Hindawi Publishing Corporation
International Journal of Analysis
Volume 2014, Article ID 684512, 4 pages
http://dx.doi.org/10.1155/2014/684512

Research Article

A New Double Sequence Space $m^2(F, \phi, p)$ Defined by a Double Sequence of Modulus Functions

Birsen Sağır, Cenap Duyar, and Oğuz Oğur

Department of Mathematics, Art and Science Faculty, Ondokuz Mayıs University, Kurupelit Campus, 55139 Samsun, Turkey

Correspondence should be addressed to Cenap Duyar; cenapd@omu.edu.tr

Received 1 November 2013; Revised 8 January 2014; Accepted 9 January 2014; Published 11 February 2014

Academic Editor: Shamsul Qamar

In this work we introduce new spaces $m^2(F, \phi, p)$ of double sequences defined by a double sequence of modulus functions, and we study some properties of this space.

1. Introduction

In this work, by w and w^2, we denote the spaces of single complex sequences and double complex sequences, respectively. $\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and complex numbers, respectively. If, for all $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $\|x_{k,l} - a\|_X < \varepsilon$ where $k > n_\varepsilon$ and $l > n_\varepsilon$, then a double sequence $\{x_{k,l}\}$ is said to be converge (in terms of Pringsheim) to $a \in \mathbb{C}$. A real double sequence $\{x_{k,l}\}$ is nondecreasing, if $x_{k,l} \le x_{p,q}$ for $(k,l) < (p,q)$. A double series is infinity sum $\sum_{k,l=1}^{\infty} x_{k,l}$ and its convergence implies the convergence by $|\cdot|$ of partial sums sequence $\{S_{n,m}\}$, where $S_{n,m} = \sum_{k=1}^{n}\sum_{l=1}^{m} x_{k,l}$ (see [1–3]).

For $1 \le p < \infty$, $\ell_p^{(2)}$ denote the space of sequences $x = \{x_{k,l}\}$ such that

$$\sum_{k,l=1}^{\infty} |x_{k,l}|^p < \infty. \tag{1}$$

(see [4]).

A double sequence $x = \{x_{k,l}\}$ is said to be bounded if and only if $\sup_{k,l}|x_{k,l}| < \infty$. The space of all bounded double sequences is denoted by $\ell_\infty^{(2)}$. It is known that $\ell_\infty^{(2)}$ is Banach space (see [5, 6]).

A double sequence space E is said to be normal if $(y_{kl}) \in E$ whenever $|y_{kl}| \le |x_{kl}|$ for all $k, l \in \mathbb{N}$ and $(x_{kl}) \in E$.

The double sequence spaces in the various forms were introduced and studies by Khan and Tabassum in [7–14], by Khan in [15], and by Khan et al. in [16, 17].

Now let φ_s be a family of subsets σ having most elements s in $\mathbb{N}$. Also $\varphi_{s,t}$ denote the class of subsets $\sigma = \sigma_1 \times \sigma_2$ in $\mathbb{N} \times \mathbb{N}$ such that the elements of σ_1 and σ_2 are most s and t, respectively. Besides $\{\phi_{k,l}\}$ is taken as a nondecreasing double sequence of the positive real numbers such that

$$k\phi_{k+1,l} \le (k+1)\,\phi_{k,l}, \qquad l\phi_{k,l+1} \le (l+1)\,\phi_{k,l}. \tag{2}$$

(see [18]).

Let $x = \{x_{k,l}\}$ be a double sequence. A set $S(x)$ is defined by

$$S(x) = \left\{\left\{x_{\pi_1(k),\pi_2(k)}\right\} : \pi_1 \text{ and } \pi_2 \text{ are permutations of } \mathbb{N}\right\}. \tag{3}$$

A double sequence space E is said to be symmetric if $u = (u_{kl}) \in E$ and $\|u\| = \|x\|$ whenever $x = (x_{kl}) \in E$ and $u \in S(x)$.

A BK-space is a Banach sequence space E in which the coordinate maps are continuous.

