Research Article

Frames of Eigenfunctions Associated with a Boundary Value Problem

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We introduce and study a redundant system of retro Banach frames consisting of eigenfunctions associated with a given boundary value problem.

1. Introduction

Duffin and Schaeffer in [1], while addressing some deep problems in nonharmonic Fourier series, abstracted Gabor's method [2], of time-frequency atomic decomposition for signal processing to define frames for Hilbert spaces.

A sequence $\{f_k\}$ in a real (or complex) separable Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ is a *frame* (or *Hilbert frame*) for $\mathcal{H}$ if there exist finite positive constants A_0 and B_0 such that

$$A_0 \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B_0 \|f\|^2, \quad \forall f \in \mathcal{H}. \quad (1)$$

The positive constants A_0 and B_0 are called *lower* and *upper* bounds of the frame, respectively. The inequality (1) is called the *frame inequality* of the frame. The operator $T : \ell^2 \rightarrow \mathcal{H}$ given by

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \quad \{c_k\} \in \ell^2 \quad (2)$$

is called the *synthesis operator* or the *preframe operator* of the frame. The adjoint operator $T^* : \mathcal{H} \rightarrow \ell^2$ of T is called the *analysis operator*. More precisely, T^* is given by

$$T^* : f \rightarrow \{\langle f, f_k \rangle\}. \quad (3)$$

Composing T and T^* , we obtain the *frame operator* $S = TT^* : \mathcal{H} \rightarrow \mathcal{H}$ which is given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad f \in \mathcal{H}. \quad (4)$$

The frame operator S is a positive continuous invertible linear operator from $\mathcal{H}$ to $\mathcal{H}$. Every vector $f \in \mathcal{H}$ can be written as

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k. \quad (5)$$

The series converges unconditionally and is called the *reconstruction formula* for the frame. The representation of f in reconstruction formula need not be unique. Thus, frames are *redundant systems* in a Hilbert space which yield one natural representation for every vector in the given Hilbert space, but which may have infinitely many different representations for a given vector.

Frames provide an appropriate mathematical framework for redundant signal expansions [3, 4]. Moreover, frames find many applications in mathematics, science, and engineering. In particular, frames are widely used in nonuniform sampling [5], wavelet theory [6, 7], wireless communication, signal processing [3, 8], filter banks [9], and many more. The reason is that frames provide both great liberties in design of vector space decompositions and quantitative measure on computability and robustness of the corresponding reconstructions. In the theoretical direction, powerful tools from

operator theory and Banach spaces are being employed to study frames. For a nice introduction to various types of frames with applications, one may refer to [10–13].

In 1986, Daubechies et al. [6] found new applications to wavelets and Gabor transforms in which frames played an important role. Coifman and Weiss [14] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchenig [15] studied the atomic decomposition via integrable group representation. Casazza et al. [16] also carried out a study of atomic decompositions and Banach frames. For recent development in Banach frames, one may refer to [17–20]. Han and Larson in [21] introduced Schauder frames in Banach spaces. Recently, various generalizations of frames and reconstruction systems in Banach spaces have been introduced and studied. Retro Banach frames were introduced in [22] and further studied in [19, 23, 24]. Casazza and Christensen [25] introduced the reconstruction property in Banach spaces. The reconstruction property in Banach spaces was further studied in [18, 26]. Actually, each vector in a Banach space which admits the reconstruction property can be reconstructed by mean of an infinite series, where linear independence is not required. Note that all the coefficients are supposed to calculate which appear in the series expansion of a certain vector. On the other hand, retro Banach frames in Banach spaces reconstruct a given vector via the preframe operator or the reconstruction operator (see Definition 1).

Consider a signal space with some reconstruction system which is not orthogonal; say nonorthogonal frames (the reconstruction property or Schauder frames). During signal processing or compression of a signal, the fast algorithms for evaluation of the concern expansion coefficients are required. Algorithms related to frames can be found in [12] and references given thereat. Note that these algorithms work efficiently according to type of the frame available. Depending on the availability of reconstruction system sometimes, it is difficult to compute the expansion coefficients. In a case when fast algorithms are not available for computing all the coefficients within suitable time, it is natural to introduce a more flexible reconstruction system for concern space.

