligned} F(Au, Bv, p, qt) &\geq \min \{F(Su, Tv, p, t), F(Au, Su, p, t), \\ &F(Bv, Tv, p, t), F(Au, Tv, p, t)\} \\ &\geq \min \{F(z, z, p, t), F(z, z, p, t), \\ &F(Bv, Tv, p, t), F(z, z, p, t)\} \\ &\geq F(Bv, Tv, p, t). \end{aligned} \quad (20)$$

That is, $F(Tv, Bv, p, qt) \geq F(Bv, Tv, p, t)$.

By Lemma 13, we have $Tv = Bv$.

Thus $Au = Su = Tv = Bv = z$.

Since the pairs (A, S) and (B, T) are weakly compatible and u and v are their points of coincidence, respectively, then

$$\begin{aligned} Az &= A(Su) = S(Au) = Sz, \\ Bz &= B(Tv) = T(Bv) = Tz. \end{aligned} \quad (21)$$

Now we prove that z is a common fixed point of A, B, S , and T .

For this purpose, putting $x = z$ and $y = v$ in (2.3), we get

$$\begin{aligned} F(Az, Bv, p, qt) &\geq \min \{F(Sz, Tv, p, t), F(Az, Sz, p, t), \\ &F(Bv, Tv, p, t), F(Az, Tv, p, t)\} \\ &\geq \min \{F(Az, Bv, p, t), F(Az, Az, p, t), \\ &F(Bv, Bv, p, t), F(Az, Bv, p, t)\} \\ &\geq F(Az, Bv, p, t). \end{aligned} \quad (22)$$

By Lemma 13, we have $Az = Bv = z$.

Hence $z = Az = Sz$, and z is a common fixed point of A and S . One can prove that $Bv = z$ is also a common fixed point of B and T .

Uniqueness. Suppose $z \neq w$ is another fixed point of A, B, S , and T .

Then, for all $p \in X$ with $p \neq z$ and $p \neq w$ and $t > 0$,

$$\begin{aligned} F(z, w, p, qt) &= F(Az, Bw, p, qt) \\ &\geq \min \{F(Sz, Tw, p, t), F(Az, Sz, p, t), \\ &F(Bw, Tw, p, t), F(Az, Tw, p, t)\} \\ &\geq \min \{F(z, w, p, t), F(z, z, p, t), \\ &F(w, w, p, t), F(z, w, p, t)\} \\ &\geq F(z, w, p, t), \end{aligned} \quad (23)$$

which implies that $F(z, w, p, qt) \geq F(z, w, p, t)$.

Hence $z = w$ is a unique common fixed point of A, B, S , and T .

This completes the proof of the theorem. $\square$

Corollary 15. Let (X, F, Δ) be a 2-Menger space with continuous t -norm Δ of H type. Let S and T be self-mappings of X . Then S and T have a unique common fixed point in X if and only if there exist two self-mappings A, B of X satisfying the following:

$$(2.5) \quad A(X) \subset S(X);$$

$$(2.6) \quad \text{pair } (A, S) \text{ is weakly compatible};$$

$$(2.7) \quad \text{there exists } q \in (0, 1) \text{ such that for every } x, y, p \in X \text{ and } t > 0,$$

$$\begin{aligned} &F(Ax, Ay, p, qt) \\ &\geq \min \{F(Sx, Sy, p, t), F(Ax, Sx, p, t), \\ &F(Ay, Sy, p, t), F(Ax, Sy, p, t)\}; \end{aligned} \quad (24)$$

$$(2.8) \quad \text{one of the subspaces } A(X) \text{ or } S(X) \text{ is a closed subspace of } X.$$

Indeed, A and S have a unique common fixed point in X .

Proof. We can easily prove the theorem by setting $B = A$ and $T = S$, in the proof of the Theorem 14. $\square$

Corollary 16. Let (X, F, Δ) be a 2-Menger space with continuous t -norm Δ of H type. Let S and T be self-mappings of X . Then S and T have a unique common fixed point in X if and only if there exist two self-mappings A, B of X satisfying (2.1), (2.2), (2.4), and the following:

$$(2.9) \text{ there exists } q \in (0, 1) \text{ such that for every } x, y, p \in X \text{ and } t > 0, F(Ax, By, p, qt) \geq F(Sx, Ty, p, t).$$

Indeed, A, B, S , and T have a unique common fixed point in X .

