Research Article

An Estimate of the Rate of Convergence of the Fourier Series in the Generalized Hölder Metric by Delayed Arithmetic Mean

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We study the rate of convergence problem of the Fourier series by Delayed Arithmetic Mean in the generalized Hölder metric $(H_p^{(w)})$ space which was earlier introduced by Das, Nath, and Ray and obtain a sharper estimate of Jackson's order.

1. Definition

Let f be a 2π -periodic function such that $f \in L_p[0, 2\pi]$, $p \geq 1$.

1. Let the Fourier series of f at $t = x$ be given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (1)$$

Let

$$\phi_x(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\}. \quad (2)$$

Let $S_n(x)$ be the n th partial sum of (1). Then it is known ([1], page 50) that

$$S_n(x) - f(x) = \frac{2}{\pi} \int_0^\pi \phi_x(t) D_n(t) dt, \quad (3)$$

where $D_n(t) = \sin(n+1/2)t/2 \sin(t/2)$ is known as Dirichlet's kernel.

Let $\mathcal{C}_{2\pi}$ denote the Banach space of all 2π -periodic continuous functions defined on $[-\pi, \pi]$ under the supremum

norm. The space $L_p[0, 2\pi]$ with $p = \infty$ reduces to $\mathcal{C}_{2\pi}$ defined over $[0, 2\pi]$. We write

$$\|f\|_c = \sup_{0 \leq t \leq 2\pi} |f(t)|, \quad (p = \infty)$$

$$\|f\|_p = \left(\int_0^{2\pi} |f(t)|^p dt \right)^{1/p}, \quad p \geq 1$$

$$\omega(\delta, f) = \sup_{0 \leq |h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_c, \quad f \in \mathcal{C}_{2\pi} \quad (4)$$

$$\omega_p(\delta, f) = \sup_{0 \leq |h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_p,$$

$$f \in L_p[0, 2\pi], \quad p \geq 1,$$

when the norm has been taken with respect to x throughout the paper. The quantities $\omega(\delta, f)$ and $\omega_p(\delta, f)$ are, respectively, called the modulus of continuity and integral modulus of continuity of f . It is known ([1], page 45) that $\omega(\delta, f)$ and $\omega_p(\delta, f)$ both tend to zero as $\delta \rightarrow 0$.

It was Prössdorf [2] who first studied the degree of approximation problems of the Fourier series in H_α ($0 < \alpha \leq 1$) space in the Hölder metric. Generalizing the Hölder metric, Leindler [3] introduced the space H^ω given by

$$H^\omega = \{f \in \mathcal{C}_{2\pi} : \omega(\delta, f) = O(\omega(\delta))\}, \quad (5)$$

where ω is a modulus of continuity; that is, ω is a positive non-decreasing continuous function on $[0, 2\pi]$ with the following property:

- (i) $\omega(0) = 0$,
- (ii) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$,
- (iii) $\omega(\lambda\delta) \leq (\lambda + 1)\omega(\delta)$, $\lambda \geq 0$.

Further Leindler [3] has introduced the following metric on H^ω space:

$$\|f\|_\omega = \|f\|_c + \sup_{x,y} \frac{|f(x) - f(y)|}{\omega(|x - y|)}, \quad x \neq y. \quad (6)$$

In the case $\omega(\delta) = \delta^\alpha$, $0 < \alpha \leq 1$ the space H^ω reduces to H_α space (the norm $\|\cdot\|_\omega$ being replaced by $\|\cdot\|_\alpha$) which is introduced by Prössdorf [2]. It is known that [2]

$$H_\alpha \subseteq H_\beta \subseteq \mathcal{C}_{2\pi}, \quad 0 \leq \beta < \alpha \leq 1. \quad (7)$$

The degree of approximation problem in H^ω space has been studied by Leindler [3], Totik [4, 5], Mazhar and Totik [6], and Mazhar [7, 8]. The space H^ω was further generalized by Das et al. [9] as follows.