In this paper, we introduce a more flexible system consisting of the redundant system of preframe operators, which can reconstruct the given Banach space (via preframe operator). Since there are numerical methods for construction of the eigenfunctions from boundary value problems [27–29], we discuss the said redundant operator system in the context of retro Banach frames which satisfies the property $\mathcal{S}$, associated with a given boundary value problem. It is proved that a retro Banach frame consisting of the eigenfunctions of a given BVP can generate a retro Banach frame which satisfies the property $\mathcal{S}$. A necessary and sufficient condition for retro Banach frames satisfying property $\mathcal{S}$ has been given. Perturbation theory is important in applied mathematics [30]; we discuss a result which deals with the block perturbation of retro Banach frames which satisfies the property $\mathcal{S}$. The retro Banach frames which satisfy the property $\mathcal{S}$ in finite product of Banach spaces are discussed.

2. Preliminaries

Throughout this paper $\mathcal{X}$ will denote an infinite dimensional separable Banach space over the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathcal{X}^*$ the conjugate space of $\mathcal{X}$. For a sequence $\{f_n\} \subset \mathcal{X}$, $[\overline{f_n}]$ denotes the closure of the linear hull of $\{f_n\}$ in the norm topology of $\mathcal{X}$. The space of all bounded linear operators from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Definition 1 (see [22]). Let $\{f_n\} \subset \mathcal{X}$ and let $\mathcal{Z}_d$ be an associated Banach space of sequences of scalars. A system $\mathcal{F} \equiv (\{f_n\}, \Theta)$ ($\Theta : \mathcal{Z}_d \rightarrow \mathcal{X}^*$) is called a *retro Banach frame* for $\mathcal{X}^*$ with respect to $\mathcal{Z}_d$ if

- (i) $\{f^*(f_n)\} \in \mathcal{Z}_d$, for each $f^* \in \mathcal{X}^*$,
- (ii) there exist positive constants $0 < A \leq B < \infty$ such that

$$A \|f^*\| \leq \| \{f^*(f_n)\} \|_{\mathcal{Z}_d} \leq B \|f^*\|, \quad \text{for each } f^* \in \mathcal{X}^*, \quad (6)$$

- (iii) Θ is a bounded linear operator such that $\Theta(\{f^*(f_n)\}) = f^*$, for all $f^* \in \mathcal{X}^*$.

The positive constants A, B are called *retro frame bounds* of $\mathcal{F}$ and the operator $\Theta : \mathcal{Z}_d \rightarrow \mathcal{X}^*$ is called the *retro preframe operator* (or simply reconstruction operator) associated with $\mathcal{F}$. The inequality in (ii) is called the *retro frame inequality*. A retro Banach frame $\mathcal{F} \equiv (\{f_n\}, \Theta)$ is said to be an *exact retro Banach frame* for $\mathcal{X}^*$ if there exists no reconstruction operator Θ_m ($m \in \mathbb{N}$) such that $(\{f_n\}_{n \neq m}, \Theta_m)$ is retro Banach frame for $\mathcal{X}^*$.

Definition 2 (see [25]). A sequence $\{f_n^*\} \subset \mathcal{X}^*$ has the *reconstruction property* for $\mathcal{X}$ with respect to $\{f_n\} \subset \mathcal{X}$ if

$$f = \sum_{n=1}^{\infty} f_n^*(f) f_n, \quad \forall f \in \mathcal{X}, \quad (7)$$

where the series converges in the norm of $\mathcal{X}$.

In short, we will also say that $(\{f_n\}, \{f_n^*\})$ has the reconstruction property for $\mathcal{X}$. More precisely, we say that $(\{f_n\}, \{f_n^*\})$ is a *reconstruction system* for $\mathcal{X}$.

Remark 3. An interesting example for the reconstruction property is given in [25]. Let $\{f_n^*\} \subset \ell^\infty$ and $\{f_n^*\}$ is unitarily equivalent to the unit vector basis of ℓ^2 . Then, $\{f_n^*\}$ has a reconstruction property with respect to its own predual (i.e., expansions with respect to the orthonormal basis). But this family cannot have the reconstruction property with respect to ℓ^1 .

Regarding existence of Banach spaces which have reconstruction system, Casazza and Christensen gave the following result.