Proof. Proof easily follows by setting $\min\{F(Sx, Ty, p, t), F(Ax, Sx, p, t), F(By, Ty, p, t), F(Ax, Ty, p, t)\} = F(Sx, Ty, p, t)$ in the proof of Theorem 14. $\square$

3. E.A. Property and Weakly Compatible Maps

Now we prove our main result for weakly compatible maps along with the E.A. property as follows.

Theorem 17. Let (X, F, Δ) be a 2-Menger space with continuous t -norm Δ of H type. Let A, B, S , and T be self-mappings of X satisfying (2.1), (2.2), (2.3), and (2.4) and the following:

$$(3.1) \text{ pair } (A, S) \text{ or pair } (B, T) \text{ satisfies E.A. property.}$$

Then A, B, S , and T have a unique common fixed point in X .

Proof. If the pair (B, T) satisfies the property E.A., then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = z$ for some $z \in X$.

Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n \rightarrow \infty} Sy_n = z$. Also $A(X) \subset T(X)$; there exists a sequence $\{y'_n\}$ in X such that $Ay'_n = Tx_n$. Hence $\lim_{n \rightarrow \infty} Ay'_n = z$.

Suppose that $S(X)$ is a complete subspace of X . Then $z = Su$ for some $u \in X$; subsequently; we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = z = \lim_{n \rightarrow \infty} Ay'_n \\ &= \lim_{n \rightarrow \infty} Sy_n = z = Su \quad \text{for some } u \in X. \end{aligned} \quad (25)$$

Next, we claim that $Au = Su$.

For this purpose, we put $x = u$ and $y = x_n$ in (2.3); this gives

$$\begin{aligned} F(Au, Bx_n, p, qt) \\ \geq \min \{F(Su, Tx_n, p, t), F(Au, Su, p, t), \\ F(Bx_n, Tx_n, p, t), F(Au, Tx_n, p, t)\}. \end{aligned} \quad (26)$$

Taking limit as $n \rightarrow \infty$

$$\begin{aligned} F(Au, z, p, qt) \\ \geq \min \{F(z, z, p, t), F(Au, z, p, t), \\ F(z, z, p, t), F(Au, z, p, t)\} \\ \geq F(Au, z, p, t). \end{aligned} \quad (27)$$

By Lemma 13, we have $Au = z$.

Hence $Au = Su = z$.

Since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $Au = z = Tv$.

Next we claim that $Tv = Bv$.

Putting $x = u$ and $y = v$ in (2.3), we have

$$\begin{aligned} F(Au, Bv, p, qt) \\ \geq \min \{F(Su, Tv, p, t), F(Au, Su, p, t), \\ F(Bv, Tv, p, t), F(Au, Tv, p, t)\} \\ \geq \min \{F(z, z, p, t), F(z, z, p, t), \\ F(Bv, Tv, p, t), F(z, z, p, t)\} \\ \geq F(Bv, Tv, p, t). \end{aligned} \quad (28)$$

That is, $F(Tv, Bv, p, qt) \geq F(Bv, Tv, p, t)$.

By Lemma 13, we have $Tv = Bv$.

Thus $Au = Su = Tv = Bv = z$.

Since the pairs (A, S) and (B, T) are weakly compatible and u and v are their points of coincidence, respectively, then $Az = A(Su) = S(Au) = Sz$ and $Bz = B(Tv) = T(Bv) = Tz$.

Now we prove that z is a common fixed point of A, B, S , and T .

For this purpose, we put $x = z$ and $y = v$ in (2.3); we get

$$\begin{aligned} F(Az, Bv, p, qt) \\ \geq \min \{F(Sz, Tv, p, t), F(Az, Sz, p, t), \\ F(Bv, Tv, p, t), F(Az, Tv, p, t)\} \\ \geq \min \{F(Az, Bv, p, t), F(Az, Az, p, t), \\ F(Bv, Bv, p, t), F(Az, Bv, p, t)\} \\ \geq F(Az, Bv, p, t). \end{aligned} \quad (29)$$

By Lemma 13, we have $Az = Bv = z$.