For $f \in L_p[0, 2\pi]$, $p \geq 1$, we write

$$A(f, \omega) = \sup_{t \neq 0} \frac{\|f(\cdot + t) - f(\cdot)\|_p}{\omega(|t|)}, \quad (8)$$

where ω is a modulus of continuity. If

$$\|f(\cdot + t) - f(\cdot)\|_p = O(\omega(|t|)), \quad (9)$$

then we say that $f \in \text{Lip}(\omega, p)$. We define

$$H_p^{(\omega)} = \{f \in L_p[0, 2\pi], p \geq 1 : A(f, \omega) < \infty\}, \quad (10)$$

$$\|f\|_p^{(\omega)} = \|f\|_p + A(f, \omega). \quad (11)$$

It can be seen that $\|f\|_p^{(\omega)}$ is a norm in $H_p^{(\omega)}$. To prove the completeness of the space $H_p^{(\omega)}$ we use the completeness of L_p ($p \geq 1$).

If we put $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, then $H_p^{(\omega)}$ reduces to $H(\alpha, p)$ space (with the norm $\|f\|_p^{(\omega)}$ replaced by $\|f\|_{(\alpha, p)}$) which is introduced earlier by Das et al. [10]. If $\omega(t)/t \rightarrow 0$ as $t \rightarrow 0$, then $f'(x)$ exists and is 0 everywhere and f is constant. Given the spaces $H_p^{(\omega)}$ and $H_p^{(v)}$, if $\omega(t)/v(t)$ is nondecreasing, then

$$H_p^{(\omega)} \subseteq H_p^{(v)} \subseteq L_p, \quad p \geq 1, \quad (12)$$

since

$$\|f\|_p^{(v)} \leq \max\left(1, \frac{\omega(2\pi)}{v(2\pi)}\right) \|f\|_p^{(\omega)}. \quad (13)$$

For $0 \leq \beta < \alpha \leq 1$, if we put $\omega(t) = t^\alpha$ and $v(t) = t^\beta$, then (12) reduces to the following:

$$H_{(\alpha, p)} \subseteq H_{(\beta, p)} \subseteq L_p, \quad p \geq 1. \quad (14)$$

Note that the space $H_{(\alpha, \infty)}$ is the familiar H_α space introduced earlier by Prössdorf [2].

1.1. The Cesàro Transformation. Let $\sum_{n=0}^{\infty} u_n$ be an infinite series and let (s_n) denote the sequence of its n th partial sums. Then the series $\sum u_n$ is said to be summable (C, μ) ($\mu > -1$) to the sum s (finite), if (see [1], page 76)

$$\sigma_n^\mu = \frac{S_n^\mu}{A_n^\mu} \rightarrow s \quad \text{as } n \rightarrow \infty, \quad (15)$$

where A_n^μ and S_n^μ are defined by the following formulae:

$$\begin{aligned} \sum_{n=0}^{\infty} A_n^\mu x^n &= (1-x)^{-1-\mu} \\ \sum_{n=0}^{\infty} S_n^\mu x^n &= (1-x)^{-\mu} \sum_{n=0}^{\infty} s_n x^n \\ &= (1-x)^{-1-\mu} \sum_{n=0}^{\infty} u_n x^n, \end{aligned} \quad (16)$$

where $(|x| < 1)$, $\mu > -1$.

From the definition of S_n^μ and A_n^μ it follows that ([1], page 77)

$$S_n^\mu = \sum_{k=0}^n A_{n-k}^{\mu-1} s_k = \sum_{k=0}^n A_{n-k}^\mu u_k \quad (17)$$

$$A_n^\mu = \binom{n+\mu}{\mu} \approx \frac{n^\mu}{\Gamma(\mu+1)} \quad (\mu \neq -1, -2, \dots).$$

The numbers S_n^μ and σ_n^μ are called, respectively, the Cesàro sums and the Cesàro means of order μ ($\mu > -1$) of the series $\sum u_n$. Applications of the Cesàro transformation can be found in engineering, for example, modal dynamics in earthquake engineering (see Chen and Hong [11], Chen et al. [12]).

1.2. Delayed Arithmetic Mean. Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with sequence of arithmetic mean $\{\sigma_n\}$. The Delayed Arithmetic Mean $\sigma_{n,k}$ of $\sum u_n$ is given by ([1], page 80)

$$\sigma_{n,k} = \left(1 + \frac{n}{k}\right) \sigma_{n+k-1} - \frac{n}{k} \sigma_{n-1}, \quad (18)$$

where k is a positive integer.