A function $f : [0,\infty) \to [0,\infty)$ is said to be a modulus function if it satisfies the following:

(1) $f(x) = 0$ if and only if $x = 0$;

(2) $f(x+y) \le f(x) + f(y)$ for all $x, y \in [0,\infty)$;

(3) f is increasing;

(4) f is continuous from right at 0.

It follows that f is continuous on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = x/(x+1)$, then $f(x)$ is bounded. But, for $0 < p < 1$, $f(x) = x^p$ is not bounded.

The BK-spaces $m(\phi)$, introduced by Sargent in [19], is in the form

$$m(\phi) = \left\{ x = \{x_k\} \in w : \|x\|_{m(\phi)} = \sup_{s\geq 1, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty \right\}. \quad (4)$$

Sargent studied some properties of this space and examined relationship between this space and l_p-space.

The space $m(\phi)$ was extended to $m(\phi, p)$ by Tripathy and Sen [20] as follows:

$$m(\phi, p) = \left\{ x = \{x_k\} \in w : \|x\|_{m(\phi,p)} = \sup_{s\geq 1, \sigma\in\varphi_s} \frac{1}{\phi_s} \left(\sum_{k\in\sigma} |x_k|^p \right)^{1/p} < \infty \right\}. \quad (5)$$

Recently, Raj et al. [21] introduced and studied the following sequence space $m(F, \phi, p)$.

Let $F = (f_k)$ be a sequence of modulus functions. Then the space $m(F, \phi, p)$ is defined by

$$m(F, \phi, p) = \left\{ x = \{x_k\} \in w : \sup_{s\geq 1, \sigma\in\varphi_s} \frac{1}{\phi_s} \left(\sum_{k\in\sigma} \left[f_k \left(\frac{|x_k|}{\rho} \right) \right]^p \right)^{1/p} < \infty, \text{for some } \rho > 0 \right\}. \quad (6)$$

In this work we introduce sequence spaces $m^2(F, \phi, p)$ defined by

$$m^2(F, \phi, p) = \left\{ x = (x_{kl}) \in w^2 : \sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} \left[f_{i,j} \left(\frac{|x_{i,j}|}{\rho} \right) \right]^p \right\}^{1/p} < \infty \text{ for } \rho > 0 \right\}, \quad (7)$$

where $F = (f_{i,j})$ is a double sequence of modulus functions.

2. Main Results

The result stated in the first theorem is not hard. So, we give it without proof.

Theorem 1. *The sequence space $m^2(F, \phi, p)$ is a linear space.*

Theorem 2. *The sequence spaces $m^2(F, \phi, p)$ are complete.*

Proof. Let $\{x^{(i)}\}$ be a double Cauchy sequence in $m^2(F, \phi, p)$ such that $x^{(i)} = \{x^{(i)}_{k,l}\}^{\infty}_{k,l=1}$ for all $i \in \mathbb{N}$. Then

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \left[f_{k,l} \left(\frac{|x^{(i)}_{k,l}|}{\rho} \right) \right]^p \right\}^{1/p} < \infty \quad (8)$$

for some $\rho > 0$ and for all $i \in \mathbb{N}$. For each $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left\| x^{(i)} - x^{(j)} \right\|_{m^2(F,\phi,p)} < \varepsilon \quad (9)$$

for all $i, j \geq n_0$. Hence

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \left[f_{k,l} \left(\frac{|x^{(i)}_{k,l} - x^{(j)}_{k,l}|}{\rho} \right) \right]^p \right\}^{1/p} < \varepsilon \quad (10)$$

for some $\rho > 0$ and for all $i, j \geq n_0$. This implies that

$$\left| x^{(i)}_{k,l} - x^{(j)}_{k,l} \right| < \varepsilon \phi_{1,1} \quad (11)$$

for all $i, j \geq n_0$ and for each fixed $(k, l) \in \mathbb{N} \times \mathbb{N}$. Hence $\{x^{(i)}\}$ is a Cauchy sequence in $\mathbb{C}$.