Hence $z = Az = Sz$ and z is a common fixed point of A and S . One can prove that $Bv = z$ is also a common fixed point of B and T .

Uniqueness. Suppose $z \neq w$ is another fixed point of A, B, S , and T .

Then, for all $p \in X$ with $p \neq z$ and $p \neq w$ and $t > 0$,

$$\begin{aligned} F(z, w, p, qt) \\ = F(Az, Bw, p, qt) \\ \geq \min \{F(Sz, Tw, p, t), F(Az, Sz, p, t), \\ F(Bw, Tw, p, t), F(Az, Tw, p, t)\} \\ \geq \min \{F(z, w, p, t), F(z, z, p, t), \\ F(w, w, p, t), F(z, w, p, t)\} \\ \geq F(z, w, p, t), \end{aligned} \quad (30)$$

which implies that $F(z, w, p, qt) \geq F(z, w, p, t)$.

Hence $z = w$ is a unique common fixed point of A, B, S , and T . $\square$

Corollary 18. Let (X, F, Δ) be a 2-Menger space with continuous t -norm Δ of H type. Let A, B, S , and T be self-mappings of X satisfying (2.1), (2.2), (2.4), and (2.9).

Then A, B, S , and T have a unique common fixed point in X .

4. CLR Property and Weakly Compatible Maps

Now we prove our main result for weakly compatible maps along with the CLR_S property as follows.

Theorem 19. Let (X, F, Δ) be a 2-Menger space with continuous t -norm Δ of H type. Let A, B, S , and T be self mapping of X satisfying (2.1), (2.2), (2.3), and the following:

- (4.1) pair (A, S) or pair (B, T) satisfies CLR_S property;
- (4.2) one of the subspaces $A(X), B(X), S(X)$, or $T(X)$ is a closed subspace of X .

Then A, B, S , and T have a unique common fixed point in X .

Proof. If the pair (A, S) satisfies the CLR_S property then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$, where $z \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = z$. Since $T(X)$ is a closed subspace of X and $A(X) \subset T(X)$, for each $\{x_n\} \subset X$ there corresponds a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$.

Therefore, $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Ax_n = z$, where $z \in S(X)$.

Thus in all, we have

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = z. \quad (31)$$

Now we are required to show that $\lim_{n \rightarrow \infty} By_n = z$.

Putting $x = x_n$ and $y = y_n$ in (2.3), we get

$$\begin{aligned} & F(Ax_n, By_n, p, qt) \\ & \geq \min \{F(Sx_n, Ty_n, p, t), F(Ax_n, Sx_n, p, t), \\ & F(By_n, Ty_n, p, t), F(Ax_n, Ty_n, p, t)\}. \end{aligned} \quad (32)$$

Let $\lim_{n \rightarrow \infty} By_n = l \neq z$ for $t > 0$. Then taking limit as $n \rightarrow \infty$, we have

$$\begin{aligned} F(z, l, p, qt) & \geq \min \{F(z, z, p, t), F(z, z, p, t), \\ & F(l, z, p, t), F(z, z, p, t)\} \\ & \geq F(l, z, p, t). \end{aligned} \quad (33)$$

By Lemma 13, we have $z = l$, then $\lim_{n \rightarrow \infty} By_n = z$.

Hence $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} By_n = z = Su$, for some $u \in X$.

Uniqueness follows from Theorem 14. $\square$

Corollary 20. Let (X, F, Δ) be a 2-Menger space with continuous t -norm Δ of H type. Let A, B, S , and T be self-mappings of X satisfying (2.1), (2.2), (2.9), (4.1), and (4.2).

Then A, B, S , and T have a unique common fixed point in X .

5. Application

As an application of Theorem 14, we prove a common fixed point theorem for four finite families of mappings which runs as follows.