And it is known [1] that

$$(C, 1) \subset \sigma_{n,k} \quad \text{if } \frac{n}{k} = O(1) \quad \text{as } k \rightarrow \infty \text{ with } n. \quad (19)$$

For $k = n$,

$$\sigma_{n,n} = 2\sigma_{2n-1} - \sigma_{n-1}, \quad (20)$$

which may be called *first type* Delayed Arithmetic Mean.

But in this present paper we take $k = 2n$ for Delayed Arithmetic Mean, which is of the form

$$\sigma_{n,2n} = \frac{1}{2} [3\sigma_{3n-1} - \sigma_{n-1}]. \quad (21)$$

This may be called *second type* Delayed Arithmetic Mean.

Let $\sigma_n(f; x)$ and $\sigma_{n,2n}(f; x)$, respectively, denote the first arithmetic mean and *second type* Delayed Arithmetic Mean of (1). It is known that (see [1], page 88 and page 89) the Fejer's kernel

$$F_n(t) = \frac{2}{n+1} \left\{ \frac{\sin\left(\frac{1}{2}\right)(n+1)t}{2 \sin\left(\frac{t}{2}\right)} \right\}^2, \quad (22)$$

$$\sigma_n(f; x) = \frac{1}{\pi} \int_0^\pi F_n(t) \{f(x+t) + f(x-t)\} dt. \quad (23)$$

It can be easily verified that

$$\sigma_{n,2n}(f; x) = \frac{1}{n\pi} \int_0^\pi \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} \{f(x+t) + f(x-t)\} dt. \quad (24)$$

We adopt the following additional notations:

$$\begin{aligned} G_{x,y}(t) &= \phi_{x+y}(t) - \phi_x(t), \\ l_n(x) &= \sigma_{n,2n}(f; x) - f(x), \\ h &= h(n) = \frac{\pi}{n}. \end{aligned} \quad (25)$$

2. Introduction

Extensive investigations of approximation by polynomials S_n and σ_n involve mediocre approximation involving $O(\log n)$ times worse than the best approximation (see the remarks by [1], page 122). The following theorems, in particular, are being quoted to vindicate that $O(\log n)$ persists in the order of approximation involving σ_n and S_n .

Das et al. [9] developed a new space $H_p^{(\omega)}$ and generalized the theorem of Leindler [3] to give the following.

Theorem A (see [9]). *Let v and ω be moduli of continuity such that ω/v is nondecreasing. If $f \in H_p^{(\omega)}$, $p \geq 1$, then*

$$\begin{aligned} \|S_n(f; x) - f(x)\|_p^{(v)} &= O\left(\frac{\omega(\pi/n)}{v(\pi/n)} \log n\right) \\ &+ O(1) \frac{1}{n} \int_{\pi/n}^\pi \frac{\omega(t)}{t^2 v(t)} dt. \end{aligned} \quad (26)$$

Theorem B (see [9]). *If $f \in Lip(\omega, p)$, $p \geq 1$, and is of monotonic type, then*

$$\|S_n(f; x) - f(x)\|_p = O(1) \frac{1}{n} \int_{1/n}^\pi \frac{\omega(t)}{t^2} dt, \quad (27)$$

where ω is a modulus of continuity.

Recently Leindler [13] proved the following.

Theorem C ([13, Theorem 1]). *Let v and ω be moduli of continuity such that ω/v is nondecreasing; moreover, for some $0 < \epsilon \leq 1$, let the function*

$$\gamma(t) := \gamma(v, \omega, \epsilon; t) := t^{-\epsilon} \frac{\omega(t)}{v(t)} \quad (28)$$

be nonincreasing. If $f \in H_p^{(\omega)}$, $p \geq 1$, then

$$\|S_n - f\|_p^{(v)} \leq \frac{\omega(1/n)}{v(1/n)} \log n \quad \forall n. \quad (29)$$

Leindler [13] has derived Theorem C from Theorem A by making use of the fact that

$$\int_{\pi/n}^\pi \frac{\omega(t)}{t^2 v(t)} dt = O\left(\frac{\omega(\pi/n)}{v(\pi/n)} \log n\right) \quad (30)$$

whenever $\omega(t)/t^\epsilon v(t)$ ($0 < \epsilon \leq 1$) is nonincreasing.

