Research Article

Geometric Properties of a Class of Analytic Functions Defined by a Differential Inequality

Manpreet Kaur, Sushma Gupta, and Sukhjit Singh

Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, Punjab 148106, India

Correspondence should be addressed to Manpreet Kaur; m.ahluwalia10@gmail.com

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Let $\mathcal{A}$ be the class of analytic functions f defined in the open unit disk E and normalized by $f(0) = f'(0) - 1 = 0$. For $f(z)/z \neq 0$ in E , let $\mathcal{M}(\alpha, \lambda) := \{f \in \mathcal{A} : |-\alpha z^2(z/f(z))'' + f'(z)(z/f(z))^2 - 1| \leq \lambda, z \in E\}$, where $\lambda > 0$ and $\alpha \in \mathbb{R} \setminus [-1/2, 0]$. In the present paper, we find conditions under which functions in the class $\mathcal{M}(\alpha, \lambda)$ are starlike of order γ , $0 \leq \gamma < 1$.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions f that are analytic in the open unit disk $E = \{z : |z| < 1\}$ and are normalized by the conditions $f(0) = f'(0) - 1 = 0$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are univalent in E and denote by $\mathcal{S}^*(\gamma)$, $0 \leq \gamma < 1$, the class of functions in $\mathcal{S}$ which are starlike of order γ in E . Analytically,

$$\mathcal{S}^*(\gamma) = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \gamma, z \in E \right\}. \quad (1)$$

Note that $\mathcal{S}^*(0)$ is the usual class of starlike (with respect to the origin) functions in E and we denote it simply by $\mathcal{S}^*$. We say that $f \in R(\beta)$, $0 < \beta \leq 1$, if $f \in \mathcal{A}$ and $|\arg f'(z)| < \beta\pi/2$, for all $z \in E$. It is well known that the functions in $R(1)$ are close-to-convex and hence univalent in E [1, 2].

In 1972, Ozaki and Nunokawa [3] studied the class $\mathcal{U}(\lambda)$, which is defined as

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, z \in E \right\}, \quad (2)$$

where $f(z)/z \neq 0$ in E . They proved that $\mathcal{U}(\lambda) \subset \mathcal{S}$ for $0 \leq \lambda \leq 1$. Several researchers studied this class (e.g., see [4–8]) and obtained many significant results.

Recently, Obradovic and Ponnusamy [9] investigated another class

$$\mathcal{M}(\lambda) = \left\{ f \in \mathcal{A} : \left| z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, z \in E \right\}, \quad (3)$$

where $\lambda > 0$ and $f(z)/z \neq 0$ in E . They obtained certain inclusion relations, characterization formula, and coefficient conditions. They also posed a question about the starlikeness of the functions in the class $\mathcal{M}(\lambda)$. The purpose of the present paper is to answer this question. In fact, we study a more general class

$$\mathcal{M}(\alpha, \lambda) = \left\{ f \in \mathcal{A} : \left| -\alpha z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, z \in E \right\}, \quad (4)$$

where $f(z)/z \neq 0$ in E , $\lambda > 0$, and $\alpha \in \mathbb{R} \setminus [-1/2, 0]$. We find conditions on α , λ , and f involved in the class $\mathcal{M}(\alpha, \lambda)$ under which members of $\mathcal{M}(\alpha, \lambda)$ are starlike of a given order γ , $0 \leq \gamma < 1$. We remark that the class $\mathcal{M}(\alpha, \lambda)$ follows essentially from the class $S(\alpha, \beta, \lambda)$ studied by Baricz and Ponnusamy [10] by taking $\beta = 0$; however, we will study those issues for the class $\mathcal{M}(\alpha, \lambda)$ which are not studied by the authors for $S(\alpha, \beta, \lambda)$.

2. Main Results

Let $\mathcal{A}_n$ denote the class of analytic functions p in E such that $p^{(k)}(0) = 0$ for $k = 0, 1, 2, \dots, n$, where $p^{(0)}(0) = p(0)$. With $w^{(0)}(0) = w(0)$, we set $\mathcal{B}_n = \{w: w \text{ is analytic, } |w(z)| \leq 1 \text{ in } E \text{ and } w^{(k)}(0) = 0 \text{ for } k = 0, 1, 2, \dots, n\}$. Functions in $\mathcal{B}_n$ are called Schwarz functions. Obviously, $w \in \mathcal{B}_k$ implies that $|w(z)| \leq |z|^{k+1}$ in E , for $k = 0, 1, 2, 3, \dots, n$.

We begin with the following result.

Lemma 1. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be in $\mathcal{M}(\alpha, \lambda)$. Then, $\operatorname{Re}[f(z)/z] \geq 1/2$ in E whenever $|a_2| \leq 1 - \lambda/|2\alpha + 1|$.

Proof. As $f \in \mathcal{M}(\alpha, \lambda)$, there exists a Schwarz function $w \in \mathcal{B}_1$ such that

$$-\alpha z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 = \lambda w(z). \quad (5)$$

Since $w \in \mathcal{B}_1$, $w(0) = w'(0) = 0$. If we set

$$\begin{aligned} p(z) &= f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \\ &= \left(\frac{z}{f(z)} \right) - z \left(\frac{z}{f(z)} \right)' - 1, \end{aligned} \quad (6)$$

where p is an analytic function in E with $p(0) = p'(0) = 0$, then (5) is equivalent to

$$\alpha z p'(z) + p(z) = \lambda w(z), \quad (7)$$

from which we get

$$p(z) = \frac{\lambda}{\alpha} \int_0^1 w(tz) t^{1/\alpha-1} dt, \quad (8)$$

or, equivalently,

$$\begin{aligned} -z \left(\frac{z}{f(z)} - 1 + a_2 z \right)' + \left(\frac{z}{f(z)} - 1 + a_2 z \right) \\ = \frac{\lambda}{\alpha} \int_0^1 w(tz) t^{1/\alpha-1} dt. \end{aligned} \quad (9)$$

Solving this equation for $z/f(z)$, we obtain

$$\frac{z}{f(z)} = 1 - a_2 z - \frac{\lambda}{\alpha} \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt. \quad (10)$$

Now, using the fact that $|w(stz)| \leq |(stz)|^2$, it follows that

$$\begin{aligned} \left| \frac{z}{f(z)} - 1 \right| &\leq |a_2| |z| + \frac{\lambda}{|\alpha|} \int_0^1 \int_0^1 t^{1/\alpha+1} |z|^2 ds dt, \\ &z \in E, \quad (11) \\ &\leq |z| \left(|a_2| + \frac{\lambda |z|}{|2\alpha + 1|} \right), \quad z \in E. \end{aligned}$$

The inequality (11) is equivalent to

$$\begin{aligned} \left| \frac{f(z)}{z} - \frac{1}{1 - |z|^2 (|a_2| + \lambda |z| / |2\alpha + 1|)} \right| \\ \leq \frac{|z| (|a_2| + \lambda |z| / |2\alpha + 1|)}{1 - |z|^2 (|a_2| + \lambda |z| / |2\alpha + 1|)^2}, \end{aligned} \quad (12)$$

which gives

$$\begin{aligned} \operatorname{Re} \left(\frac{f(z)}{z} \right) &\geq \frac{1}{1 + |z| (|a_2| + \lambda |z| / |2\alpha + 1|)} \\ &> \frac{1}{1 + (|a_2| + \lambda / |2\alpha + 1|)} \geq \frac{1}{2}, \end{aligned} \quad (13)$$

for all $z \in E$ whenever $|a_2| \leq 1 - \lambda/|2\alpha + 1|$. $\square$

In the next result, we find the range of values of λ for which $f \in \mathcal{M}(\alpha, \lambda)$ implies that $f \in R(\beta)$, $0 < \beta \leq 1$.

Theorem 2. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be in $\mathcal{M}(\alpha, \lambda)$. Then, $f \in R(\beta)$ for $0 < \lambda \leq \lambda_\beta$, where λ_β satisfies the inequality

$$\begin{aligned} \sqrt{1 - \frac{\lambda^2}{|2\alpha + 1|^2}} \sin \frac{\beta\pi}{2} \\ - 2 \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right) \sqrt{1 - \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right)^2} \\ \geq \frac{\lambda}{|2\alpha + 1|} \cos \frac{\beta\pi}{2}. \end{aligned} \quad (14)$$

Proof. From (8), we get

$$|p(z)| < \frac{\lambda}{|2\alpha + 1|}, \quad z \in E, \quad (15)$$

where

$$p(z) = f'(z) \left(\frac{z}{f(z)} \right)^2 - 1. \quad (16)$$

Therefore,

$$\left| \arg f'(z) \left(\frac{z}{f(z)} \right)^2 \right| \leq \arcsin \left(\frac{\lambda}{|2\alpha + 1|} \right), \quad z \in E. \quad (17)$$

Also, in view of (11), we obtain

$$\left| \arg \left(\frac{z}{f(z)} \right) \right| \leq \arcsin \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right), \quad z \in E. \quad (18)$$

