Research Article

Initial Coefficient Bounds for a General Class of Biunivalent Functions

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We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $B(\lambda, \mu, \phi)$. The results presented in this paper improve or generalize the recent work of Srutha Keerthi and Raja (2013).

1. Introduction and Definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\} \quad (1)$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (2)$$

and let S be the class of all functions from A which are univalent in U .

The Koebe one-quarter theorem [1] states that the image of U under every function f from S contains a disk of radius $1/4$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U), \quad (3)$$

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (4)$$

A function $f(z) \in A$ is said to be biunivalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U .

If the functions f and g are analytic in U , then f is said to be subordinate to g , written as $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$.

Let Σ denote the class of biunivalent functions defined in the unit disk U . For a brief history and interesting examples in the class Σ , (see [2]).

Lewin [3] studied the class of biunivalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu [5] showed that $\max |a_2| = 4/3$ if $f(z) \in \Sigma$.

Brannan and Taha [6] introduced certain subclasses of the biunivalent function class Σ similar to the familiar subclasses, $\delta^*(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α ($0 < \alpha \leq 1$), respectively (see [5]). Thus, following Brannan and Taha [6], a function $f \in A$ is in the class $K_{\Sigma}(\alpha)$ of strongly biconvex functions of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left| \arg \left(\frac{z^2 f''(z) + z f'(z)}{z f'(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, z \in U),$$

$$\left| \arg \left(\frac{w^2 g''(w) + w g'(w)}{w g'(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, w \in U), \quad (5)$$

where g is the extension of f^{-1} to U . The classes $\delta_{\Sigma}^*(\alpha)$ and $K_{\Sigma}(\alpha)$ of bistarlike functions of order α and biconvex functions of order α , corresponding to the function classes $\delta^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $\delta_{\Sigma}^*(\alpha)$ and $K_{\Sigma}(\alpha)$, they found nonsharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of biunivalent functions ([2, 7, 8]). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, \dots\}$ is presumably still an open problem.

In this paper, by using the method [9] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of biunivalent functions considered by Srutha Keerthi and Raja and get more accurate estimates than that given in [10].

2. Coefficient Estimates

In the following, let ϕ be an analytic function with positive real part in U , with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \quad (B_1 > 0). \quad (6)$$

Suppose that $u(z)$, and $v(z)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n \quad (|z| < 1). \quad (7)$$

It is well known that

$$\begin{aligned} |b_1| &\leq 1, & |b_2| &\leq 1 - |b_1|^2, \\ |c_1| &\leq 1, & |c_2| &\leq 1 - |c_1|^2. \end{aligned} \quad (8)$$

Next, (6) and (7) lead to

$$\begin{aligned} \phi(u(z)) &= 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \dots, \quad |z| < 1, \\ \phi(v(w)) &= 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + \dots, \quad |w| < 1. \end{aligned} \quad (9)$$

Definition 1. A function $f \in \Sigma$ is said to be in the class $B(\lambda, \mu, \phi)$, $\lambda \geq 0$, $0 \leq \mu \leq \lambda \leq 1$, if the following subordinations hold:

$$\begin{aligned} \frac{\lambda \mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu) z^2 f''(z) + z f'(z)}{\lambda \mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)} &< \phi(z), \\ \frac{\lambda \mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu) w^2 g''(w) + w g'(w)}{\lambda \mu w^2 g''(w) + (\lambda - \mu) w g'(w) + (1 - \lambda + \mu) g(w)} &< \phi(w), \end{aligned} \quad (10)$$

where $g(w) = f^{-1}(w)$.

Theorem 2. Let f given by (2) be in the class $B(\lambda, \mu, \phi)$. Then

$$\begin{aligned} |a_2| &\leq B_1 \sqrt{B_1} \\ &\times \left(\left| 2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 \right. \right. \\ &\quad \left. \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2) \right| \right. \\ &\quad \left. + (1 + \lambda - \mu + 2\lambda\mu)^2 B_1 \right)^{-1/2}, \end{aligned} \quad (11)$$

$$|a_3| \leq \begin{cases} \frac{B_1}{2(1 + 2\lambda - 2\mu + 6\lambda\mu)}, \\ \text{if } B_1 \leq \frac{(1 + \lambda - \mu + 2\lambda\mu)^2}{2(1 + 2\lambda - 2\mu + 6\lambda\mu)}, \\ \left(\left| 2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 \right. \right. \\ \quad \left. \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2) \right| B_1 \right. \\ \quad \left. + 2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^3 \right) \\ \quad \times \left(\left| 2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 \right. \right. \\ \quad \left. \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2) \right| \right. \\ \quad \left. + (1 + \lambda - \mu + 2\lambda\mu)^2 B_1 \right) \\ \quad \times (1 + 2\lambda - 2\mu + 6\lambda\mu)^{-1}, \\ \text{if } B_1 > \frac{(1 + \lambda - \mu + 2\lambda\mu)^2}{2(1 + 2\lambda - 2\mu + 6\lambda\mu)}. \end{cases} \quad (12)$$

Proof. Let $f \in B(\lambda, \mu, \phi)$, $\lambda \geq 0$, and $0 \leq \mu \leq \lambda \leq 1$. Then there are analytic functions $u, v : U \rightarrow U$ given by (7) such that

$$\begin{aligned} \frac{\lambda \mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu) z^2 f''(z) + z f'(z)}{\lambda \mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)} \\ = \phi(u(z)), \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\lambda \mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu) w^2 g''(w) + w g'(w)}{\lambda \mu w^2 g''(w) + (\lambda - \mu) w g'(w) + (1 - \lambda + \mu) g(w)} \\ = \phi(v(w)), \end{aligned}$$

where $g(w) = f^{-1}(w)$. Since

$$\begin{aligned} \frac{\lambda \mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu) z^2 f''(z) + z f'(z)}{\lambda \mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)} \\ = 1 + (1 + \lambda - \mu + 2\lambda\mu) a_2 z \\ + \left[2(1 + 2\lambda - 2\mu + 6\lambda\mu) a_3 \right. \\ \quad \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 a_2^2 \right] z^2 + \dots, \\ \frac{\lambda \mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu) w^2 g''(w) + w g'(w)}{\lambda \mu w^2 g''(w) + (\lambda - \mu) w g'(w) + (1 - \lambda + \mu) g(w)} \\ = 1 - (1 + \lambda - \mu + 2\lambda\mu) a_2 w \\ + \left[2(1 + 2\lambda - 2\mu + 6\lambda\mu) (2a_2^2 - a_3) \right. \\ \quad \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 a_2^2 \right] w^2 + \dots, \end{aligned} \quad (14)$$

it follows from (9) and (13) that

$$(1 + \lambda - \mu + 2\lambda\mu) a_2 = B_1 b_1, \quad (15)$$

$$\begin{aligned} 2(1 + 2\lambda - 2\mu + 6\lambda\mu) a_3 - (1 + \lambda - \mu + 2\lambda\mu)^2 a_2^2 \\ = B_1 b_2 + B_2 b_1^2, \end{aligned} \quad (16)$$

$$-(1 + \lambda - \mu + 2\lambda\mu) a_2 = B_1 c_1, \quad (17)$$

$$\begin{aligned} 2(1 + 2\lambda - 2\mu + 6\lambda\mu) (2a_2^2 - a_3) - (1 + \lambda - \mu + 2\lambda\mu)^2 a_2^2 \\ = B_1 c_2 + B_2 c_1^2. \end{aligned} \quad (18)$$

From (15) and (17) we obtain

$$c_1 = -b_1. \quad (19)$$

By adding (18) to (16), further computations using (15) to (19) lead to

$$\begin{aligned} [4(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 \\ - 2(1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2)] a_2^2 = B_1^3 (b_2 + c_2). \end{aligned} \quad (20)$$

Equations (19) and (20), together with (8), give

$$\begin{aligned} |2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2)| \\ \times |a_2|^2 \leq B_1^3 (1 - |b_1|^2). \end{aligned} \quad (21)$$

From (15) and (21) we get

$$\begin{aligned} |a_2| \leq B_1 \sqrt{B_1} \\ \times \left([2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 \right. \\ \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2)] \right. \\ \left. + (1 + \lambda - \mu + 2\lambda\mu)^2 B_1 \right)^{-1/2}. \end{aligned} \quad (22)$$

Next, in order to find the bound on $|a_3|$, by subtracting (18) from (16), we obtain

$$\begin{aligned} 4(1 + 2\lambda - 2\mu + 6\lambda\mu) a_3 - 4(1 + 2\lambda - 2\mu + 6\lambda\mu) a_2^2 \\ = B_1 (b_2 - c_2). \end{aligned} \quad (23)$$

Then, in view of (8) and (19), we have

$$\begin{aligned} 2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1 |a_3| \\ \leq [2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1 \\ - (1 + \lambda - \mu + 2\lambda\mu)^2] |a_2|^2 + B_1^2. \end{aligned} \quad (24)$$