Quade [14] however proved the following.

Theorem D ([14, Theorem 5]). *Let $f \in Lip(\alpha, p)$, $0 < \alpha \leq 1$; then*

- (i) $\|f - S_n(f, \cdot)\|_p = O(1/n^\alpha)$, $p > 1$,
- (ii) $\|f - S_n(f, \cdot)\|_1 = O((\log n)/n^\alpha)$, $p = 1$.

Theorem E ([14, Theorem 6]). *If $f \in Lip(\alpha, p)$, $0 < \alpha \leq 1$, then*

- (i) if $p > 1$ or if $p = 1, \alpha < 1$,

$$\|f - \sigma_n\|_p = O\left(\frac{1}{n^\alpha}\right); \quad (31)$$

- (ii) if $p = \alpha = 1$,

$$\|f - \sigma_n\|_1 = O\left(\frac{\log n}{n}\right). \quad (32)$$

Prössdorf [2] obtained the following result concerning the degree of approximation of the Fourier series using Fejer's mean in the Hölder metric.

Theorem F (see [2]). *Let $f \in H_\alpha$ ($0 < \alpha \leq 1$) and $0 \leq \beta < \alpha \leq 1$; then*

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} \frac{1}{n^{\alpha-\beta}}, & 0 < \alpha < 1 \\ \frac{(1 + \log n)^{1-\beta}}{n^{1-\beta}}, & \alpha = 1. \end{cases} \quad (33)$$

The objective of the present paper is to establish theorems, which involve Jackson's order (see [15], page 56) not involving $O(\log n)$.

3. Main Results

We prove the following theorems.

Theorem 1. *Let v and ω be moduli of continuity such that ω/v is nondecreasing. If $f \in H_p^{(\omega)}$, $p \geq 1$, then*

$$(I) \quad \|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_p^{(v)} = O\left(\frac{1}{n}\right) + O(1) \frac{1}{n^2} \int_{\pi/n}^\pi \frac{\omega(t)}{t^3 v(t)} dt. \quad (34)$$

If in addition $\omega(t)/tv(t)$ is nonincreasing, then

$$(II) \quad \|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_p^{(v)} = O(1) \frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3 v(t)} dt \quad (35)$$

and a fortiori

$$(III) \quad \|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_p^{(v)} = O\left(\frac{\omega(\pi/n)}{v(\pi/n)}\right). \quad (36)$$

Theorem 2. If $f \in Lip(\omega, p)$, $p \geq 1$, then

$$(I) \quad \|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_p = O\left(\frac{1}{n}\right) + O(1) \frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3} dt. \quad (37)$$

If in addition $\omega(t)/t$ is nonincreasing, then

$$(II) \quad \|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_p = O(1) \frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3} dt \quad (38)$$

and a fortiori

$$(III) \quad \|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_p = O\left(\omega\left(\frac{\pi}{n}\right)\right). \quad (39)$$

We need the following lemma.

Lemma 3 ([9, Lemma 1]). Let $v(t)$ and $\omega(t)$ be defined as in Theorem 1. Then for $0 < t \leq \pi$ and $p \geq 1$

- (i) $\|\phi(t)\|_p \leq 2\omega(t)$,
- (ii) $\|G_{\cdot,y}(t)\|_p = O(1) \left\{ \frac{\omega(t)}{\omega(|y|)} \right\}$,
- (iii) $\|G_{\cdot,y}(t)\|_p = O(1)v(|y|)(\omega(t)/v(t))$,
- (iv) $\|G_{\cdot,y}(t) - G_{\cdot,y}(t+h)\|_p = O(1)v(|y|)(\omega(h)/v(h))$,
where $h = \pi/n$.

4. Proof of Theorem 1

It may be easily verified that

$$\frac{2}{n\pi} \int_0^{\pi} \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} dt = 1, \quad (40)$$

and hence

$$\begin{aligned} l_n(x) &= \sigma_{n,2n}(f; x) - f(x) \\ &= \frac{2}{n\pi} \int_0^{\pi} \phi_x(t) \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} dt. \end{aligned} \quad (41)$$

By definition

$$\|l_n\|_p^{(\omega)} = \|l_n\|_p + \sup_{y \neq 0} \frac{\|l_n(\cdot) - l_n(\cdot + y)\|_p}{\omega(|y|)}. \quad (42)$$

Hence

$$\begin{aligned} l_n(x+y) - l_n(x) &= \frac{2}{n\pi} \int_0^{\pi} \{\phi_{x+y}(t) - \phi_x(t)\} \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} dt \\ &= \frac{2}{n\pi} \int_0^{\pi} \{G_{x,y}(t)\} \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} dt \\ &= \frac{2}{n\pi} \int_0^h \{G_{x,y}(t)\} \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} dt \\ &\quad + \frac{2}{n\pi} \int_h^{\pi} \{G_{x,y}(t)\} \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} dt \\ &= I + J. \end{aligned} \quad (43)$$

Since

$$\frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} = O(n^2), \quad 0 < t < \pi, \quad (44)$$

now, by the generalized Minkowski inequality and for $p \geq 1$, we have

$$\begin{aligned} \|I\|_p &\leq \frac{2}{n\pi} \int_0^h \|G_{\cdot,y}(t)\|_p \left| \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} \right| dt \\ &= O(1)nv(|y|) \int_0^h \frac{\omega(t)}{v(t)} dt, \quad (\text{by Lemma 3}) \\ &= O(1)nv(|y|)h \frac{\omega(h)}{v(h)} \\ &= O(1)v(|y|) \frac{\omega(h)}{v(h)}. \end{aligned} \quad (45)$$

Now,

$$\begin{aligned} J &= \frac{2}{n\pi} \int_h^{\pi} G_{x,y}(t) \frac{\sin 2nt \sin nt}{(2 \sin(t/2))^2} dt \\ &= \frac{2}{n\pi} \int_h^{\pi} G_{x,y}(t) \sin 2nt \sin nt \left\{ \frac{1}{(2 \sin(t/2))^2} - \frac{1}{t^2} \right\} dt \\ &\quad + \frac{2}{n\pi} \int_h^{\pi} G_{x,y}(t) \frac{\sin 2nt \sin nt}{t^2} dt \\ &= J_1 + J_2. \end{aligned} \quad (46)$$

Since $\{1/(2 \sin(t/2))^2 - 1/t^2\}$ is bounded and by Lemma 3, we get

$$\begin{aligned} \|J_1\|_p &\leq \frac{2}{n\pi} \int_h^{\pi} \|G_{\cdot,y}(t)\|_p dt \\ &= O(1) \frac{1}{n} v(|y|) \int_h^{\pi} \frac{\omega(t)}{v(t)} dt \\ &= O(1) \frac{1}{n} v(|y|). \end{aligned} \quad (47)$$

Now,

$$J_2 = \frac{2}{n\pi} \int_h^\pi G_{x,y}(t) \frac{\sin 2nt \sin nt}{t^2} dt. \quad (48)$$

Replacing t with $t + h$ in (48), we obtain

$$J_2 = -\frac{2}{n\pi} \int_0^{\pi-h} G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt. \quad (49)$$

From (48) and (49), it follows that

$$\begin{aligned} J_2 &= \frac{1}{n\pi} \int_h^\pi G_{x,y}(t) \frac{\sin 2nt \sin nt}{t^2} dt \\ &\quad - \frac{1}{n\pi} \int_0^{\pi-h} G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt \\ &= \frac{1}{n\pi} \int_h^\pi G_{x,y}(t) \frac{\sin 2nt \sin nt}{t^2} dt \\ &\quad - \frac{1}{n\pi} \int_0^h G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt \\ &\quad - \frac{1}{n\pi} \int_h^\pi G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt \\ &\quad + \frac{1}{n\pi} \int_{\pi-h}^\pi G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt \\ &= \frac{1}{n\pi} \int_h^\pi \left\{ \frac{G_{x,y}(t)}{t^2} - \frac{G_{x,y}(t+h)}{(t+h)^2} \right\} \sin 2nt \sin nt dt \\ &\quad - \frac{1}{n\pi} \int_0^h G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt \\ &\quad + \frac{1}{n\pi} \int_{\pi-h}^\pi G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt \\ &= K - L + M. \end{aligned} \quad (50)$$

Since $\sin 2nt \sin nt = O(n^2 t^2)$, $0 < t < \pi$, it follows that

$$\begin{aligned} \|L\|_p &\leq \frac{1}{n\pi} \int_0^h \|G_{x,y}(t+h)\|_p \frac{n^2 t^2}{(t+h)^2} dt \\ &= O(1) \frac{1}{n} v(|y|) \int_0^h \frac{\omega(t+h)}{v(t+h)} n^2 dt \\ &= O(1) n v(|y|) \int_0^h \frac{\omega(t+h)}{v(t+h)} dt \\ &= O(1) n v(|y|) \frac{\omega(2h)}{v(2h)} \int_0^h dt \\ &= O(1) n v(|y|) \frac{\omega(h)}{v(h)} h \\ &= O(1) v(|y|) \frac{\omega(h)}{v(h)} \end{aligned} \quad (51)$$

as $\omega(2h) \leq 3\omega(h)$ and $v(2h) \geq v(h)$.

Since $|\sin 2nt \sin nt| \leq 1$, for $t \in (\pi - h, \pi)$, it follows that

$$\begin{aligned} \|M\|_p &\leq \frac{1}{n\pi} \int_{\pi-h}^\pi \|G_{x,y}(t+h)\|_p \frac{dt}{(t+h)^2} \\ &= O(1) \frac{1}{n} v(|y|) \int_{\pi-h}^\pi \frac{\omega(t+h)}{v(t+h)} \frac{dt}{(t+h)^2} \\ &= O(1) \frac{1}{n} v(|y|) \int_\pi^{\pi+h} \frac{\omega(\theta)}{v(\theta)} \frac{d\theta}{\theta^2} \\ &= O(1) \frac{1}{n^2} v(|y|). \end{aligned} \quad (52)$$

Now,

$$\begin{aligned} K &= \frac{1}{n\pi} \left\{ \int_h^\pi G_{x,y}(t) \frac{\sin 2nt \sin nt}{t^2} dt \right. \\ &\quad - \int_h^\pi G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{t^2} dt \\ &\quad + \int_h^\pi G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{t^2} dt \\ &\quad \left. - \int_h^\pi G_{x,y}(t+h) \frac{\sin 2nt \sin nt}{(t+h)^2} dt \right\} \\ &= \frac{1}{n\pi} \int_h^\pi \{G_{x,y}(t) - G_{x,y}(t+h)\} \frac{\sin 2nt \sin nt}{t^2} dt \\ &\quad + \frac{1}{n\pi} \int_h^\pi G_{x,y}(t+h) \sin 2nt \sin nt \left\{ \frac{1}{t^2} - \frac{1}{(t+h)^2} \right\} dt \\ &= K_1 + K_2. \end{aligned} \quad (53)$$

Since $|\sin 2nt \sin nt| \leq 1$, for $t \in (h, \pi)$, using Lemma 3 (iv), we get

$$\begin{aligned} \|K_1\|_p &\leq \frac{1}{n\pi} \int_h^\pi \|G_{x,y}(t) - G_{x,y}(t+h)\|_p \frac{dt}{t^2} \\ &= O(1) \frac{1}{n} v(|y|) \frac{\omega(h)}{v(h)} \int_h^\pi \frac{dt}{t^2} \\ &= O(1) v(|y|) \frac{\omega(h)}{v(h)}. \end{aligned} \quad (54)$$

Similarly,

$$\begin{aligned} \|K_2\|_p &\leq \frac{1}{n\pi} \int_h^\pi \|G_{x,y}(t+h)\|_p \left\{ \frac{1}{t^2} - \frac{1}{(t+h)^2} \right\} dt \\ &= O(1) \frac{1}{n} v(|y|) \int_h^\pi \frac{\omega(t+h)}{v(t+h)} \frac{(t+h)^2 - t^2}{t^2(t+h)^2} dt \\ &= O(1) \frac{1}{n^2} v(|y|) \int_h^\pi \frac{\omega(t+h)}{v(t+h)} \frac{dt}{t^3} \\ &= O(1) \frac{1}{n^2} v(|y|) \int_h^\pi \frac{\omega(t)}{v(t)} \frac{dt}{t^3}. \end{aligned} \quad (55)$$