Proposition 4 (see [25]). *There exists a Banach space $\mathcal{X}$ with the following properties:*

- (i) *there is a sequence $\{f_n\}$ such that each $f \in \mathcal{X}$ has an expansion $f = \sum_{n=1}^{\infty} f_n^*(f) f_n$;*
- (ii) *$\mathcal{X}$ does not have the reconstruction property with respect to any pair $(\{h_n\}, \{h_n^*\})$.*

The notion of the reconstruction property is related to the bounded approximation property (BAP). If $(\{f_n\}, \{f_n^*\})$ has the reconstruction property for $\mathcal{X}$, then $\mathcal{X}$ has the bounded approximation property. So, $\mathcal{X}$ is isomorphic to a complemented subspace of a Banach space with a basis. It is also used to study geometry of Banach spaces. For more results and basics on the bounded approximation property, one may refer to [16, 31] and references therein.

3. Boundary Value Problem and Frames in Banach Spaces

Let $\mathcal{X} = L^2(a, b)$. Consider a boundary value problem (BVP) with a set of n boundary conditions

$$(\mathfrak{x}) \equiv \nabla(f) = \lambda f, \quad \Lambda(f) = 0, \quad (8)$$

where $\nabla(\bullet) = (\bullet)^n + \Phi_1(\xi)(\bullet)^{n-1} + \dots + \Phi_n(\xi)(\bullet)$ is a linear differential operator with $\Phi_j \in C^{n-k}[a, b]$ and $\Lambda(f) = 0$ denotes the set of n boundary conditions given by

$$\Lambda_j(\Phi) = \sum_{i=1}^n [\alpha_{j,k} \Phi^{k-1}(a) + \beta_{j,k} \Phi^{k-1}(b)] = 0, \quad (9)$$

$$1 \leq j \leq n.$$

The BVP $(\mathfrak{x})$ admits system $\{\Phi_n(\xi)\}$ and $\{\Psi_n(\xi)\}$ consisting of eigenfunctions associated with $(\mathfrak{x})$ (see [29, page 66]) such that

$$\begin{aligned} \Phi_n(\xi) &= A_n \left[\cos \frac{2\pi n \xi}{b-a} + O\left(\frac{1}{n}\right) \right], \\ \Psi_m(\xi) &= B_m \left[\sin \frac{2\pi m \xi}{b-a} + O\left(\frac{1}{m}\right) \right], \end{aligned} \quad (10)$$

where $n, m \in \mathbb{N} \cup \{0\}$.

Note that

$$\begin{aligned} \left\| \Phi_n(\xi) - A_n \cos \frac{2\pi n \xi}{b-a} \right\|^2 &< 1, \\ \left\| \Psi_m(\xi) - B_m \sin \frac{2\pi m \xi}{b-a} \right\|^2 &< 1. \end{aligned} \quad (11)$$

Choose $\{Y_k\} = \{\Phi_n\} \sqcup \{\Psi_m\}$. By using Paley and Weiner Theorem [32, page 208], there exists $\{Y_k^*\} \subset \mathcal{X}^*$ such that $\{Y_k\}^*$ has a reconstruction system for $\mathcal{X}$ with respect to $\{Y_k\}$. More precisely, every element of $L^2(a, b)$ can write a linear combination (infinite) of the atoms Φ_n and Ψ_m , where computation of all coefficients which involve these atoms is

required. In order to reduce such computation, naturally a more flexible reconstruction system is required. This new flexible reconstruction system proposed in this paper is a retro Banach frame which satisfies the property $\mathcal{S}$. Recall that a retro Banach frame can reconstruct the given Banach space by a bounded linear operator (retro preframe operator). This is useful in the sense that we can choose a suitable set (index) which controls such complicated computation (via operator theory).

Definition 5. A retro Banach frame $\mathcal{F} \equiv (\{f_n\}, \Theta)$ for $\mathcal{X}^*$ is said to satisfy the property $\mathcal{S}$ if, for each infinite subset $\sigma \subset \mathbb{N}$, there exists a reconstruction operator Θ_σ such that $\mathcal{F}_\sigma \equiv (\{f_k\}_{k \in \sigma}, \Theta_\sigma)$ is a retro Banach frame for $\mathcal{X}^*$.