Then, there exists $x_{k,l} \in \mathbb{C}$ such that $x^{(i)}_{k,l} \to x_{k,l}$ as $i \to \infty$ and let us define $x = (x_{k,l})$. From (10), for each fixed (s, t),

$$\left\{ \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \left[f_{k,l} \left(\frac{|x^{(i)}_{k,l} - x^{(j)}_{k,l}|}{\rho} \right) \right]^p \right\} < \varepsilon^p \phi^p_{st} \quad (12)$$

for some $\rho > 0$, for all $i, j \geq n_0$ and $\sigma_1 \times \sigma_2 \in \varphi_{st}$.

Letting $j \to \infty$, we get

$$\left\{ \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \left[f_{k,l} \left(\frac{|x^{(i)}_{k,l} - x_{k,l}|}{\rho} \right) \right]^p \right\} < \varepsilon^p \phi^p_{st} \quad (13)$$

for some $\rho > 0$, for all $i, j \geq n_0$, and $\sigma_1 \times \sigma_2 \in \varphi_{st}$. Thus we obtain

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \left[f_{k,l} \left(\frac{|x^{(i)}_{k,l} - x_{k,l}|}{\rho} \right) \right]^p \right\}^{1/p} < \varepsilon \quad (14)$$

for some $\rho > 0$ and for all $i, j \geq n_0$. This implies that $x^{(i)} - x \in m^2(F, \phi, p)$ for all $i, j \geq n_0$.

Hence $x = x^{(n_0)} + x - x^{(n_0)} \in m^2(F,\phi,p)$. By (14),

$$\left\|x^{(i)} - x\right\|_{m^2(F,\phi,p)} < \varepsilon \tag{15}$$

for all $i \geq n_0$. This means that $x^{(i)} \to x$ as $i \to \infty$. Hence $m^2(F,\phi,p)$ is a Banach space. □

Theorem 3. *The space $m^2(F,\phi,p)$ is a BK-space.*

Proof. Suppose that $\{x^{(i)}\} \in m^2(F,\phi,p)$ with $\|x^{(i)} - x\|_{m^2(F,\phi,p)} \to 0$ as $i \to \infty$. For each $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\left\|x^{(i)} - x\right\|_{m^2(F,\phi,p)} < \varepsilon \tag{16}$$

for all $i \geq n_0$. Thus

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left[f_{k,l}\left(\frac{\left|x_{k,l}^{(i)} - x_{k,l}\right|}{\rho}\right)\right]^p \right\}^{1/p} < \varepsilon \tag{17}$$

for some $\rho > 0$ and for all $i \geq n_0$. Hence we obtain

$$\left|x_{k,l}^{(i)} - x_{k,l}\right| < \varepsilon\phi_{1,1} \tag{18}$$

for all $i \geq n_0$ and for all $(k,l) \in \mathbb{N}\times\mathbb{N}$. This implies $|x_{k,l}^{(i)} - x_{k,l}| \to 0$ as $i \to \infty$. This completes the proof. □

Corollary 4. *The space $m^2(F,\phi,p)$ is a symmetric space.*

Proof. Let $x = \{x_{k,l}\} \in m^2(F,\phi,p)$ and let $y = \{y_{k,l}\} \in S(x)$. Then we can write $y_{k,l} = x_{m_k,m_l}$. Thus we obtain

$$\|x\|_{m^2(F,\phi,p)} = \|y\|_{m^2(F,\phi,p)}. \tag{19}$$

□

Corollary 5. *The space $m^2(F,\phi,p)$ is a normal space.*

Proof. It is obvious. □

Theorem 6. *Consider*

$$m^2(\phi) \subseteq m^2(F,\phi,p). \tag{20}$$

Proof. Suppose that $x \in m^2(\phi)$. Then we have

$$\|x\|_{m^2(\phi)} = \sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} |x_{k,l}| \right\} = K < \infty. \tag{21}$$