Therefore,

$$\begin{aligned} |\arg f'(z)| &\leq \left| \arg f'(z) \left(\frac{z}{f(z)} \right)^2 \right| + 2 \left| \arg \left(\frac{z}{f(z)} \right) \right| \\ &\leq \arcsin \left(\frac{\lambda}{|2\alpha + 1|} \right) + 2 \arcsin \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right) \\ &= \arcsin \left(\frac{\lambda}{|2\alpha + 1|} \right) + \arcsin \left(2 \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right) \right) \\ &\quad \cdot \sqrt{1 - \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right)^2}. \end{aligned} \quad (19)$$

Now, the desired result follows, if

$$\begin{aligned} &\arcsin \left(\frac{\lambda}{|2\alpha + 1|} \right) + \arcsin \left(2 \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right) \right) \\ &\quad \cdot \sqrt{1 - \left(|a_2| + \frac{\lambda}{|2\alpha + 1|} \right)^2} \leq \frac{\beta\pi}{2}. \end{aligned} \quad (20)$$

By using $\arcsin(x) + \arcsin(y) = \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2})$, $x, y \in [-1, 1]$, and $x^2 + y^2 \leq 1$ and carrying out some simplifications, we conclude that (20) is equivalent to (14). $\square$

If $a_2 = 0$ and $\beta = 1$, then Theorem 2 gives the following.

Corollary 3. *If $f(z) = z + a_3 z^3 + \dots$ is in $\mathcal{M}(\alpha, \lambda)$, and then $\operatorname{Re} f'(z) > 0$ in E , whenever $0 < \lambda \leq |2\alpha + 1|/2$.*

Setting $a_2 = 0$, $\beta = 1$, and $\alpha = -1$ in Theorem 2, we obtain the following.

Corollary 4. *If $f(z) = z + a_3 z^3 + \dots$ is in $\mathcal{M}(-1, \lambda)$, and then $\operatorname{Re} f'(z) > 0$ in E , whenever $0 < \lambda \leq 1/2$.*

In the following theorem, we find conditions under which functions in the class $\mathcal{M}(\alpha, \lambda)$ belong to $\mathcal{S}^*(\gamma)$, $0 \leq \gamma < 1$.

Theorem 5. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be in $\mathcal{M}(\alpha, \lambda)$ and let $f''(0) = 0$. Then, for $0 \leq \gamma < 1$, $f \in \mathcal{S}^*(\gamma)$ provided $0 < \lambda \leq (1 - \gamma)|2\alpha + 1|/(\sqrt{2} + \gamma)$.*

Proof. As $a_2 = f''(0)/2 = 0$, from (8) and (10), we get

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{f'(z)(z/f(z))^2}{z/f(z)} \\ &= \frac{1 + (\lambda/\alpha) \int_0^1 w(tz) t^{1/\alpha-1} dt}{1 - (\lambda/\alpha) \int_0^1 \int_0^1 (w(stz)/s^2) t^{1/\alpha-1} ds dt}. \end{aligned} \quad (21)$$

Now, $\operatorname{Re}(zf'(z)/f(z)) > \gamma$ is equivalent to

$$\frac{zf'(z)/f(z) - \gamma}{1 - \gamma} \neq iT, \quad T \in \mathbb{R}. \quad (22)$$

Using (21) in (22), we get

$$\begin{aligned} &\frac{\lambda}{\alpha} \int_0^1 w(tz) t^{1/\alpha-1} dt \\ &\quad - \gamma(1 - iT) \left(1 - \frac{\lambda}{\alpha} \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right) \\ &\quad + \frac{\lambda}{\alpha} iT \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \neq -(1 - iT), \end{aligned} \quad (23)$$

or, equivalently,

$$\begin{aligned} &\frac{\lambda}{2\alpha} \left[\int_0^1 w(tz) t^{1/\alpha-1} dt - \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right] \\ &\quad - \gamma \left[1 - \frac{\lambda}{\alpha} \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right] + \frac{\lambda}{2\alpha} \\ &\quad \cdot \frac{1 + iT}{1 - iT} \left[\int_0^1 w(tz) t^{1/\alpha-1} dt \right. \\ &\quad \left. + \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right] \neq -1. \end{aligned} \quad (24)$$

If we denote the left-hand side of (24) by $H(w, T, z)$ and let

$$M = \sup_{T \in \mathbb{R}, w \in \mathcal{B}_1, z \in E} |H(w, T, z)|, \quad (25)$$

then, in view of the rotation invariance property of the set $\mathcal{B}_1$, we obtain that (22) holds if $M \leq 1$.

A simple calculation gives

$$\begin{aligned} M &\leq \sup_{w \in \mathcal{B}_1, z \in E} \frac{\lambda}{2|\alpha|} \left| \int_0^1 w(tz) t^{1/\alpha-1} dt \right. \\ &\quad \left. - \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right| \\ &\quad + \frac{\lambda}{2|\alpha|} \left| \int_0^1 w(tz) t^{1/\alpha-1} dt \right. \\ &\quad \left. + \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right| + \gamma \left| 1 - \frac{\lambda}{\alpha} \right. \\ &\quad \left. \cdot \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right|. \end{aligned} \quad (26)$$

Now, by the parallelogram Law, $|z_1 + z_2| + |z_1 - z_2| \leq 2\sqrt{|z_1|^2 + |z_2|^2}$, we have

$$M \leq \frac{\lambda}{|\alpha|} \sup_{w \in \mathcal{B}_1, z \in E} \left[\sqrt{\left| \int_0^1 w(tz) t^{1/\alpha-1} dt \right|^2 + \left| \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right|^2} \right] + \gamma + \frac{\lambda\gamma}{|\alpha|} \left| \int_0^1 \int_0^1 \frac{w(stz)}{s^2} t^{1/\alpha-1} ds dt \right|. \quad (27)$$

Using the fact that $|w(z)| \leq |z|^2$ in E , we get

$$\begin{aligned} M &\leq \frac{\lambda}{|\alpha|} \\ &\cdot \sup_{z \in E} \sqrt{\left(\int_0^1 |z|^2 t^{1/\alpha+1} dt \right)^2 + \left(\int_0^1 \int_0^1 |z|^2 t^{1/\alpha+1} dt \right)^2} \\ &+ \gamma + \frac{\lambda\gamma}{|\alpha|} \int_0^1 \int_0^1 |z|^2 t^{1/\alpha+1} dt \\ &= \frac{\lambda(\sqrt{2} + \gamma)}{|2\alpha + 1|} + \gamma. \end{aligned} \quad (28)$$

Thus, $M \leq 1$, if $\lambda \leq (1 - \gamma)|2\alpha + 1|/(\sqrt{2} + \gamma)$. $\square$

Taking $\gamma = 0$, in Theorem 5, we get the following result.

Corollary 6. *Let f be as in Theorem 5. Then, $f \in S^*$ provided $0 < \lambda \leq |2\alpha + 1|/\sqrt{2}$.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] K. Noshiro, "On the theory of Schlicht functions," *Journal of the Faculty of Science, Hokkaido University*, vol. 2, no. 3, pp. 129–155, 1934-1935.
- [2] S. E. Warchawski, "On the higher derivatives at the boundary in conformal mapping," *Transactions of the American Mathematical Society*, vol. 38, no. 2, pp. 310–340, 1935.
- [3] S. Ozaki and M. Nunokawa, "The Schwarzian derivative and univalent functions," *Proceedings of the American Mathematical Society*, vol. 33, no. 2, pp. 392–394, 1972.
- [4] M. Obradović, "A class of univalent functions," *Hokkaido Mathematical Journal*, vol. 27, no. 2, pp. 329–335, 1998.
- [5] M. Obradovic, S. Ponnusamy, V. Singh, and P. Vasundhara, "Univalence, starlikeness and convexity applied to certain classes of rational functions," *Analysis: International Mathematical Journal of Analysis and its Applications*, vol. 22, no. 3, pp. 225–242, 2002.
- [6] S. Ponnusamy and P. Vasundhara, "Univalent functions with missing Taylor coefficients," *Hokkaido Mathematical Journal*, vol. 33, no. 2, pp. 341–355, 2004.
- [7] S. Ponnusamy and P. Vasundhara, "Criteria for univalence, starlikeness and convexity," *Annales Polonici Mathematici*, vol. 85, no. 2, pp. 121–133, 2005.
- [8] V. Singh, "On a class of univalent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 23, no. 12, pp. 855–857, 2000.

[9] M. Obradovic and S. Ponnusamy, "A class of univalent functions defined by a differential inequality," *Kodai Mathematical Journal*, vol. 34, no. 2, pp. 169–178, 2011.

[10] Á. Baricz and S. Ponnusamy, "Differential inequalities and Bessel functions," *Journal of Mathematical Analysis and Applications*, vol. 400, no. 2, pp. 558–567, 2013.