Example 6. Let $\mathcal{X} = L^2(a, b)$. Consider the system of eigenfunctions $\{Y_k\} = \{\Phi_n\} \sqcup \{\Psi_m\}$ associated with the BVP $(\mathfrak{x})$. Then, $\mathcal{X}_d = \{\{Y^*(Y_k)\} : Y^* \in \mathcal{X}^*\}$ is a Banach space of sequences of scalars with the norm given by

$$\|(\bullet)(Y_k)\|_{\mathcal{X}_d} = \|\bullet\|_{\mathcal{X}^*}. \quad (12)$$

Define $\Theta : \mathcal{X}_d \rightarrow \mathcal{X}^*$ by

$$\Theta(\{Y^*(Y_k)\}) = Y^*. \quad (13)$$

Then, $\Theta \in \mathcal{B}(\mathcal{X}_d, \mathcal{X}^*)$. Therefore, $\mathcal{F} \equiv (\{Y_k\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{X}_d$ and with bounds $A = B = 1$. The retro Banach frame $\mathcal{F} \equiv (\{Y_k\}, \Theta)$ is called a retro Banach frame associated with BVP $(\mathfrak{x})$.

Remark 7. One may observe that we can find an infinite set $\sigma \subset \mathbb{N}$ for which there is no reconstruction operator Θ_σ such that $(\{Y_k\}_{k \in \sigma}, \Theta_\sigma)$ is a retro Banach frame for $\mathcal{X}^*$. Hence, $\mathcal{F}$ does not satisfy the property $\mathcal{S}$.

Example 8. Let $\mathcal{X} = (L^1(\Omega), \mu)$, where $\Omega = \mathbb{N}$ and μ is the counting measure. Choose $f_n = \{e^{-(1+1/n)\ell}\}_{\ell=1}^{\infty}$, $n \in \mathbb{N}$. Then, there exists a reconstruction operator $\Theta_o : \mathcal{X}_{d_o} = \{\{f^*(f_n)\} : f^* \in \mathcal{X}^*\}$ such that $(\{f_n\}, \Theta_o)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{X}_{d_o}$ and which satisfies the property $\mathcal{S}$.

We can generate a system in $\mathcal{X}$ consisting of the eigenfunctions associated with the BVP given in $(\mathfrak{x})$, which can generate the retro Banach frame which satisfies the property $\mathcal{S}$. This is given in the following theorem.

Theorem 9. Let $\mathcal{X} = L^2(a, b)$ and let $\mathcal{F} \equiv (\{Y_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$, consisting of eigenfunctions of the BVP $(\mathfrak{x})$. Then, there exists a retro Banach frame $(\{\widehat{Y}_n\}, \widehat{\Theta})$ for $\mathcal{X}^*$ (generated by $\mathcal{F}$) which satisfies the property $\mathcal{S}$.

Proof. Since $\mathcal{F} \equiv (\{Y_k\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to an associated Banach space of scalar-valued sequences $\mathcal{X}_d$. Then, there exist positive constants $0 < A \leq B < \infty$ such that

$$A\|\Phi^*\|_{\mathcal{X}^*} \leq \|(\Phi^*(Y_k))\|_{\mathcal{X}_d} \leq B\|\Phi^*\|_{\mathcal{X}^*}, \quad \forall \Phi^* \in \mathcal{X}^*. \quad (14)$$

Define a system $\{\widehat{Y}_n\} \subset \mathcal{X}$ by $\widehat{Y}_n = \sum_{i=1}^n (1/i^n) Y_i$, $n \in \mathbb{N}$.

Then, $\mathcal{Z}_{d_0} = \{\{Y^*(\widehat{Y}_n)\} : Y^* \in \mathcal{X}^*\}$ is a Banach space of sequences of scalars with the norm given by

$$\|\{Y^*(\widehat{Y}_n)\}\|_{\mathcal{Z}_{d_0}} = \|Y^*\|_{\mathcal{X}^*}, \quad Y^* \in \mathcal{X}^*. \quad (15)$$

Define $\widehat{\Theta} : \mathcal{Z}_{d_0} \rightarrow \mathcal{X}^*$ by $\widehat{\Theta}(\{Y^*(\widehat{Y}_n)\}) = Y^*, Y^* \in \mathcal{X}^*$. Then, $\widehat{\Theta} \in \mathcal{B}(\mathcal{Z}_{d_0}, \mathcal{X}^*)$. Therefore, $\mathcal{F}_0 \equiv (\{\widehat{Y}_n\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_{d_0}$.