Collecting the results from (45)–(55), we obtain

$$\sup_{y \neq 0} \frac{\|I_n(\cdot) - I_n(\cdot + y)\|_p}{v(|y|)} = O\left(\frac{1}{n}\right) + O(1) \frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3 v(t)} dt. \quad (56)$$

Proceeding in the similar lines, it can be proved that

$$\|I_n\|_p = O\left(\frac{1}{n}\right) + O(1) \frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3} dt. \quad (57)$$

Hence collecting results of (42), (56), and (57), we obtain

$$\|I_n\|_p^{(v)} = O\left(\frac{1}{n}\right) + O(1) \frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3 v(t)} dt, \quad (58)$$

which ensures (I) of Theorem 1.

If in addition $\omega(t)/tv(t)$ is nonincreasing, we have

$$\frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3 v(t)} dt \geq \frac{1}{n^2} \frac{\omega(\pi)}{\pi v(\pi)} \int_{\pi/n}^{\pi} \frac{dt}{t^2} \geq \frac{C}{n}, \quad (59)$$

where C is some positive constant. Further

$$\frac{1}{n^2} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^3 v(t)} dt \leq \frac{1}{n^2} \frac{n \omega(\pi/n)}{\pi v(\pi/n)} \int_{\pi/n}^{\pi} \frac{dt}{t^2} = O\left(\frac{\omega(\pi/n)}{v(\pi/n)}\right). \quad (60)$$

Using (59) and (60) in (58), we get (II) and (III) of Theorem 1. This completes the proof.

5. Proof of Theorem 2

We adopt the technique used in the proof of Theorem 1 and use the fact that $\|\phi_\cdot(t)\|_p = O(\omega(t))$ whenever $f \in \text{Lip}(\omega, p)$, $p \geq 1$, to prove Theorem 2.

6. Remarks and Corollaries

Comparing Theorem 1 with Theorem A and Theorem C we observe that trigonometric polynomial $\sigma_{n,2n}(f; x)$ provides sharper estimation than those obtained by sequence of partial sums of a Fourier series. The comparison of Theorem 2 with Theorem B also exhibits the advantage of $\sigma_{n,2n}(f; x)$ over the sequence of partial sums of the Fourier series in the degree of approximation problem.

We specialize $\omega(t)$ and $v(t)$ to obtain the following corollaries from Theorems 1 and 2. Taking $\omega(t) = t^\alpha$ and $v(t) = t^\beta$, $0 \leq \beta < \alpha \leq 1$ in Theorem 1, we get the following.

Corollary 4. If $f \in H(\alpha, p)$, $p \geq 1$, $0 \leq \beta < \alpha \leq 1$, then

$$\|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_{(\beta, p)} = O\left(\frac{1}{n^{\alpha-\beta}}\right). \quad (61)$$

Taking $p = \infty$ in Corollary 4, we obtain the following.

Corollary 5. If $f \in H_\alpha$, $0 \leq \beta < \alpha \leq 1$, then

$$\|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_\beta = O\left(\frac{1}{n^{\alpha-\beta}}\right). \quad (62)$$

The above corollary can be compared with Theorem F.

Taking $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$ in Theorem 2, we obtain the following.

Corollary 6. If $f \in \text{Lip}(\alpha, p)$, $0 < \alpha \leq 1$ and $p \geq 1$, then

$$\|\sigma_{n,2n}(f; \cdot) - f(\cdot)\|_p = O\left(\frac{1}{n^\alpha}\right). \quad (63)$$

For the space $\text{Lip}(1, 1)$ Quade has provided the result (32) of Theorem E using $\sigma_n(f; x)$ trigonometric polynomial. Estimate given in (32) certainly is not of Jackson's order. Therefore in Corollary 6 a more general trigonometric polynomial (namely, $\sigma_{n,2n}(f; x)$) has been taken to ensure Jackson's order, which is sharper than the one given in (32).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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