Theorem 21. Let $\{A_1, A_2, \dots, A_m\}$, $\{B_1, B_2, \dots, B_n\}$, $\{S_1, S_2, \dots, S_p\}$, and $\{T_1, T_2, \dots, T_q\}$ be four finite families of self-mappings of a 2-Menger space (X, F, Δ) with continuous t -norm of Hadzic type such that $A = A_1 A_2 \cdots A_m$, $B = B_1 B_2 \cdots B_n$, $S = S_1 S_2 \cdots S_p$, and $T = T_1 T_2 \cdots T_q$ satisfy condition (2.1), (2.2), (2.3), and (2.4); then

(i) A and S have a point of coincidence;

(ii) B and T have a point of coincidence.

Moreover, if $A_i A_j = A_j A_i$, $B_k B_l = B_l B_k$, $S_r S_s = S_s S_r$, $T_t T_u = T_u T_t$, $A_i S_r = S_r A_i$, and $B_k T_t = T_t B_k$ for all $i, j \in I_1 = \{1, 2, \dots, m\}$, $k, l \in I_2 = \{1, 2, \dots, n\}$, $r, s \in I_3 = \{1, 2, \dots, p\}$, and $t, u \in I_4 = \{1, 2, \dots, q\}$, then (for all $i \in I_1, k \in I_2, r \in I_3$, and $t \in I_4$) A_i, S_r, B_k , and T_t have a common fixed point.

Proof. The conclusions (i) and (ii) are immediate as A, S, B , and T satisfy all the conditions of Theorem 14. Now appealing to component wise commutativity of various pairs, one can immediately prove that $AS = SA$ and $BT = TB$, and hence, obviously both pairs (A, S) and (B, T) are coincidentally commuting. Note that all the conditions of Theorem 14 (for mappings A, S, B , and T) are satisfied ensuring the existence of a unique common fixed point, say z . Now one needs to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned} A(A_i z) &= ((A_1 A_2 \cdots A_m) A_i) z \\ &= (A_1 A_2 \cdots A_{m-1}) ((A_m A_i) z) \\ &= (A_1 A_2 \cdots A_{m-1}) (A_i A_m z) \\ &= (A_1 A_2 \cdots A_{m-2}) (A_{m-1} A_i (A_m z)) \\ &= (A_1 A_2 \cdots A_{m-2}) (A_i A_{m-1} (A_m z)) \\ &= \cdots = A_1 A_i (A_2 A_3 A_4 \cdots A_m z) \\ &= A_i A_1 (A_2 A_3 A_4 \cdots A_m z) = A_i (Az) = A_i z. \end{aligned} \quad (34)$$

Similarly, one can show that

$$\begin{aligned} A(S_r z) &= S_r (Az) = S_r z, & S(S_r z) &= S_r (Sz) = S_r z, \\ S(A_i z) &= A_i (Sz) = A_i z, & B(B_k z) &= B_k (Bz) = B_k z, \\ B(T_t z) &= T_t (Bz) = T_t z, & T(T_t z) &= T_t (Tz) = T_t z, \\ T(B_k z) &= B_k (Tz) = B_k z, \end{aligned} \quad (35)$$

which show that (for all i, r, k , and t) $A_i z$ and $S_r z$ are other fixed points of the pair (A, S) whereas $B_k z$ and $T_t z$ are other fixed points of the pair (B, T) . Now appealing to the

uniqueness of common fixed points of both pairs separately, we get

$$z = A_i z = S_r z = B_k z = T_t z, \quad (36)$$

which shows that z is a common fixed point of A_i , S_r , B_k , and T_t for all i, r, k , and t . $\square$

By setting $A = A_1 = A_2 = \cdots = A_m$, $B = B_1 = B_2 = \cdots = B_n$, $S = S_1 = S_2 = \cdots = S_p$, and $T = T_1 = T_2 = \cdots = T_q$, one deduces the following for certain iterates of maps, which run as follows.

Corollary 22. *Let A, B, S , and T be four self-mappings of a 2-Menger space (X, F, Δ) such that A^m , B^n , S^p , and T^q satisfy the conditions (2.1) and (2.3). If one of $A^m(X)$, $B^n(X)$, $S^p(X)$, or $T^q(X)$ is a closed subset of X , then A, B, S , and T have a unique common fixed point provided (A, S) and (B, T) commute.*

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