Now, we show that $\mathcal{F}_0$ satisfies the property $\mathcal{S}$. Assume that $\mathcal{F}_0$ does not satisfy the property $\mathcal{S}$. Then, there exists no reconstruction operator Θ_σ corresponding to some $\sigma = \{n_k\} \subset \mathbb{N}$ such that $(\{\widehat{Y}_{n_k}\}, \Theta_\sigma)$ is a retro Banach frame for $\mathcal{X}^*$. Therefore, by the Hahn-Banach theorem there exists a nonzero functional $Y_0^* \in \mathcal{X}^*$ such that $Y_0^*(\widehat{Y}_{n_k}) = 0$, for all $k \in \mathbb{N}$.

This gives

$$\sum_{i=1}^{n_k} \frac{1}{i^{m_k}} Y_0^*(Y_i) = 0, \quad k \in \mathbb{N}. \quad (16)$$

Thus, $Y_0^*(Y_k) = 0$, for all $k \in \mathbb{N}$. By using retro frame inequality of $\mathcal{F}$, we obtain $Y_0^* = 0$, a contradiction. Hence, $\mathcal{F}_0$ satisfies the property $\mathcal{S}$. $\square$

Corollary 10. Let $\mathcal{X} = L^2(a, b)$. Then, $\mathcal{X}^*$ has a retro Banach frame if and only if it admits a retro Banach frame which satisfies the property $\mathcal{S}$.

Perturbation theory is a very important tool in various areas of applied mathematics [18, 25, 30]. In frame theory, it began with the fundamental perturbation result of Young [13]. The basic of Paley and Wiener was that a system that is sufficiently close to an orthonormal system (basis) in a Hilbert space is also from an orthonormal system (basis). Since then, a number of variations and generalizations of perturbations to the atomic decompositions, Hilbert frames, and Banach frames have been studied. One of such perturbation is a block perturbation. We observe that behavior of the retro Banach frames which satisfies the property $\mathcal{S}$ under block perturbation is invariant.

Definition 11. Let $\mathcal{F} \equiv (\{Y_n\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ associated with BVP $(\mathfrak{H})$ and let $\{m_n\}, \{p_n\}$ be increasing sequences of positive integers, where $m_0 = 0$ and $m_{n-1} + 1 \leq p_n \leq m_n, n \in \mathbb{N}$.

Define $\{g_n\} \subset \mathcal{X}$ by

$$g_k = \begin{cases} Y_k, & \text{if } k \neq p_n, \\ Y_{p_n} + h_n & \text{if } k = p_n, \end{cases} \quad n \in \mathbb{N}, \quad (17)$$

where $h_n = \sum_{i=m_{n-1}+1}^{p_n-1} \alpha_i Y_i + \sum_{i=p_n+1}^{m_n} \alpha_i Y_i$, for all $n \in \mathbb{N}$. Then, $\{g_n\}$ is called the *block perturbation* of $\{Y_n\}$.

By Example 6 and the result given in Theorem 9, we observe that the eigenfunctions associated with BVP $(\mathfrak{H})$ can generate a retro Banach frame for $\mathcal{X}^*$ which satisfy the property $\mathcal{S}$, where $\mathcal{X} = L^2(a, b)$. The following theorem shows that the retro Banach frames which satisfy the property $\mathcal{S}$ are invariant under block perturbation.

Theorem 12. Let $\mathcal{X} = L^2(a, b)$ and let $\mathcal{F} \equiv (\{Y_k\}, \Theta)$ be a retro Banach frame (for $\mathcal{X}^*$) consisting of eigenfunctions of the BVP $(\mathfrak{H})$. If $\mathcal{F}$ satisfies the property $\mathcal{S}$, then its block perturbation also satisfies the property $\mathcal{S}$.