Notice that (11), we get

$$|a_3| \leq \begin{cases} \frac{B_1}{2(1 + 2\lambda - 2\mu + 6\lambda\mu)}, \\ \text{if } B_1 \leq \frac{(1 + \lambda - \mu + 2\lambda\mu)^2}{2(1 + 2\lambda - 2\mu + 6\lambda\mu)}, \\ \left([2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 \right. \\ \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2)] B_1 \right. \\ \left. + 2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^3 \right) \\ \times \left(2 [2(1 + 2\lambda - 2\mu + 6\lambda\mu) B_1^2 \right. \\ \left. - (1 + \lambda - \mu + 2\lambda\mu)^2 (B_1^2 + B_2)] \right. \\ \left. + (1 + \lambda - \mu + 2\lambda\mu)^2 B_1 \right) \\ \times (1 + 2\lambda - 2\mu + 6\lambda\mu)^{-1}, \\ \text{if } B_1 > \frac{(1 + \lambda - \mu + 2\lambda\mu)^2}{2(1 + 2\lambda - 2\mu + 6\lambda\mu)}. \end{cases} \quad (25)$$

Remark 3. If we let

$$\phi(z) = \left(\frac{1+z}{1-z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1), \quad (26)$$

then inequalities (11) and (12) become

$$\begin{aligned} |a_2| \leq 2\alpha \\ \times \left([4(1 + 2\lambda - 2\mu + 6\lambda\mu) \right. \\ \left. - 3(1 + \lambda - \mu + 2\lambda\mu)^2] \alpha \right. \\ \left. + (1 + \lambda - \mu + 2\lambda\mu)^2 \right)^{-1/2}, \end{aligned}$$

$$|a_3| \leq \begin{cases} \frac{\alpha}{1 + 2\lambda - 2\mu + 6\lambda\mu}, \\ \text{if } 0 < \alpha \leq \frac{(1 + \lambda - \mu + 2\lambda\mu)^2}{4(1 + 2\lambda - 2\mu + 6\lambda\mu)}, \\ \left([4(1 + 2\lambda - 2\mu + 6\lambda\mu) \right. \\ \left. - 3(1 + \lambda - \mu + 2\lambda\mu)^2] \alpha^2 \right) \\ \times \left([4(1 + 2\lambda - 2\mu + 6\lambda\mu) \right. \\ \left. - 3(1 + \lambda - \mu + 2\lambda\mu)^2] \alpha \right. \\ \left. + (1 + \lambda - \mu + 2\lambda\mu)^2 \right) \\ \times (1 + 2\lambda - 2\mu + 6\lambda\mu)^{-1}, \\ \text{if } \frac{(1 + \lambda - \mu + 2\lambda\mu)^2}{4(1 + 2\lambda - 2\mu + 6\lambda\mu)} < \alpha \leq 1. \end{cases} \quad (27)$$

The bounds on $|a_2|$ and $|a_3|$ given by (27) are more accurate than those given in Theorem 2.2 in [10].

Remark 4. If we let

$$\begin{aligned} \phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)z + 2(1 - \alpha)^2 z^2 + \dots \\ (0 < \alpha \leq 1), \end{aligned} \quad (28)$$

then inequalities (11) and (12) become

$$\begin{aligned}
 |a_2| &\leq 2(1-\alpha) \\
 &\quad \times \left(\left| 4(1-\alpha)(1+2\lambda-2\mu+6\lambda\mu) \right. \right. \\
 &\quad \left. \left. - (3-2\alpha)(1+\lambda-\mu+2\lambda\mu)^2 \right| \right. \\
 &\quad \left. + (1+\lambda-\mu+2\lambda\mu)^2 \right)^{-1/2}, \\
 |a_3| &\leq \begin{cases} \frac{1-\alpha}{1+2\lambda-2\mu+6\lambda\mu}, \\ \text{if } \frac{4(1+2\lambda-2\mu+6\lambda\mu) - (1+\lambda-\mu+2\lambda\mu)^2}{4(1+2\lambda-2\mu+6\lambda\mu)} \\ \leq \alpha < 1, \\ \left(\left[4(1-\alpha)(1+2\lambda-2\mu+6\lambda\mu) \right. \right. \\ \left. \left. - (3-2\alpha)(1+\lambda-\mu+2\lambda\mu)^2 \right] \right) \\ \times (1-\alpha) + 4(1-\alpha)^2(1+2\lambda-2\mu+6\lambda\mu) \\ \times \left(\left[\left| 4(1-\alpha)(1+2\lambda-2\mu+6\lambda\mu) \right. \right. \right. \\ \left. \left. - (3-2\alpha)(1+\lambda-\mu+2\lambda\mu)^2 \right| \right. \\ \left. \left. + (1+\lambda-\mu+2\lambda\mu)^2 \right] \right) \\ \times (1+2\lambda-2\mu+6\lambda\mu) \right)^{-1}, \\ \text{if } 0 \leq \alpha \\ < \frac{4(1+2\lambda-2\mu+6\lambda\mu) - (1+\lambda-\mu+2\lambda\mu)^2}{4(1+2\lambda-2\mu+6\lambda\mu)}. \end{cases} \quad (29)
 \end{aligned}$$

The bounds on $|a_2|$ and $|a_3|$ given by (29) are more accurate than those given in Theorem 3.3 in [10]. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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