Thus for each fixed (s,t) and for $\sigma_1\times\sigma_2 \in \varphi_{st}$,

$$\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} |x_{k,l}| \leq K\phi_{st} \tag{22}$$

for some $\rho > 0$. Hence

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left[f_{k,l}\left(\frac{|x_{k,l}|}{\rho}\right)\right]^p \right\}^{1/p} \leq K \tag{23}$$

for some $\rho > 0$. This implies that $x \in m^2(F,\phi,p)$. Hence $m^2(\phi) \subseteq m^2(F,\phi,p)$. □

Theorem 7. *$m^2(F,\phi,p) \subseteq m^2(F,\psi,p)$ if and only if $\sup_{(s,t)\geq(1,1)}(\phi_{st}/\psi_{st}) < \infty$.*

Proof. Let $K = \sup_{(s,t)\geq(1,1)}(\phi_{st}/\psi_{st}) < \infty$. Then $\phi_{st} \leq K\psi_{st}$ for all $(s,t) \geq (1,1)$. If $x \in m^2(F,\phi,p)$, then

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left[f_{k,l}\left(\frac{|x_{k,l}|}{\rho}\right)\right]^p \right\}^{1/p} < \infty \tag{24}$$

for some $\rho > 0$. Thus

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{K\psi_{st}} \left\{ \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left[f_{k,l}\left(\frac{|x_{k,l}|}{\rho}\right)\right]^p \right\}^{1/p} < \infty \tag{25}$$

for some $\rho > 0$. Hence $x \in m^2(F,\psi,p)$. This shows that $m^2(F,\phi,p) \subseteq m^2(F,\psi,p)$. Conversely, let $m^2(F,\phi,p) \subseteq m^2(F,\psi,p)$. We define $\gamma_{s,t} = \phi_{st}/\psi_{st}$. Let $\sup_{(s,t)\geq(1,1)}\gamma_{s,t} = \infty$. Then there exists a subsequence $\{\gamma_{s_i,t_i}\}$ of $\{\gamma_{s,t}\}$ such that $\gamma_{s_i,t_i} \to \infty$ as $i \to \infty$. Then for $x \in m^2(F,\phi,p)$ we have

$$\begin{aligned} &\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\psi_{st}} \left\{ \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left[f_{k,l}\left(\frac{|x_{k,l}|}{\rho}\right)\right]^p \right\}^{1/p} \\ &\quad\geq \sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{\gamma_{s,t}}{\phi_{st}} \left\{ \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left[f_{k,l}\left(\frac{|x_{k,l}|}{\rho}\right)\right]^p \right\}^{1/p} \\ &\quad= \infty \end{aligned} \tag{26}$$

for some $\rho > 0$. This is a contradiction as $x \notin m^2(F,\psi,p)$ and this completes the proof. □

Proposition 8. *Consider*

$$\ell_p^{(2)} \subseteq m^2(F,\phi,p) \subseteq \ell_\infty^{(2)}. \tag{27}$$

Proof. Clearly, $\ell_p^{(2)} = m^2(F,\psi,p)$, where $\psi_{st} = 1$ for $s,t = 1,2,\ldots$ when $f_{k,l}(x) = x$ and $\sup_{(s,t)\geq(1,1)}(\psi_{st}/\phi_{st}) < \infty$ by nondecreasing (ϕ_{st}). Then, by Theorem 7, first inclusion is obtained. Suppose $x \in m^2(F, \phi, p)$. Then

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \left\{ \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \left[f_{k,l} \left(\frac{|x_{k,l}|}{\rho} \right) \right]^p \right\}^{1/p} = K < \infty \tag{28}$$

for some $\rho > 0$. Hence we obtain

$$|x_{kl}| \leq K\phi_{1,1} \tag{29}$$

for all $k, l \in \mathbb{N}$. Thus $x \in \ell_\infty^{(2)}$ and proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] T. Apostol, *Mathematical Analysis*, Addison-Wesley, London, UK, 1978.

[2] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, e-books, Monographs, İstanbul, Turkey, 2012.