Proof. Let $\{g_k\}$ be the block perturbation of $\mathcal{F} \equiv (\{Y_k\}, \Theta)$ and let $\mathcal{Z}_0 = \{\{Y^*(g_k)\} : Y^* \in \mathcal{X}^*\}$. Then, $\mathcal{Z}_0$ is a Banach space of scalar-valued sequences with norm given by

$$\|\{(\cdot)(g_k)\}\|_{\mathcal{Z}_0} = \|\cdot\|_{\mathcal{X}^*}. \quad (18)$$

Define $\widetilde{\Theta} : \mathcal{Z}_0 \rightarrow \mathcal{X}^*$ by $\widetilde{\Theta}(\{f^*(g_k)\}) = f^*, f^* \in \mathcal{X}^*$. Therefore, there exists a reconstruction operator $\widetilde{\Theta}$ such that $\mathcal{G} \equiv (\{g_k\}, \widetilde{\Theta})$ is a normalized tight retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_0$.

Define $\{g_{n_k}\} \subset \mathcal{X}$ by

$$g_{n_k} = \begin{cases} Y_{n_k}, & \text{if } n_k \neq p_{n_k}, \\ Y_{p_{n_k}} + h_{n_k} & \text{if } n_k = p_{n_k}, \end{cases} \quad k \in \mathbb{N}, \quad (19)$$

where $h_{n_k} = \sum_{n_i=m_{n_k-1}+1}^{p_{n_k}-1} \alpha_{n_i} Y_{n_i} + \sum_{n_i=p_{n_k}+1}^{m_{n_k}} \alpha_{n_i} Y_{n_i}$, for all $k \in \mathbb{N}$. Then, $\{g_{n_k}\}$ is a block perturbation of $\{Y_{n_k}\}$.

Also note that $[g_{n_k}] = \mathcal{X}$. Indeed, let $[g_{n_k}] \neq \mathcal{X}$. Then, by the Hahn-Banach Theorem there exists a nonzero functional $\Phi^* \in \mathcal{X}^*$ such that $\Phi^*(g_{n_k}) = 0$, for all $k \in \mathbb{N}$.

Therefore,

$$\begin{aligned} \Phi^*(Y_{n_i}) &= 0, \quad n_i \neq p_{n_k}, \\ \Phi^*(Y_{p_{n_k}}) &= -\Phi^*(h_{n_k}), \quad k \in \mathbb{N}. \end{aligned} \quad (20)$$

This gives $\Phi^*(Y_{n_i}) = 0$, for all $i \in \mathbb{N}$, since $(\{Y_k\}, \Theta)$ is satisfying the property $\mathcal{S}$. Therefore, corresponding to $\sigma = \{n_i\}$, there exists a reconstruction operator Θ_σ such that $(\{Y_{n_i}\}, \Theta_\sigma)$ is a retro Banach frame for $\mathcal{X}^*$. So, by using retro frame inequality of $(\{Y_{n_i}\}, \Theta_\sigma)$, we obtain $\Phi^* = 0$. This is a contradiction. Therefore, we can find an associated Banach space of scalar-valued sequences, say $\mathcal{Z}^0$, and a bounded linear operator Θ_0 such that $(\{g_{n_k}\}, \Theta_0)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}^0$. Hence, $(\{g_n\}, \widetilde{\Theta})$ is a retro Banach frame for $\mathcal{X}^*$ which satisfies the property $\mathcal{S}$. $\square$

Let us come back to BVP $(\mathfrak{H})$, which is not multidimensional. If domain of the said problem is multidimensional (which is standard nowadays), then reconstruction can be controlled with such level of liberty. More precisely, in multidimensional domain, retro Banach frames enjoy the property $\mathcal{S}$. This is what the concluding theorem of this paper says and can be generalized to any multidimensional domain (finite).

Theorem 13. Let $(\{Y_k^+\}, \Theta^+)$ and $(\{Y_k^-\}, \Theta^-)$ be retro Banach frames satisfying the property $\mathcal{S}$ for Banach spaces $\mathcal{X}^*$ and $\mathcal{Y}^*$, respectively. Then, there exists a system $\{\widehat{Y}_k\} \subset (\mathcal{X} \times \mathcal{Y})$ and a reconstruction operator $\Theta_0 : \mathcal{Z}(\mathcal{X} \times \mathcal{Y}) \rightarrow (\mathcal{X} \times \mathcal{Y})$ such that $(\{\widehat{Y}_k\}, \Theta_0)$ is a normalized tight retro Banach frame for $(\mathcal{X} \times \mathcal{Y})^*$ which satisfies the property $\mathcal{S}$.

Proof. Define a system $\{\widehat{Y}_k\} \subset (\mathcal{X} \times \mathcal{Y})$ by

$$\begin{aligned}\widehat{Y}_{2k} &= (0, Y_k^-), \quad k \in \mathbb{N}, \\ \widehat{Y}_{2k-1} &= (Y_k^+, 0), \quad k \in \mathbb{N}.\end{aligned}\quad (21)$$

Then, $[\widehat{Y}_k] = (\mathcal{X} \times \mathcal{Y})$. Otherwise, by Hahn-Banach theorem there exists a nonzero $\Phi^* \in (\mathcal{X} \times \mathcal{Y})^*$ such that $\Phi^*(\widehat{Y}_k) = 0$, for all $k \in \mathbb{N}$. By the nature of the system $\{\widehat{Y}_k\}$ and by use of retro frame inequalities of $(\{Y_k^+\}, \Theta^+)$ and $(\{Y_k^-\}, \Theta^-)$, we obtain $\Phi = 0$, a contradiction.

Therefore, $\mathcal{Z}_{(\mathcal{X} \times \mathcal{Y})} = \{\{\Psi^*(\widehat{Y}_k)\} \mid \Psi^* \in (\mathcal{X} \times \mathcal{Y})^*\}$ is an associated Banach space of scalar-valued sequences with the norm given by

$$\|\{\Psi^*(\widehat{Y}_k)\}\|_{\mathcal{Z}_{(\mathcal{X} \times \mathcal{Y})}} = \|\Psi^*\|_{(\mathcal{X} \times \mathcal{Y})^*}. \quad (22)$$

Define $\Theta_0 : \mathcal{Z}_{(\mathcal{X} \times \mathcal{Y})} \rightarrow (\mathcal{X} \times \mathcal{Y})^*$ by

$$\Theta_0(\{\Psi^*(\widehat{Y}_k)\}) = \Psi^*. \quad (23)$$

Then, Θ_0 is the bounded linear operator such that $(\{\widehat{Y}_k\}, \Theta_0)$ is a normalized tight retro Banach frame for $(\mathcal{X} \times \mathcal{Y})^*$ with respect to $\mathcal{Z}(\mathcal{X} \times \mathcal{Y})$.

Fix $\sigma = \{n_k\} \subset \mathbb{N}$. Then, similar to a construction given in the proof of Theorem 9, we can construct a system $\{\widehat{Y}_{n_k}\} \subset (\mathcal{X} \times \mathcal{Y})$ for which there exists a reconstruction operator $\Theta_\sigma : (\mathcal{X} \times \mathcal{Y})^* \rightarrow (\mathcal{X} \times \mathcal{Y})$ such that $(\{\widehat{Y}_{n_k}\}, \Theta_\sigma)$ is a normalized tight retro Banach frame for $(\mathcal{X} \times \mathcal{Y})^*$. Hence, $(\{\widehat{Y}_n\}, \Theta_0)$ is a retro Banach frame for $(\mathcal{X} \times \mathcal{Y})^*$ which satisfies the property $\mathcal{S}$. $\square$

4. Concluding Remark

Given a Hilbert frame $\{f_n\}$ for a separable Hilbert space $\mathcal{H}$ with frame operator S , every element in $\mathcal{H}$ can be represented as linear combination (infinite) of the frame vectors, where computation of all the frame coefficients $\langle S^{-1}f, f_k \rangle$ is required. So, a popular choice of frame coefficients is missing (in the said reconstruction). On the other hand, in case of retro Banach frames each element can be recovered by the retro preframe operator. The result given in Theorem 9 gives the construction of a more flexible system (of preframe operators), where it is observed that, corresponding to an index (appropriate) set, the retro preframe operator reconstructs the underlying space. Overall, the retro Banach frames which satisfy the property $\mathcal{S}$ provide the *redundant preframe operator system* or *overfilled systems of preframe operators* which can reconstruct the underlying space.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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