[3] B. V. Limaye and M. Zeltser, "On the Pringsheim convergence of double series," *Proceedings of the Estonian Academy of Sciences*, vol. 58, no. 2, pp. 108–121, 2009.

[4] F. Başar and Y. Sever, "The space L_q of double sequences," *Mathematical Journal of Okayama University*, vol. 51, pp. 149–157, 2009.

[5] F. Móricz and B. E. Rhoades, "Almost convergence of double sequences and strong regularity of summability matrices," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 104, no. 2, pp. 283–294, 1988.

[6] F. Móricz, "Extensions of the spaces c and c_0 from single to double sequences," *Acta Mathematica Hungarica*, vol. 57, no. 1-2, pp. 129–136, 1991.

[7] V. A. Khan and S. Tabassum, "Statistically convergent double sequence spaces in 2-normed spaces defined by Orlicz function," *Applied Mathematics*, vol. 2, no. 4, pp. 398–402, 2011.

[8] V. A. Khan and S. Tabassum, "On some new quasi almost Δ^m-lacunary strongly P-convergent double sequences defined by Orlicz functions," *Journal of Mathematics and Applications*, vol. 34, pp. 45–52, 2011.

[9] V. A. Khan and S. Tabassum, "On ideal convergent difference double sequence spaces in 2-normed spaces defined by Orlicz function," *International Journal of Mathematics and Applications*, vol. 1, no. 2, pp. 26–34, 2011.

[10] V. A. Khan and S. Tabassum, "Some vector valued multiplier difference double sequence spaces in 2-normed spaces defined by Orlicz functions," *Journal of Mathematical and Computational Science*, vol. 1, no. 1, pp. 126–139, 2011.

[11] V. A. Khan and S. Tabassum, "On some new double sequence spaces of invariant means defined by Orlicz functions," *Communications de la Faculté des Sciences de l'Université d'Ankara. Séries A1*, vol. 60, no. 2, pp. 11–21, 2011.

[12] V. A. Khan and S. Tabassum, "The strongly summable generalized difference double sequence spaces in 2-normed spaces defined by Orlicz functions," *Journal of Mathematical Notes*, vol. 7, no. 2, pp. 45–58, 2011.

[13] V. A. Khan and S. Tabassum, "On some new almost double lacunary Δ^m-sequence spaces defined by Orlicz functions," *Journal of Mathematical Notes*, vol. 6, no. 2, pp. 80–94, 2011.

[14] V. A. Khan and S. Tabassum, "Statistically pre-Cauchy double sequences," *Southeast Asian Bulletin of Mathematics*, vol. 36, no. 2, pp. 249–254, 2012.

[15] V. A. Khan, "On Δ_ν^m-Cesáro summable double sequences," *Thai Journal of Mathematics*, vol. 10, no. 3, pp. 535–539, 2012.

[16] V. A. Khan, S. Tabassum, and A. Esi, "Statistically convergent double sequence spaces in n-normed spaces," *Journal of Science and Technology*, vol. 2, no. 10, pp. 991–995, 2012.

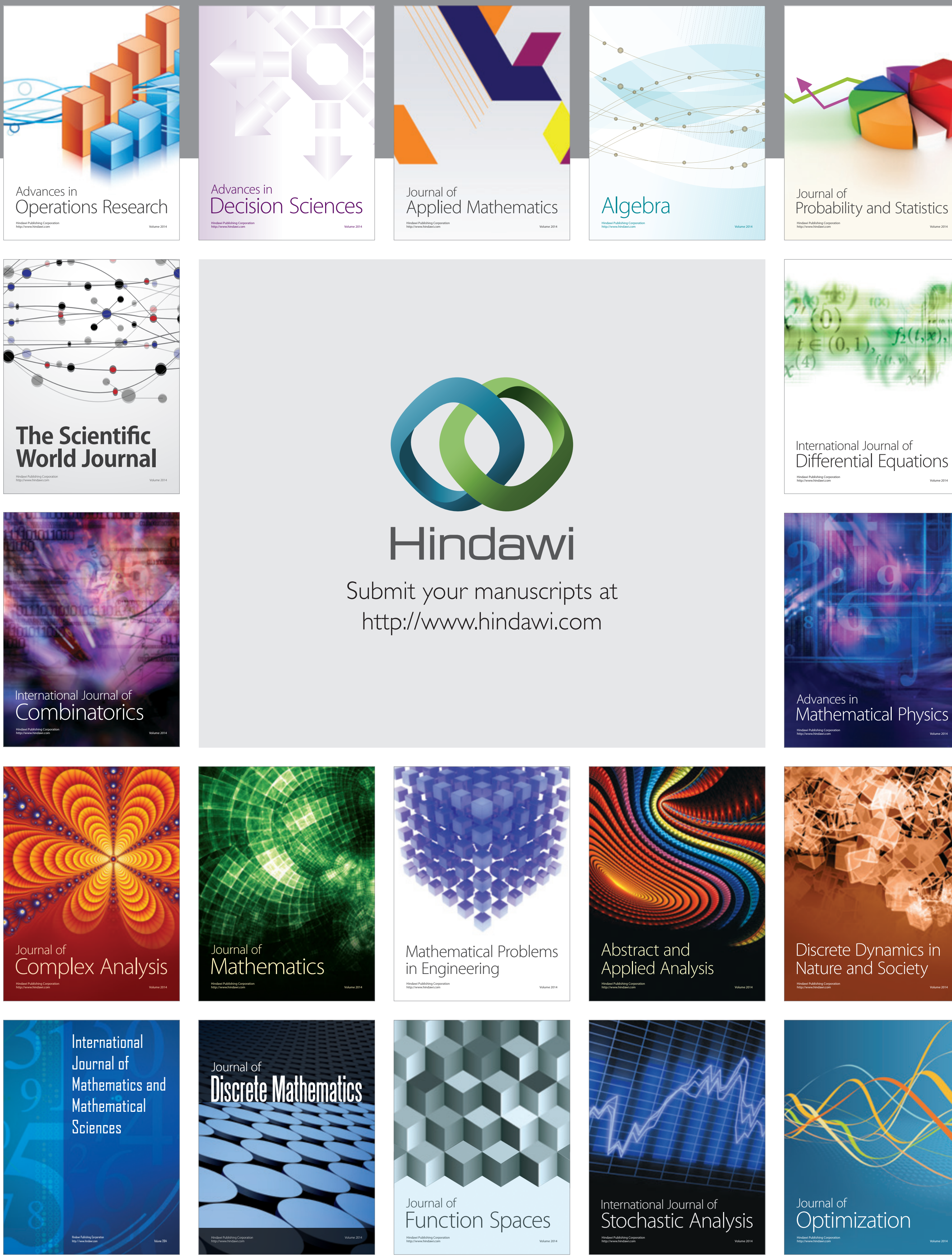
[17] V. A. Khan, S. Tabassum, and A. Esi, "A_σ double sequence spaces and statistical convergence in 2-normed spaces dened by Orlicz functions," *Theory and Applications of Mathematics and Computer Science*, vol. 2, no. 1, pp. 61–71, 2012.

[18] C. Duyar and O. Oğur, "On a new space $m^2(M, A, \varphi, p)$ of double sequences," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 509613, 8 pages, 2013.

[19] W. L. C. Sargent, "Some sequence spaces related to the l^p spaces," *Journal of the London Mathematical Society*, vol. 35, pp. 161–171, 1960.

[20] B. C. Tripathy and M. Sen, "On a new class of sequences related to the space l^p," *Tamkang Journal of Mathematics*, vol. 33, no. 2, pp. 167–171, 2002.

[21] K. Raj, S. K. Sharma, A. Gupta, and A. Kumar, "A sequence space defined by a sequence of modulus functions," *International Journal of Mathematical Analysis*, vol. 5, no. 29-32, pp. 1569–1574, 2011.



Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization