Review Article

Mathematical Fundamentals of Modern Linear Optics

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Received 25 August 2011; Accepted 13 January 2012

Academic Editor: Charles Bunting

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All known quantum-mechanical approaches to wave and statistical optics are united into a single theory, using Feynman's path integral as a fundamental principle. In short-wave approximations, this principle, the Fourier transformations, and concepts of the theory reproduce Fermat's principle, the Legendre transformations, and concepts of Hamilton's optics and radiometry in a one-to-one fashion.

1. Introduction

There is a fruitful analogy between mechanics and optics. For example, geometrical optics and classical mechanics can be expressed in the same mathematical language of Hamilton's theory [1]. Moreover, this analogy played a substantial role in the development of wave and statistical optics as a theory that has the same relations to geometric optics and radiometry as quantum mechanics has to classical mechanics. The advent of quantum mechanics radically changed the classical way of physical thinking [2]. Physics that incorporates elements of quantum mechanics is said to be modern physics [3]. Thus, classical optics based on mathematical fundamentals of quantum and statistical mechanics can be called "modern linear optics." The adjective "linear" emphasizes its difference from modern optics that embraces quantum optical phenomena.

Although the traditional wave optics is based on Maxwell's electrodynamics [4], in the literature there are four approaches to wave optics which are based on the analogy with quantum mechanics.

- (1) Generalizing the Huygens-Fresnel principle [5], Feynman has created a new mathematical tool: the path integral [6] which can be used as a fundamental principle of wave and statistical optics [7]. In the stationary-phase approximation Feynman's integral over all paths transforms into an integral over one

path which is satisfied by Fermat's principle [7], that is, the fundamental principle of Hamilton's optics.

- (2) It is known that the Legendre transformations are a framework of Hamilton's optics [1, 4, 8], and the Fourier transformations are a framework of wave optics [9–14]. Using the stationary phase approximation Walther proved [12–14] that these transformations which before were considered independent, are connected to each other: if the complex wave functions are connected by the Fourier transformations, then the phases of these functions are connected by the Legendre transformation.
- (3) It is known that geometrical optics is only a short-wave approximation of wave optics and cannot describe optical phenomena as completely as wave optics does, because all equations of geometrical optics can be derived from the wave equation by neglecting certain terms. But Gloge and Marcuse [15, 16] proved that the wave equation can indeed be recovered from the Hamilton equation of geometrical optics by quantization, that is, by formal replacing of all of the variables by corresponding operators.
- (4) In 1968 Walther proved [17] that the two divisions of optics, which were considered before independently—the theory of the partial spatial coherence (statistical optics) and radiometry of non-Lambertian sources—have in fact the general subject for study,

namely, the radiation energy transfer. The basic laws of radiometry can be derived from the theory of partial coherence by using the Wigner definition function (WDF) [17, 18].

Note that according to Bohr's correspondence principle [19] modern optics in short-wave approximation must reproduce Hamilton's optics and radiometry.

The purpose of the present paper is to unite these four approaches to wave and statistical optics in a single theory, modern optics. In the short-wave approximation the Huygens-Fresnel principle (in form of Feynman's path integral), the Fourier transformations, and concepts (e.g., WDF) of this modern optics must reproduce Fermat's principle, Legendre transformations, and concepts (e.g., radiance) of Hamilton's optics [8] and radiometry [20] in a one-to-one fashion.

2. Feynman's Path Integral Reproduces Fermat's Principle

It is known that all results of geometrical optics can be derived from only one principle—the Fermat principle [1, 4, 8], and all results of wave optics (in the scalar quasi-monochromatic approximation) can be derived from the another principle—the Huygens-Fresnel principle, which can be mathematically expressed in the form of Feynman's path integral [5]. Let us show that the Fermat principle can be deduced from the Feynman path integral by the method of stationary phase [5, 7].

In a scalar quasi-monochromatic approach, that is, neglecting polarisation and spectral structure of radiation, the distribution of a wave field in the plane z can be described by the complex amplitude $U(x, y; z)$. Consider the propagation of a scalar quasi-monochromatic field with wavelength λ_o in vacuum along the z -direction through a layer of an inhomogeneous optical medium with the refractive index distribution $n(x, y, z)$ restricted by the input $\{(x, y)\} = \{(x, y, z) : z = 0\}$ and the output $\{(x', y')\} = \{(x, y, z) : z = Z\}$ planes (see Figure 1). This layer can be described as a linear system which transforms the input complex amplitude $U(x, y)$ into the output complex amplitude $U'(x', y')$ by the superposition integral

$$U'(x', y') = \iint_{R^2} U(x, y) \cdot K_S(x, y; x', y') dx dy. \quad (1)$$

The kernel $K_S(x, y; x', y')$ of this input-output relation is called an optical propagator. Following Feynman [6, 7], the propagator can be written as the integral over all possible virtual paths connecting points (x, y) and (x', y') :

$$K_S(x, y; x', y') = \int_{\text{all path}} A \cdot \exp [ik_o S_{\text{path}}(x, y; x', y')] D(\text{path}), \quad (2)$$

where the symbol $D(\text{path})$ is the “path differential measure” signifying integration over all paths, $S_{\text{path}}(x, y; x', y')$ is called the optical length of the virtual path connecting points (x, y)

and (x', y') , $k_o \equiv 2\pi/\lambda_o$ is the wave number of the scalar quasi-monochromatic wave in vacuum, and A is a normalization constant.

Let us consider the simplest case. The collection of all virtual paths connecting points (x, y) and (x', y') can be labelled with a parameter u [8]. All these paths can be projected onto the z axis so that they can be described by functions $x(z)$ and $y(z)$. Then the entire assembly of paths running from the input point (x, y) to the output point (x', y') is given by two functions of two parameters: $\gamma_u(z) = \{x(z, u), y(z, u)\}$ (see Figure 1). Following Luneburg [1], we use the z -axis as an independent variable, similar to the time axis in analytical mechanics. So the direction in space can be characterized by “velocities”:

$$\begin{aligned} \dot{x} &= \frac{dx}{dz} = \tan \theta \cdot \sin \varphi, \\ \dot{y} &= \frac{dy}{dz} = \tan \theta \cdot \cos \varphi, \end{aligned} \quad (3)$$

in terms of the angles (θ, φ) of the spherical coordinate system (see Figure 2). In this case the optical length of the path takes the following form $S_u(x, y; x', y')$:

$$S_u(x, y; x', y') = \int_{\gamma_u} L(x, y; \dot{x}, \dot{y}; z) dz, \quad (4)$$

where

$$L(x, y; \dot{x}, \dot{y}; z) = n(x, y, z) \sqrt{1 + \dot{x}^2 + \dot{y}^2} \quad (5)$$

is the optical Lagrangian [1, 8]. In this case the optical propagator (2) can be rewritten as

$$K_S(x, y; x', y') = \int_{-\infty}^{\infty} A \cdot \exp[ik_o S_u(x, y; x', y')] du. \quad (6)$$

Note that for sufficiently large k_o (the short-wavelength approximation), the exponential functions $\exp[ik_o S_u(x, y; x', y')]$ fluctuate so rapidly that they compensate each other, and the final amplitude becomes close to zero. Compensation does not occur in the neighbourhood of the stationary point $\bar{u}$ (see Figure 1):

$$\left. \frac{\partial S_u}{\partial u} \right|_{u=\bar{u}} = 0, \quad (7)$$

where the phase function $S_u(x, y; x', y')$ varies relatively slowly, the exponential function constructively interferes, and the final amplitude, the optical propagator equation (6), is rather large.

Since outside the neighbourhood of the stationary point $\bar{u}$ the resulting amplitude nearly equals zero the function $S_u(x, y; x', y')$ can be approximated by a parabola with the vertex at point $(\bar{u}, S_{\bar{u}})$ (see Figure 1):

$$S_u(x, y; x', y') \approx S_{\bar{u}}(x, y; x', y') + \frac{1}{2} \alpha (u - \bar{u})^2, \quad (8)$$

where

$$\alpha \equiv \left. \frac{\partial^2 S_u(x, y; x', y')}{\partial u^2} \right|_{u=\bar{u}}. \quad (9)$$

FIGURE 1: Illustration of the path integral applied to a layer of an inhomogeneous medium. The path integral contains three elements. (1) A collection of virtual paths connecting an input point (x, y) with an output point (x', y') can be labelled with a parameter $u : \gamma : (z, u) \rightarrow (x, y; z) \in R^3$, that is, $\gamma_u(0) = (x, y; 0) = (x, y)$ and $\gamma_u(Z) = (x, y; Z) = (x', y')$. (2) All virtual paths going through the optical inhomogeneous medium $n(x, y, z)$ from the input point (x, y) to the output point (x', y') have the optical length $S : u \rightarrow \int_{\gamma_u} n(x, y, z) \sqrt{1 + \dot{x}^2 + \dot{y}^2} dz$. (3) The path integral $K_S(x, y; x', y') = \int_{-\infty}^{\infty} A \cdot \exp[ik_o S_u(x, y; x', y')] du$ is the optical propagator. The phase function $S_u(x, y; x', y')$ in the neighbourhood of the stationary point $\bar{u}$, where $\partial S_u / \partial u|_{u=\bar{u}} = 0$ can be approximated by a parabola with the vertex at point $(\bar{u}, S_{\bar{u}})$. In this case the optical propagator can be approximated by the function $K_S(x, y; x', y') = C \cdot |K_S(x, y; x', y')| \cdot \exp[ik_o S_{\bar{u}}(x, y; x', y')]$, where $S_{\bar{u}}(x, y; x', y')$ is a point eikonal, that is, the optical length of the path which satisfies Fermat's principle.

FIGURE 2: Direction of the tangent to a path $\gamma_u(z)$ at a given point $P^*(\gamma_u(\zeta), \zeta)$ can be characterized by angles (θ, φ) of the spherical coordinate system ($\varphi \in (0, 2\pi)$, $\theta \in (0, \pi/2)$), or by "velocities" $(\dot{x}, \dot{y})$.

Thus, in (6) we can substitute the phase function $S_u(x, y; x', y')$ by its approximation (8):

$$K_S(x, y; x', y') = A \cdot \exp[ik_o S_{\bar{u}}(x, y; x', y')] \cdot \int_{-\infty}^{\infty} \exp\left[ik_o \frac{\alpha}{2} (u - \bar{u})^2\right] du. \quad (10)$$

As the integral has the closed form [10, 21, 22]

$$\int_{-\infty}^{\infty} \exp\left[ik_o \frac{\alpha}{2} (u - \bar{u})^2\right] du = \sqrt{\frac{2\pi}{k_o \alpha}} \exp\left[i\frac{\pi}{4}\right]. \quad (11)$$

Equation (8) can be rewritten as

$$K_S(x, y; x', y') = |K_S(x, y; x', y')| \cdot \exp[ik_o S_{\bar{u}}(x, y; x', y')]. \quad (12)$$

Equation (12) is the stationary phase approximation of the propagator (6) [5].

Note that (12) creates the impression that the wave runs along the path $\gamma_{\bar{u}}(z)$ whose optical length is stationary with respect to small perturbations of u , Equation (6), that is, whose optical length satisfies Fermat's principle and the phase function $S_{\bar{u}}(x', y'; x, y)$ is a point eikonal [5]. Thus, using the method of stationary phase, we have shown that Feynman's path integral reproduces Fermat's principle and the point eikonal (see sections 2 and 4 in [8]).

3. k_o -Fourier Transformations Reproduce the Legendre Transformations

For true understanding of problems, it is necessary to consider them from various points of view. This possibility is given by one-to-one mappings such as the Legendre transformations in geometrical optics and the Fourier transformations in wave optics. Note that to understand optics as a whole, one needs to know not only these transformations alone, but also the one-to-one mapping between them which is given by the method of stationary phase [5, 12, 13, 21, 22].

The direct k_o -Fourier transformation of the complex-valued function $U(x, y)$ of real variables (x, y) is a new

complex-valued function $\tilde{U}(p, q)$ of new real variables (p, q) , which is defined as

$$\begin{aligned}\tilde{U}(p, q) &= {}^{k_o}F_{x \rightarrow p, y \rightarrow q}^{-1} \{U(x, y)\} \\ &\equiv \left(\frac{k_o}{2\pi i}\right) \iint_{R^2} U(x, y) \cdot \exp[-ik_o(x \cdot p + y \cdot q)] dx dy.\end{aligned}\quad (13a)$$

The inverse k_o -Fourier transformation of the complex-valued function $\tilde{U}(p, q)$ of real variables (p, q) is a new complex-valued function $U(x, y)$ of new real variables (x, y) , which is defined as

$$\begin{aligned}U(x, y) &= {}^{k_o}F_{p \rightarrow x, q \rightarrow y}^{-1} \{\tilde{U}(p, q)\} \\ &\equiv \left(-\frac{k_o}{2\pi i}\right) \iint_{R^2} \tilde{U}(p, q) \cdot \exp[ik_o(x \cdot p + y \cdot q)] dp dq.\end{aligned}\quad (13b)$$

The composition of the direct k_o -Fourier transformation and the inverse k_o -Fourier transformation (or the composition of the inverse k_o -Fourier transformation and the direct k_o -Fourier transformation) is the identity transformation I :

$$\begin{aligned}{}^{k_o}F_{x \rightarrow p, y \rightarrow q} \circ {}^{k_o}F_{p \rightarrow x, q \rightarrow y}^{-1} &= I \\ {}^{k_o}F_{p \rightarrow x, q \rightarrow y}^{-1} \circ {}^{k_o}F_{x \rightarrow p, y \rightarrow q} &= I.\end{aligned}\quad (14)$$

If the complex functions are connected by the k_o -Fourier transformations, the phases of these functions are connected by the Legendre transformations [21, 22].

Note that if the complex functions $U(x, y)$ and $\tilde{U}(p, q)$ are factorized into the amplitude and phase factors:

$$\begin{aligned}U(x, y) &= |U(x, y)| \cdot \exp[ik_o\Psi(x, y)], \\ \tilde{U}(p, q) &= |\tilde{U}(p, q)| \cdot \exp[ik_o\tilde{\Psi}(p, q)],\end{aligned}\quad (15)$$

then their direct and inverse k_o -Fourier transformations (13a), and (13b) take the forms

$$\begin{aligned}\tilde{U}(p, q) &= \left(\frac{k_o}{2\pi i}\right) \iint_{R^2} |U(x, y)| \\ &\quad \cdot \exp[ik_o(\Psi(x, y) - x \cdot p - y \cdot q) dx dy], \\ U(x, y) &= \left(-\frac{k_o}{2\pi i}\right) \iint_{R^2} |\tilde{U}(p, q)| \\ &\quad \cdot \exp[ik_o(\tilde{\Psi}(p, q) + x \cdot p + y \cdot q)] dp dq.\end{aligned}\quad (16)$$

The phase functions $(x, y) - xp - yq\Psi$ and $\tilde{\Psi}(p, q) + xp + yq$ have stationary points (p, q) and (x, y) , respectively, that is,

$$\begin{aligned}\frac{\partial(\Psi(x, y) - x \cdot p - y \cdot q)}{\partial x} &= 0, \\ \frac{\partial(\Psi(x, y) - x \cdot p - y \cdot q)}{\partial y} &= 0 \\ \Rightarrow (p, q) &= \left(\frac{\partial\Psi(x, y)}{\partial x}, \frac{\partial\Psi(x, y)}{\partial y}\right) \\ \frac{\partial(\tilde{\Psi}(p, q) + x \cdot p + y \cdot q)}{\partial p} &= 0, \\ \frac{\partial(\tilde{\Psi}(p, q) + x \cdot p + y \cdot q)}{\partial q} &= 0 \\ \Rightarrow (x, y) &= \left(-\frac{\partial\tilde{\Psi}(p, q)}{\partial p}, -\frac{\partial\tilde{\Psi}(p, q)}{\partial q}\right).\end{aligned}\quad (17)$$

These stationary points are nondegenerate: $\det\begin{pmatrix} \Psi''_{xx} & \Psi''_{xy} \\ \Psi''_{yx} & \Psi''_{yy} \end{pmatrix} \neq 0$ and $\det\begin{pmatrix} \tilde{\Psi}''_{pp} & \tilde{\Psi}''_{pq} \\ \tilde{\Psi}''_{qp} & \tilde{\Psi}''_{qq} \end{pmatrix} \neq 0$.

Using the stationaryphase approximation (see Appendix A), the Fourier transformations can be estimated by the equations

$$\begin{aligned}{}^{k_o}F_{x \rightarrow p, y \rightarrow q} \{ |U(x, y)| \exp[ik_o\Psi(x, y)] \} \\ \approx {}^{k_o} \left| \det \begin{pmatrix} \Psi''_{xx} & \Psi''_{xy} \\ \Psi''_{yx} & \Psi''_{yy} \end{pmatrix} \right|^{1/2} \cdot \left| U \left(-\frac{\partial\Psi}{\partial p}, -\frac{\partial\Psi}{\partial q} \right) \right| \\ \cdot \exp\{ik_o\tilde{\Psi}(p, q)\},\end{aligned}\quad (18a)$$

$$\begin{aligned}{}^{k_o}F_{p \rightarrow x, q \rightarrow y}^{-1} \{ |\tilde{U}(p, q)| \exp[ik_o\tilde{\Psi}(p, q)] \} \\ \approx {}^{k_o} \left| \det \begin{pmatrix} \Psi''_{xx} & \Psi''_{xy} \\ \Psi''_{yx} & \Psi''_{yy} \end{pmatrix} \right|^{1/2} \cdot \left| \tilde{U} \left(\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y} \right) \right| \\ \cdot \exp\{ik_o\Psi(x, y)\}.\end{aligned}\quad (18b)$$

The phase function $\tilde{\Psi}(p, q)$ is the direct Legendre transformation of the continuously differentiable real-valued phase function $\Psi(x, y)$, that is, a new real-valued phase function $\tilde{\Psi}(p, q)$ of new real variables (p, q) , which is defined as

$$\begin{aligned}\tilde{\Psi}(p, q) &= L_{x \rightarrow p, y \rightarrow q} \{\Psi(x, y)\} \equiv \Psi(x, y) - xp - yq, \\ \text{where } p &\equiv \frac{\partial\Psi}{\partial x}, \quad q \equiv \frac{\partial\Psi}{\partial y}.\end{aligned}\quad (19a)$$

The phase function $\Psi(x, y)$ is the direct Legendre transformation of the continuously differentiable real-valued phase

FIGURE 3: A layer of an optically homogeneous medium as a linear system.

function, that is, a new real-valued phase function $\tilde{\Psi}(p, q)$ of new real variables (x, y) , which is defined as

$$\Psi(x, y) = L_{p \rightarrow x}^{-1} \left\{ \tilde{\Psi}(p, q) \right\} \equiv \tilde{\Psi}(p, q) + xp + yq, \quad (19b)$$

where $x \equiv \frac{\partial \tilde{\Psi}}{\partial p}$, $y \equiv \frac{\partial \tilde{\Psi}}{\partial q}$.

If the function $\Psi(x, y)$ is positive definite, $\det \begin{pmatrix} \Psi''_{xx} & \Psi''_{xy} \\ \Psi''_{yx} & \Psi''_{yy} \end{pmatrix} > 0$, (or negative definite, $\det \begin{pmatrix} \Psi''_{xx} & \Psi''_{xy} \\ \Psi''_{yx} & \Psi''_{yy} \end{pmatrix} < 0$), then the function $\tilde{\Psi}(p, q)$ is negative definite, $\det \begin{pmatrix} \tilde{\Psi}''_{pp} & \tilde{\Psi}''_{pq} \\ \tilde{\Psi}''_{qp} & \tilde{\Psi}''_{qq} \end{pmatrix} < 0$ (or positive definite, $\det \begin{pmatrix} \tilde{\Psi}''_{pp} & \tilde{\Psi}''_{pq} \\ \tilde{\Psi}''_{qp} & \tilde{\Psi}''_{qq} \end{pmatrix} > 0$). In this case

$$\det \begin{pmatrix} \Psi''_{xx} & \Psi''_{xy} \\ \Psi''_{yx} & \Psi''_{yy} \end{pmatrix} \cdot \det \begin{pmatrix} \tilde{\Psi}''_{pp} & \tilde{\Psi}''_{pq} \\ \tilde{\Psi}''_{qp} & \tilde{\Psi}''_{qq} \end{pmatrix} = -1. \quad (20)$$

If the functions $\Psi(x, y)$ and $\tilde{\Psi}(p, q)$ are positive or negative definite functions, then the composition of the direct Legendre transformation and the inverse Legendre transformation (or the composition of the inverse Legendre transformation and the direct Legendre transformation) is the identity transformation I :

$$L_{x \rightarrow p} \circ L_{y \rightarrow q}^{-1} = I, \dots L_{p \rightarrow x}^{-1} \circ L_{q \rightarrow y} = I. \quad (21)$$

4. Dual Meaning of Spatial Frequencies

In Fourier optics, the spatial frequency has a dual meaning. On the one hand it is a sinusoidal grating, and on the other hand it is the momentum of the monochromatic plane wave which goes through this grating. To analyze this in detail, let us consider the simplest optical system, that is, a layer of an optically homogeneous medium with refractive index n and thickness Z bounded by parallel input $(x, y) = (x, y, 0)$ and output planes $(x_z, y_z) = (x, y, Z)$ (see Figure 3).

It is known that a scalar quasi-monochromatic plane wave travelling through this layer in a direction specified by

FIGURE 4: Direction of the “ray vector” can be characterized by angles (θ, φ) of the spherical coordinate system ($\varphi \in (0, 2\pi)$, $\theta \in (0, \pi/2)$), or by momenta (p, q) .

the vector $(p, q, \sqrt{n^2 - p^2 - q^2})$ (see Figure 4) can be described by the equation

$$\exp \left[ik_0 \left(xp + yq + z\sqrt{n^2 - p^2 - q^2} \right) \right]. \quad (22)$$

Using the direction cosines p and q as two independent variables, we can describe the complex amplitude on a plane Z , $U(x, y; Z)$, as a linear superposition of plane waves with weight function $\tilde{U}_{in}(p, q)$ [9–13]:

$$U(x, y; Z) = \left(-\frac{k_0}{2\pi i} \right) \iint_{R^2} \tilde{U}_{in}(p, q) \exp \left[ik_0 \left(xp + yq + Z\sqrt{n^2 - p^2 - q^2} \right) \right] dp dq. \quad (23)$$

Let us consider the layer of a homogeneous optical medium as a linear system (see Figure 3) [9–13] which transforms the input complex amplitude $U_{in}(x, y) \equiv U(x, y; 0)$ to the output complex amplitude $U_{out}(x_z, y_z) \equiv U(x, y; Z)$. Then using the inverse k_0 -Fourier transformation (14), (24a) and (24b) can be rewritten twice

$$U_{in}(x, y) = {}^{k_0}F_{p \rightarrow x}^{-1} \left\{ \tilde{U}_{in}(p, q) \right\}, \quad \text{if } z = 0, \quad (24a)$$

$$U_{out}(x_z, y_z) = {}^{k_0}F_{p \rightarrow x_z}^{-1} \left\{ \tilde{U}_{in}(p, q) \cdot \exp[-ik_0 Z \cdot H(p, q)] \right\}, \quad (24b)$$

if $z = Z$,

where the function

$$H(p, q) = -\sqrt{n^2 - p^2 - q^2} \quad (25)$$

is known as the optical Hamiltonian [1, 5, 8].

Thus, the weight function $\tilde{U}_{\text{in}}(p, q) = {}^{k_o}F_{x \rightarrow p, y \rightarrow q} \{U_{\text{in}}(x, y)\}$ can be interpreted as an input frequency spectrum, and the variables $(p, q) \in R^2$ —as spatial frequencies. According to (24b), the input frequency spectrum and the output frequency spectrum $\tilde{U}_{\text{out}}(p, q) = {}^{k_o}F_{x_z \rightarrow p, y_z \rightarrow q} \{U_{\text{out}}(x_z, y_z)\}$ are related by the equation

$$\tilde{U}_{\text{out}}(p, q) = \tilde{U}_{\text{in}}(p, q) \cdot \tilde{K}_Z(p, q), \quad (26)$$

where $\tilde{K}_Z(p, q) = \exp[-ik_o Z \cdot H(p, q)]$ is the transfer function of the layer of a homogeneous optical medium with refractive index n of thickness Z .

Thus, in region $p^2 + q^2 \leq n^2$ variables (p, q) can be interpreted not only as spatial frequencies but also as the direction cosines of the wave vector. This manifests the so called dual meaning of spatial frequencies [11].

According to the convolution theorem [9, 10], Equation (26) can be rewritten as

$$\begin{aligned} U_{\text{out}}(x_z, y_z) &= \iint_{R^2} U_{\text{in}}(x_z - Z\dot{x}, y_z - Z\dot{y}) \cdot G(\dot{x}, \dot{y}) d\dot{x} d\dot{y} \\ &= \iint_{R^2} U_{\text{in}}(x, y) \cdot G_z(x_z - x, y_z - y) dx dy, \end{aligned} \quad (27)$$

where

$$G_z(x_z - x, y_z - y) = \frac{1}{Z^2} G\left(\frac{x_z - x}{Z}, \frac{y_z - y}{Z}\right) \quad (28a)$$

is the impulse response function of the layer of a homogeneous optical medium with refractive index n of thickness Z , and

$$G(\dot{x}, \dot{y}) \equiv \left(\frac{k_o Z}{2\pi i}\right) \cdot {}^{k_o}F_{p \rightarrow \dot{x}, q \rightarrow \dot{y}}^{-1} \{\exp[-ik_o Z \cdot H(p, q)]\} \quad (28b)$$

is the impulse response function of the layer of a homogeneous optical medium with refractive index n of unit thickness. Using the stationary phase approximation of the inverse Fourier transformation (18b), the impulse response function (28b) can be rewritten as:

$$\begin{aligned} G(\dot{x}, \dot{y}) &\equiv \left(\frac{k_o Z}{2\pi i}\right) \cdot {}^{k_o}F_{p \rightarrow \dot{x}, q \rightarrow \dot{y}}^{-1} \{\exp[-i(k_o Z) \cdot H(p, q)]\} \\ &\approx_{k_o Z \rightarrow \infty} \left(\frac{k_o Z}{2\pi i}\right) \cdot \left| \det \begin{pmatrix} L''_{\dot{x}\dot{x}} & L''_{\dot{x}\dot{y}} \\ L''_{\dot{y}\dot{x}} & L''_{\dot{y}\dot{y}} \end{pmatrix} \right|^{1/2} \\ &\quad \cdot \exp\{ik_o Z \cdot L(\dot{x}, \dot{y})\}. \end{aligned} \quad (29)$$

Here

$$L(\dot{x}, \dot{y}) = -L_{p \rightarrow \dot{x}, q \rightarrow \dot{y}}^{-1} \{H(p, q)\} = \dot{x} \cdot p + \dot{y} \cdot q - H(p, q), \quad (30)$$

where

$$\dot{x} = \frac{\partial H(p, q)}{\partial p}, \quad \dot{y} = \frac{\partial H(p, q)}{\partial q}. \quad (31)$$

Note that (31) can be interpreted as Hamilton's equations which describe propagation of a light ray through an inhomogeneous optical medium with the refractive index distribution $n(x, y, z)$.

In the optically homogeneous medium ($n = \text{const}$) the Legendre transformation (30) of the Hamiltonian (25) is the optical Lagrangian $L(\dot{x}, \dot{y}) = n\sqrt{1 + \dot{x}^2 + \dot{y}^2}$ [1, 8, 15, 16]. Since

$$\begin{aligned} &\det \begin{pmatrix} L''_{\dot{x}\dot{x}} & L''_{\dot{x}\dot{y}} \\ L''_{\dot{y}\dot{x}} & L''_{\dot{y}\dot{y}} \end{pmatrix} \\ &= \det \begin{pmatrix} n \frac{1 + \dot{y}}{(1 + \dot{x}^2 + \dot{y}^2)^{3/2}} & n \frac{-\dot{x}\dot{y}}{(1 + \dot{x}^2 + \dot{y}^2)^{3/2}} \\ n \frac{-\dot{x}\dot{y}}{(1 + \dot{x}^2 + \dot{y}^2)^{3/2}} & n \frac{1 + \dot{x}}{(1 + \dot{x}^2 + \dot{y}^2)^{3/2}} \end{pmatrix} \\ &= \left(\frac{n}{1 + \dot{x}^2 + \dot{y}^2} \right)^2, \end{aligned} \quad (32)$$

the impulse response function (29) takes the form

$$G(\dot{x}, \dot{y}) \approx \left(\frac{nk_o Z}{2\pi i}\right) \frac{\exp\left[ink_o Z \sqrt{1 + \dot{x}^2 + \dot{y}^2}\right]}{1 + \dot{x}^2 + \dot{y}^2}, \quad (33)$$

or, according to (28a),

$$\begin{aligned} &G_z(x_z - x, y_z - y) \\ &\approx \left(\frac{nk_o Z}{2\pi i}\right) \frac{\exp\left[ink_o \sqrt{Z^2 + (x_z - x)^2 + (y_z - y)^2}\right]}{Z^2 + (x_z - x)^2 + (y_z - y)^2}. \end{aligned} \quad (34)$$

Let us compare the approximation (34) to the well-known exact solution of (28a), and (28b), the so-called Rayleigh-Sommerfeld impulse response function given by [9–13]

$$\begin{aligned} &G_z(x_z - x, y_z - y) \\ &= -\frac{1}{2\pi} \frac{d}{dZ} \left(\frac{\exp\left[ink_o \sqrt{Z^2 + (x_z - x)^2 + (y_z - y)^2}\right]}{\sqrt{Z^2 + (x_z - x)^2 + (y_z - y)^2}} \right) \\ &= \left(1 - \frac{1}{ink_o \sqrt{Z^2 + (x_z - x)^2 + (y_z - y)^2}} \right) \cdot \left(\frac{nk_o Z}{2\pi i} \right) \\ &\quad \times \frac{\exp\left[ink_o \sqrt{Z^2 + (x_z - x)^2 + (y_z - y)^2}\right]}{Z^2 + (x_z - x)^2 + (y_z - y)^2}. \end{aligned} \quad (35)$$

For large values of Z , the second term between the parentheses is smaller than the first term, and in this most important case approximation (34) is very good.

Using the method of stationary phase we have shown that the Fourier transformations between the transfer function and the impulse response function of the layer of a homogeneous optical medium with refractive index n of the unit thickness reproduce the Legendre transformations between the Hamiltonian and the Lagrangian (see Section 3 in [8]).

5. Input-Output Transformations of Wave Optics Reproduce Eikonals Theory

From the possibility of both coordinate and frequency descriptions of the wave field at the input and output of the optical system follows the existence of four equivalent ways of wave description of this system in terms of “input-output transformations.” Using the method of stationary phase, Walter showed that each of these transformations of the wave field reproduces the eikonal which describes the transformation in terms of geometrical optics.

Consider a linear optical system bounded by parallel input (x, y) and output (x', y') planes lying in layers of optically homogeneous media with refractive indices n and n' , respectively, and a layer of an optically inhomogeneous medium between them (see Figure 5). Variables with primes refer to the output plane.

As the radiation field in the input and output planes of the linear optical system can be specified either in the coordinate representations $U(x, y)$ and $U'(x', y')$ or in the frequency representations $\tilde{U}(p, q)$ and $\tilde{U}'(p', q')$, according to the superposition principle, the linear optical system can be described by any of the four integral equations [5, 12–14]:

$$U'(x', y') = \iint_{R^2} K_S(x, y; x' y') \cdot U(x, y) dx dy, \quad (36a)$$

$$\tilde{U}'(p', q') = \iint_{R^2} K_V(x, y; p' q') \cdot U(x, y) dx dy, \quad (36b)$$

$$U'(x', y') = \iint_{R^2} K_{V'}(p, q; x' y') \cdot \tilde{U}(p, q) dp dq, \quad (36c)$$

$$\tilde{U}'(p', q') = \iint_{R^2} K_T(p, q; p' q') \cdot \tilde{U}(p, q) dp dq. \quad (36d)$$

The kernels of these linear input-output transformations can be called the point $K_S(x, y; x', y')$, point-angle $K_V(x, y; p', q')$, angle-point $K_{V'}(p, q; x', y')$, and angle $K_T(p, q; p', q')$ propagators.

The complex amplitudes, $U(x, y)$ and $U'(x', y')$, and frequency spectra, $\tilde{U}(p, q)$ and $\tilde{U}'(p', q')$, are connected by k_0 -Fourier transformations. As a result, the propagators of the optical system are connected by the Fourier transformations too [5, 12–14]:

$$K_V(x, y; p', q') = {}^{k_0}F_{x' \rightarrow p'}^{-1} \{K_S(x, y; x' y')\},$$

$$K_{V'}(p, q; x', y') = {}^{k_0}F_{x \rightarrow p}^{-1} \{K_S(x, y; x' y')\}, \quad (37)$$

$$K_T(p, q; p', q') = {}^{k_0}F_{x \rightarrow p}^{-1} {}^{k_0}F_{x' \rightarrow p'} \{K_S(x, y; x' y')\}.$$

FIGURE 5: Linear optical system.

In these expressions, the frequency representations are transformed into the coordinate representations by the direct Fourier transformations in the input plane and by the inverse Fourier transformations in the output plane.

Note that propagators can be factorized into amplitude and phase factors:

$$K_S(x, y; x', y') = |K_S(x, y; x', y')| \cdot \exp\{ik_0 S(x, y; x', y')\}, \quad (38a)$$

$$K_V(x, y; p', q') = |K_V(x, y; p', q')| \cdot \exp\{ik_0 V(x, y; p', q')\}, \quad (38b)$$

$$K_{V'}(p, q; x', y') = |K_{V'}(p, q; x', y')| \cdot \exp\{ik_0 V'(p, q; x', y')\}, \quad (38c)$$

$$K_T(p, q; p', q') = |K_T(p, q; p', q')| \cdot \exp\{ik_0 T(p, q; p', q')\}. \quad (38d)$$

Since the propagators of the optical system are connected by the Fourier transformations (37), then according to (18a), and (18b), their phase functions are connected by the Legendre transformations:

$$V(x, y; p', q') = L_{x' \rightarrow p'}^{-1} \{S(x, y; x' y')\},$$

$$V'(p, q; x', y') = L_{x \rightarrow p}^{-1} \{S(x, y; x' y')\}, \quad (39)$$

$$T(p, q; p', q') = L_{x \rightarrow p}^{-1} L_{x' \rightarrow p'} \{S(x, y; x' y')\}.$$

Because the phase function $S(x, y; x', y')$ is a point eikonal $S_{\bar{u}}(x, y; x', y')$ (4), the phase functions $V(x, y; p', q')$, $V'(p, q; x', y')$, and $T(p, q; p', q')$ can be interpreted as the point-angle, angle-point and angle eikonals [8, 12–14].

It is known that the point-angle eikonal of the perfect optical system has the form

$$V^{\text{perfect}}(x, y; p', q') = -m(xp' + yq'), \quad (40)$$

and the point-angle eikonal of the isoplanar (or shift invariant) optical system has the form

$$V^{\text{isoplanar}}(x, y; p', q') = -m(xp' + yq') + W(p', q'), \quad (41)$$

FIGURE 6: An isoplanar optical system with telecentric stop in object space.

where $W(p', q')$ is the wave aberration [23, 24]. Note that in an optical system with telecentric stop in object space (see Figure 6) the wave aberration is determined in the region of moments limited by the telecentric stop with the radius R : $\sqrt{p'^2 + q'^2} \leq R/f$, where f is the back focal length (distance between front focal F and front nodal points N of the optical system). In the isoplanar optical system, the point eikonal $S^{\text{isoplanar}}(x, y; x', y')$ and the point propagator $K_S^{\text{isoplanar}}(x, y; x', y')$ depend only on the difference variables:

$$\begin{aligned}
 S^{\text{isoplanar}}(x, y; x', y') &= L_{p' \rightarrow x', q' \rightarrow y'}^{-1} \{ V^{\text{isoplanar}}(x, y; p', q') \} \\
 &= [-m(xp' + yq') + W(p', q')] + x'p' + y'q' \\
 &= W(p', q') + (x' - mx)p' + (y' - my)q' \\
 &= L_{p' \rightarrow (x' - mx), q' \rightarrow (y' - my)}^{-1} \{ W(p', q') \} \\
 &= S(x' - mx, y' - my),
 \end{aligned} \tag{42}$$

$$\begin{aligned}
 K_S^{\text{isoplanar}}(x, y; x', y') &= {}^{k_0}F_{p' \rightarrow x', q' \rightarrow y'}^{-1} \{ K_V(x, y; p', q') \} \\
 &= {}^{k_0}F_{p' \rightarrow x', q' \rightarrow y'}^{-1} \{ \tilde{K}(p', q') \} \\
 &\quad \cdot \exp[ik_0 V^{\text{isoplanar}}(x, y; p', q')] \} \\
 &= \left(-\frac{k_0}{2\pi i} \right) \iint_{R^2} |\tilde{K}(p', q')| \cdot \exp\{ik_0[(-m(xp' + yq') \\
 &\quad + W(p', q') + x'p' + y'q')] \} dp' dq' \\
 &= \left(-\frac{k_0}{2\pi i} \right) \iint_{R^2} |\tilde{K}(p', q')| \cdot \exp\{ik_0[(W(p', q') \\
 &\quad + (x' - mx)p' + (y' - my)q')] \} dp' dq'
 \end{aligned}$$

$$\begin{aligned}
 &= {}^{k_0}F_{p' \rightarrow (x' - mx), q' \rightarrow (y' - my)}^{-1} \{ |\tilde{K}(p', q')| \} \\
 &\quad \cdot \exp\{ik_0 W(p', q')\} \} \\
 &\equiv K(x' - mx, y' - my).
 \end{aligned} \tag{43}$$

In this case the functions $\tilde{K}(p', q')$ and $K(x' - mx, y' - my)$ are called the transfer function and the impulse response function of the isoplanar optical system, respectively.

Using the method of stationary phase, we have shown that the Fourier transformations between propagators of optical system (37) reproduce the Legendre transformations between its eikonals (39) (see Section 4 in [8]). In the same way, Fourier transformations between the transfer function and the impulse response function of an isoplanar optical system (43) reproduce the Legendre transformations between its wave aberration function and its point eikonal (42).

6. Quantization: Hamiltonian Reproduces Helmholtz Equation

According to Gloge and Marcuse a formal quantization of the Hamiltonian of geometrical optics leads to the Helmholtz equation of wave optics [15, 16]. Let us show that this formal procedure can be substantiated by using the relations between the Fourier and the Legendre transformations, (18a), and (18b).

Note that the Fourier transformations between input-output relations (37) and the Legendre transformations between eikonals (39) are valid for any position of the input and output planes. Let the input and output planes be separated by a layer of a medium of small thickness ε . Then the differential equation of wave optics (the Helmholtz equation) can be derived from the integral superposition (2).

Consider the particular case where the input plane ζ and the output plane $\zeta + \varepsilon$ are separated by a plane-parallel layer of medium with smooth homogeneities (see Figure 7). Let the thickness ε be less than the characteristic size of the optical homogeneities, but substantially greater than the radiation wavelength λ_0/n [5]. In this case the medium in a neighbourhood of a point (x, y) can be considered as quasi-homogeneous and the integral superposition equation (36a) with the propagator (38a) can be approximated by the equation [5, 6]

$$\begin{aligned}
 U(x', y') &\equiv U(x, y; \zeta + \varepsilon) \\
 &= \text{const} \cdot \left(\frac{k_0}{2\pi i} \right) \int_{R^2} \exp[i \cdot \varepsilon k_0 L(\dot{x}, \dot{y}, x, y; \zeta)] \\
 &\quad \times U(x - \varepsilon \dot{x}, y - \varepsilon \dot{y}; \zeta) d\dot{x} d\dot{y},
 \end{aligned} \tag{44}$$

because $S(x, y, x', y') = S(x' - x, y' - y) = \varepsilon L((x' - x)/\varepsilon, (y' - y)/\varepsilon, x, y)$.

FIGURE 7: Thin layer of an optically quasi-homogeneous medium.

Using the operator representation of the Taylor series (Appendix B), the function $U(x - \varepsilon\dot{x}, y - \varepsilon\dot{y}; \zeta)$ can be given as

$$U(x - \varepsilon\dot{x}, y - \varepsilon\dot{y}; \zeta) = \exp\left(-\varepsilon\dot{x}\frac{\partial}{\partial x} - \varepsilon\dot{y}\frac{\partial}{\partial y}\right)U(x, y; \zeta). \quad (45)$$

Substituting this expression into the left-hand side of (44), we get

$$\begin{aligned} & U(x, y; \zeta + \varepsilon) \\ &= \text{const} \cdot \left(\frac{k_o}{2\pi i}\right) \cdot \iint_{R^2} \exp\left\{i\varepsilon k_o \left[L(\dot{x}, \dot{y}, x, y; \zeta) \right. \right. \\ &\quad \left. \left. - \dot{x}\left(-\frac{i}{k_o}\frac{\partial}{\partial x}\right) \right. \right. \\ &\quad \left. \left. - \dot{y}\left(-\frac{i}{k_o}\frac{\partial}{\partial y}\right) \right] \right\} \\ &\quad \times U(x, y; \zeta) d\dot{x} d\dot{y} \end{aligned} \quad (46)$$

or, using the definition of the Fourier transformation (13a),

$$\begin{aligned} & U(x, z; \zeta + \varepsilon) \\ &= \text{const} \cdot {}^{k_o}F_{\dot{x} \rightarrow (-i/k_o \partial/\partial x)} \left\{ \exp[i\varepsilon k_o L(\dot{x}, \dot{y}, x, y; \zeta)] \right\} \\ &\quad \times U(x, y; \zeta). \end{aligned} \quad (47)$$

If $L(\dot{x}, \dot{y}, x, y; \zeta)$ is a smoothly varying function, then, according to the asymptotic (18a), the phases of the functions connected by the Fourier transformation are connected by the Legendre transformation:

$$\begin{aligned} & {}^{k_o}F_{\dot{x} \rightarrow (-i/k_o \partial/\partial x)} \left\{ \exp[i\varepsilon k_o L(\dot{x}, \dot{y}, x, y; \zeta)] \right\} \\ & \quad \dot{y} \rightarrow (-i/k_o \partial/\partial y) \\ & \cong \text{const}' \cdot \exp\left[i\varepsilon k_o \hat{H}\left(-\frac{i}{k_o}\frac{\partial}{\partial x}, -\frac{i}{k_o}\frac{\partial}{\partial y}, x, y; \zeta\right)\right], \end{aligned} \quad (48)$$

where

$$\begin{aligned} & \hat{H}\left(-\frac{i}{k_o}\frac{\partial}{\partial x}, -\frac{i}{k_o}\frac{\partial}{\partial y}, x, y; \zeta\right) \\ &= L_{\dot{x} \rightarrow (-i/k_o \partial/\partial x)} \left\{ L(\dot{x}, \dot{y}, x, y; \zeta) \right\} \\ & \quad \dot{y} \rightarrow (-i/k_o \partial/\partial y) \\ &= \dot{x}\left(-\frac{i}{k_o}\frac{\partial}{\partial x}\right) + \dot{y}\left(-\frac{i}{k_o}\frac{\partial}{\partial y}\right) - L(\dot{x}, \dot{y}, x, y; \zeta). \end{aligned} \quad (49)$$

The approximation of (48) makes it possible to simplify the right-hand side of (47) and to write it in the differential form

$$\left[\frac{\partial}{\partial \zeta} + i k_o \hat{H}\left(-\frac{i}{k_o}\frac{\partial}{\partial x}, -\frac{i}{k_o}\frac{\partial}{\partial y}, x, y; \zeta\right) \right] U(x, y; \zeta) = 0. \quad (50)$$

Comparing (49) and the optical Hamiltonian (25) [1, 8, 15, 16], we can see that the operator $\hat{H}((-i/k_o)(\partial/\partial x), (-i/k_o)(\partial/\partial y), x, y; \zeta)$ is obtained from the optical Hamiltonian by replacing coordinates x, y and momenta p and q with the corresponding operators in the position representation [5, 25]

$$\hat{x} = x, \quad \hat{y} = y, \quad \hat{p} = -\frac{i}{k_o}\frac{\partial}{\partial x}, \quad \hat{q} = -\frac{i}{k_o}\frac{\partial}{\partial y}, \quad (51)$$

that is,

$$\begin{aligned} & \hat{H}\left(-\frac{i}{k_o}\frac{\partial}{\partial x}, -\frac{i}{k_o}\frac{\partial}{\partial y}, x, y; \zeta\right) \\ &= -\sqrt{n^2(x, y; \zeta) + \left(\frac{1}{k_o}\right)^2 \frac{\partial^2}{\partial x^2} + \left(\frac{1}{k_o}\right)^2 \frac{\partial^2}{\partial y^2}}. \end{aligned} \quad (52)$$

This makes it possible to interpret (50) as the symbolic representation of the Helmholtz equation [5, 15, 16]

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial \zeta^2} + k_o^2 n^2(x, y; \zeta) U = 0. \quad (53)$$

So we substantiate the “quantum theory of light rays” (see Appendix C) [15, 16]: the operator representation of the Helmholtz equation can be obtained from the optical Hamiltonian by replacing the momenta p and q with the corresponding momentum operators in the position representation.

Note that the symbolic representation of the Helmholtz equation (50) has the complex conjugation analog

$$\left[\frac{\partial}{\partial \zeta} + ik_o \hat{H}^* \left(-\frac{i}{k_o} \frac{\partial}{\partial x}, -\frac{i}{k_o} \frac{\partial}{\partial y}, x, y; \zeta \right) \right] U^*(x, y; \zeta) = 0, \quad (54)$$

where the asterisk $*$ denotes complex conjugation.

It is shown that formal quantization, that is, replacing coordinates x , y and momenta p and q with the corresponding operators, (52), can be substantiated by using the relation between the Fourier and the Legendre transformations.

7. Position Representation and Momentum Representation of Operators

It is known that a wave field of radiation can be described in the position and the momentum representations which are connected to each other by Fourier transformations [25].

The fundamental symmetry of the operator theory is the duality principle: the k_o -Fourier transformation (13a) connects the complex amplitude $U(x, y; z)$ of the wave field with its spectrum $\tilde{U}(p, q; z)$, and also replaces the operators in position representation (51) with the operators in the momentum representation [25]:

$$\check{x} = \frac{i}{k_o} \frac{\partial}{\partial p}, \quad \check{y} = \frac{i}{k_o} \frac{\partial}{\partial q}, \quad \check{p} = p, \quad \check{q} = q. \quad (55)$$

More precisely,

$$\begin{aligned} k_o F_{x \rightarrow p} \{x U(x, y; \zeta)\} &= \frac{i}{k_o} \frac{\partial}{\partial p} k_o F_{x \rightarrow p} \{U(x, y; \zeta)\}, \\ k_o F_{y \rightarrow q} \{y U(x, y; \zeta)\} &= \frac{i}{k_o} \frac{\partial}{\partial q} k_o F_{y \rightarrow q} \{U(x, y; \zeta)\}, \\ k_o F_{x \rightarrow p} \left\{ -\frac{i}{k_o} \frac{\partial}{\partial x} U(x, y; \zeta) \right\} &= p \cdot k_o F_{x \rightarrow p} \{U(x, y; \zeta)\}, \\ k_o F_{y \rightarrow q} \left\{ -\frac{i}{k_o} \frac{\partial}{\partial y} U(x, y; \zeta) \right\} &= q \cdot k_o F_{y \rightarrow q} \{U(x, y; \zeta)\}. \end{aligned} \quad (56)$$

Therefore the k_o -Fourier transformation (13a) replaces the Hamilton operator in the position representation (52) with the Hamilton operator in the momentum representation:

$$\begin{aligned} k_o F_{x \rightarrow p} \left\{ \hat{H} \left(-\frac{i}{k_o} \frac{\partial}{\partial x}, -\frac{i}{k_o} \frac{\partial}{\partial y}; x, y; \zeta \right) U(x, y; \zeta) \right\} \\ = \check{H} \left(p, q; \frac{i}{k_o} \frac{\partial}{\partial p}, \frac{i}{k_o} \frac{\partial}{\partial q}; \zeta \right) k_o F_{x \rightarrow p} \{U(x, y; \zeta)\}. \end{aligned} \quad (57a)$$

That is true for the complex conjugation Hamilton operator as well:

$$\begin{aligned} k_o F_{x \rightarrow p} \left\{ \hat{H}^* \left(-\frac{i}{k_o} \frac{\partial}{\partial x}, -\frac{i}{k_o} \frac{\partial}{\partial y}; x, y; \zeta \right) U(x, y; \zeta) \right\} \\ = \check{H}^* \left(-p, -q; -\frac{i}{k_o} \frac{\partial}{\partial p}, -\frac{i}{k_o} \frac{\partial}{\partial q}; \zeta \right) \\ \times k_o F_{y \rightarrow q} \{U(x, y; \zeta)\}. \end{aligned} \quad (57b)$$

Thus, the Hamilton operator in the momentum representation

$$\begin{aligned} \check{H} \left(p, q; \frac{i}{k_o} \frac{\partial}{\partial p}, \frac{i}{k_o} \frac{\partial}{\partial q}; \zeta \right) \\ = -\sqrt{n^2 \left(\frac{i}{k_o} \frac{\partial}{\partial p}, \frac{i}{k_o} \frac{\partial}{\partial q}; \zeta \right) - p^2 - q^2} \end{aligned} \quad (58)$$

can be derived from the optical Hamiltonian (25) by replacing coordinates x , y and momenta p and q with the corresponding operators in the momentum representation (55). If the Helmholtz equation in the position representation has form (50), then in the momentum representation it takes the following form:

$$\left\{ \frac{\partial}{\partial \zeta} + ik_o \check{H} \left(p, q; \frac{i}{k_o} \frac{\partial}{\partial p}, \frac{i}{k_o} \frac{\partial}{\partial q}; \zeta \right) \right\} \tilde{U}(p, q; \zeta) = 0. \quad (59)$$

8. Transport Equation for the Wigner Distribution Function

It is known that in a plane $z = \zeta$ the spatial coherence of a scalar quasi-monochromatic wave is characterized by two-point correlation functions—the mutual intensity

$$\Gamma(x_1, y_1, x_2, y_2; \zeta) \equiv \langle U(x_1, y_1; \zeta) U^*(x_2, y_2; \zeta) \rangle \quad (60a)$$

or the mutual coherence spectrum

$$\tilde{\Gamma}(p_1, q_1, p_2, q_2; \zeta) \equiv \langle \tilde{U}(p_1, q_1; \zeta) \tilde{U}^*(p_2, q_2; \zeta) \rangle. \quad (60b)$$

Here, the angle brackets denote averaging over an ensemble.

By introducing mean variables $x = (x_1 + x_2)/2$, $y = (y_1 + y_2)/2$, $p = (p_1 + p_2)/2$, $q = (q_1 + q_2)/2$ and difference variables $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta p = p_2 - p_1$, $\Delta q = q_2 - q_1$ and using the Fourier transform from difference variables, it is easy to obtain from the correlation functions (60a) and (60b) the mathematically equivalent but more convenient expressions of the spatial coherence of radiation—the WDF [17, 26–34]:

$$\begin{aligned} W(x, y; p, q; \zeta) \\ = k_o F_{\Delta x \rightarrow p} \left\{ \left\langle U \left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}; \zeta \right) \right. \right. \\ \left. \left. \times U^* \left(x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}; \zeta \right) \right\rangle \right\} \end{aligned} \quad (61a)$$

or

$$\begin{aligned}
 W(x, y; p, q; \zeta) &= {}^{k_o}F_{\Delta p \rightarrow x, \Delta q \rightarrow y}^{-1} \left\{ \left\langle \tilde{U}\left(p + \frac{\Delta p}{2}, q + \frac{\Delta q}{2}; \zeta\right) \right. \right. \\
 &\quad \left. \left. \times \tilde{U}^*\left(p - \frac{\Delta p}{2}, q - \frac{\Delta q}{2}; \zeta\right) \right\rangle \right\}. \quad (61b)
 \end{aligned}$$

Thus, the WDF describes the partially coherent radiation field in the position and in the momentum representations simultaneously.

The fundamental equations of optics are the equations describing propagation of light through an optically inhomogeneous medium. They are Hamilton's equations in geometrical optics, (31), and the Helmholtz equation in wave optics, (53). Propagation of the WDF through an optically inhomogeneous (and random) media is described by the so-called transport equation [32, 33]. Let us show that the duality between the position and momentum representations of the Helmholtz equation, (50) and (59), allows us to prove it.

If the WDF (61a) is differentiated with respect to the longitudinal variable ζ , then we have

$$\begin{aligned}
 \frac{\partial W(x, y; p, q; \zeta)}{\partial \zeta} &= {}^{k_o}F_{x \rightarrow p, y \rightarrow q} \left\{ \left\langle \frac{\partial U(x + \Delta x/2, y + \Delta y/2; \zeta)}{\partial \zeta} \right. \right. \\
 &\quad \times U^*\left(x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}; \zeta\right) \\
 &\quad + U\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}; \zeta\right) \\
 &\quad \left. \left. \times \frac{\partial U^*(x - \Delta x/2, y - \Delta y/2; \zeta)}{\partial \zeta} \right\rangle \right\}. \quad (62)
 \end{aligned}$$

Using the Helmholtz equation in the position representation (50) and its complex conjugate analogue (54) we can rewrite (62) in the following form:

$$\begin{aligned}
 \frac{\partial W(p, q; x, y; \zeta)}{\partial \zeta} &= -i {}^{k_o}F_{\Delta x \rightarrow p, \Delta y \rightarrow q} \left\{ k_o \left[\hat{\mathbf{H}} \left(-\frac{i}{k_o} \frac{\partial}{\partial(x + \Delta x/2)}, \right. \right. \right. \\
 &\quad \left. \left. -\frac{i}{k_o} \frac{\partial}{\partial(y + \Delta y/2)} x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}; \zeta \right) \right. \\
 &\quad \left. - \hat{\mathbf{H}}^* \left(-\frac{i}{k_o} \frac{\partial}{\partial(x - \Delta x/2)}, \right. \right. \\
 &\quad \left. \left. -\frac{i}{k_o} \frac{\partial}{\partial(y - \Delta y/2)} x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}; \zeta \right) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 &\times \left\langle U\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}; \zeta\right) U^* \right. \\
 &\quad \left. \times \left(x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}; \zeta\right) \right\rangle \Bigg\}. \quad (63)
 \end{aligned}$$

According to Appendix D

$$\begin{aligned}
 \frac{\partial}{\partial(x \pm \Delta x/2)} &= \frac{1}{2} \frac{\partial}{\partial x} \pm \frac{\partial}{\partial \Delta x}, \\
 \frac{\partial}{\partial(y \pm \Delta y/2)} &= \frac{1}{2} \frac{\partial}{\partial y} \pm \frac{\partial}{\partial \Delta y}, \quad (64)
 \end{aligned}$$

so (65) takes the following form:

$$\begin{aligned}
 \frac{\partial W(p, q; x, y; \zeta)}{\partial \zeta} &= -i {}^{k_o}F_{\Delta x \rightarrow p, \Delta y \rightarrow q} \left\{ k_o \left[\hat{\mathbf{H}} \left(-\frac{i}{k_o} \left(\frac{1}{2} \frac{\partial}{\partial x} + \frac{\partial}{\partial \Delta x} \right), \right. \right. \right. \\
 &\quad \left. \left. -\frac{i}{k_o} \left(\frac{1}{2} \frac{\partial}{\partial y} + \frac{\partial}{\partial \Delta y} \right); \right. \right. \\
 &\quad \left. \left. x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}; \zeta \right) \right. \\
 &\quad \left. - \hat{\mathbf{H}}^* \left(-\frac{i}{k_o} \left(\frac{1}{2} \frac{\partial}{\partial x} - \frac{\partial}{\partial \Delta x} \right), \right. \right. \\
 &\quad \left. \left. -\frac{i}{k_o} \left(\frac{1}{2} \frac{\partial}{\partial y} - \frac{\partial}{\partial \Delta y} \right); \right. \right. \\
 &\quad \left. \left. x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}; \zeta \right) \right] \\
 &\quad \left\langle U\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}; \zeta\right) U^* \right. \\
 &\quad \left. \times \left(x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}; \zeta\right) \right\rangle \Bigg\}. \quad (65)
 \end{aligned}$$

Using (57a), (57b) and (65) can be written as

$$\begin{aligned}
 \frac{\partial W(p, q; x, y; \zeta)}{\partial \zeta} &= -i k_o \left\{ \hat{\mathbf{H}} \left(p - \frac{i}{2k_o} \frac{\partial}{\partial x}, q - \frac{i}{2k_o} \frac{\partial}{\partial y}; x + \frac{i}{2k_o} \frac{\partial}{\partial p}, \right. \right. \\
 &\quad \left. \left. y + \frac{i}{2k_o} \frac{\partial}{\partial q}; \zeta \right) \right. \\
 &\quad \left. - \hat{\mathbf{H}}^* \left(p - \frac{i}{2k_o} \frac{\partial}{\partial x}, q - \frac{i}{2k_o} \frac{\partial}{\partial y}; \right. \right. \\
 &\quad \left. \left. x + \frac{i}{2k_o} \frac{\partial}{\partial p}, y + \frac{i}{2k_o} \frac{\partial}{\partial q}; \zeta \right) \right\} \\
 &\quad \times W(p, q; x, y; \zeta). \quad (66)
 \end{aligned}$$

Using the identity

$$-i\{\check{\mathbf{H}} - \check{\mathbf{H}}^*\} = 2 \text{Im} \check{\mathbf{H}}, \quad (67)$$

where the symbol $\text{Im} \check{\mathbf{H}}$ denotes the imaginary part of the Hamilton operator $\check{\mathbf{H}}$, (66) can be rewritten in the symmetric form [32, 33]:

$$\begin{aligned} & \frac{\partial W(p, q; x, y; \zeta)}{\partial \zeta} \\ &= 2k_o \text{Im} \left\{ \check{\mathbf{H}} \left(p - \frac{i}{2k_o} \frac{\partial}{\partial x}, q - \frac{i}{2k_o} \frac{\partial}{\partial y}; x + \frac{i}{2k_o} \frac{\partial}{\partial p}, \right. \right. \\ & \quad \left. \left. y + \frac{i}{2k_o} \frac{\partial}{\partial q}; \zeta \right) \right\} W(p, q; x, y; \zeta). \end{aligned} \quad (68)$$

We can approximate the Hamiltonian in (68) via a Taylor series expansion by the Appendix B

$$\begin{aligned} & \check{\mathbf{H}} \left(p - \frac{i}{2k_o} \frac{\partial}{\partial x}, q - \frac{i}{2k_o} \frac{\partial}{\partial y}; x + \frac{i}{2k_o} \frac{\partial}{\partial p}, y + \frac{i}{2k_o} \frac{\partial}{\partial q}; \zeta \right) \\ &= H(p, q; x, y; \zeta) \cdot \exp \left[\frac{i}{2} \left(\frac{\partial H}{\partial x} \frac{\partial}{\partial p} + \frac{\partial H}{\partial y} \frac{\partial}{\partial q} \right. \right. \\ & \quad \left. \left. - \frac{\partial H}{\partial p} \frac{\partial}{\partial x} - \frac{\partial H}{\partial q} \frac{\partial}{\partial y} \right) \right]. \end{aligned} \quad (69)$$

In that case the transport equation (68) takes the form

$$\begin{aligned} & \frac{\partial W(p, q; x, y; \zeta)}{\partial \zeta} \\ &= -2 \text{Im} \left\{ H(p, q; x, y; \zeta) \right. \\ & \quad \cdot \exp \left[\frac{i}{2} \left(\frac{\partial H}{\partial x} \frac{\partial}{\partial p} + \frac{\partial H}{\partial y} \frac{\partial}{\partial q} - \frac{\partial H}{\partial p} \frac{\partial}{\partial x} - \frac{\partial H}{\partial q} \frac{\partial}{\partial y} \right) \right] \left. \right\} \\ & \quad \times W(p, q; x, y; \zeta) \end{aligned} \quad (70)$$

or, in the symbolic notation of Besieris and Tappert [34],

$$\begin{aligned} & \frac{\partial W(p, q; x, y; \zeta)}{\partial \zeta} \\ &= -2 \text{Im} \left\{ H(p, q; x, y; \zeta) \right. \\ & \quad \cdot \exp \left[\frac{i}{2} \left(\frac{\overrightarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} + \frac{\overrightarrow{\partial}}{\partial y} \frac{\overrightarrow{\partial}}{\partial q} - \frac{\overrightarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial x} - \frac{\overrightarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial y} \right) \right] \left. \right\} \\ & \quad \times W(p, q; x, y; \zeta). \end{aligned} \quad (71)$$

In this expression the arrows over the partial derivatives indicate the direction in which the derivatives act, that is,

those with a left arrow act on $H(p, q; x, y; \zeta)$, and those with a right arrow act on $W(p, q; x, y; \zeta)$.

In a weakly inhomogeneous medium the transport equation (71) reduces to

$$\begin{aligned} & \frac{\partial W(p, q; x, y; \zeta)}{\partial \zeta} \\ &= -2 \text{Im} \left\{ H(p, q; x, y; \zeta) \right. \\ & \quad \times \left[1 + \frac{i}{2} \left(\frac{\overrightarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} + \frac{\overrightarrow{\partial}}{\partial y} \frac{\overrightarrow{\partial}}{\partial q} - \frac{\overrightarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial x} - \frac{\overrightarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial y} \right) \right] \left. \right\} \\ & \quad \times W(p, q; x, y; \zeta). \end{aligned} \quad (72)$$

Thus, in the usual notation, this equation takes the form of the Liouville equation [27–31]

$$\frac{\partial W}{\partial \zeta} = -2(\text{Im} H) \cdot W - [\text{Re} H, W], \quad (73)$$

where Re denotes the real part and

$$\begin{aligned} [\text{Re} H, W] &= \left(\frac{\partial \text{Re} H}{\partial x} \frac{\partial W}{\partial p} + \frac{\partial \text{Re} H}{\partial y} \frac{\partial W}{\partial q} \right. \\ & \quad \left. - \frac{\partial \text{Re} H}{\partial p} \frac{\partial W}{\partial x} - \frac{\partial \text{Re} H}{\partial q} \frac{\partial W}{\partial y} \right) \end{aligned} \quad (74)$$

is the “Poisson bracket”. The Liouville equation (73) reduces to the ordinary differential equation [27–31]

$$\frac{dW}{d\zeta} - 2(\text{Im} \hat{H}) \cdot W = 0. \quad (75)$$

In the special case that $H(p, q; x, y; \zeta)$ is a real function of x, y, p , and q , Equation (75) can be formulated in geometric optical terms as follows: along a light ray the WDF has a constant value:

$$\frac{dW}{d\zeta} = 0. \quad (76)$$

It is known that the WDF with some natural assumptions can be interpreted as a wave analogue of the radiance [1]. So equation (76) is the theorem about invariance of radiance along the light ray, that is, the most fundamental theorem of radiometry. Note that this theorem in the form of the Liouville equation (73) can be derived from the point of view of Hamilton’s optics, too [20].

9. Conclusion

We have unite all known quantum-mechanical approaches to wave and statistical optics in a single theory using Feynman’s path integral as a fundamental principle. In particular, we have substantiated the quantization which allows one to translate the Hamiltonian of geometrical optics into the Helmholtz equation of wave optics.

Appendices

A. The Stationary-Phase Approximation

Wave optics often takes account of oscillatory integrals of the type

$$I = \iint_{R^2} g(x, y) \exp\{ik_0 \Psi(x, y)\} dx dy \quad (\text{A.1})$$

in which $g(x, y)$ is a slowly varying complex-valued function and $\Psi(x, y)$ is a slowly varying real-valued function of the real variables (x, y) . We assume that the phase function $\Psi(x, y)$ has one stationary point (x_0, y_0) in the interior of the region of integration:

$$\left. \frac{\partial \Psi(x, y)}{\partial x} \right|_{x=x_0, y=y_0} = 0, \quad \left. \frac{\partial \Psi(x, y)}{\partial y} \right|_{x=x_0, y=y_0} = 0. \quad (\text{A.2})$$

The stationary point (x_0, y_0) is nondegenerate

$$\det \begin{pmatrix} \Psi''_{xx}(x_0, y_0) & \Psi''_{xy}(x_0, y_0) \\ \Psi''_{yx}(x_0, y_0) & \Psi''_{yy}(x_0, y_0) \end{pmatrix} \neq 0. \quad (\text{A.3})$$

Then, according to the stationary phase method [5, 12–14, 21, 22], the integral (A.1) can be approximated by the equation

$$\begin{aligned} & \iint_{R^2} g(x, y) \exp\{ik_0 \Psi(x, y)\} dx dy \\ & \approx_{k_0 \rightarrow \infty} \frac{2\pi i}{k_0} \cdot \left| \det \begin{pmatrix} \Psi''_{xx}(x_0, y_0) & \Psi''_{xy}(x_0, y_0) \\ \Psi''_{yx}(x_0, y_0) & \Psi''_{yy}(x_0, y_0) \end{pmatrix} \right|^{-1/2} \\ & \cdot g(x_0, y_0) \exp\{ik_0 \Psi(x_0, y_0)\}. \end{aligned} \quad (\text{A.4})$$

B. The Taylor Series in Operator Form

Using the Taylor series, a function $U(x - \Delta x, y - \Delta y)$ can be expanded in a power series up to the linear term:

$$U(x - \Delta x, y - \Delta y; \zeta) \cong U(x, y; \zeta) - \frac{\partial U}{\partial x} \Delta x - \frac{\partial U}{\partial y} \Delta y. \quad (\text{B.1})$$

We can rewrite (B.1) in the operator form [5, 6]

$$\begin{aligned} U(x - \Delta x, y - \Delta y; \zeta) & \cong U(x, y; \zeta) - \frac{\partial U}{\partial x} \Delta x - \frac{\partial U}{\partial y} \Delta y \\ & = \left[1 - \frac{\partial}{\partial x} \Delta x - \frac{\partial}{\partial y} \Delta y \right] U(x, y; \zeta). \end{aligned} \quad (\text{B.2})$$

Using the Taylor series, we can expand the exponent $\exp(-(\partial U/\partial x)\Delta x, -(\partial U/\partial y)\Delta y)$ in a power series up to the linear term:

$$\exp\left(-\frac{\partial}{\partial x} \Delta x - \frac{\partial}{\partial y} \Delta y\right) \cong 1 - \frac{\partial}{\partial x} \Delta x - \frac{\partial}{\partial y} \Delta y. \quad (\text{B.3})$$

Comparing (B.2) with (B.3), we see that the right-hand sides of these equations are equal, so we can write the approximate operator equation [5, 35]:

$$U(x - \Delta x, y - \Delta y; \zeta) = \exp\left(-\frac{\partial}{\partial x} \Delta x - \frac{\partial}{\partial y} \Delta y\right) U(x, y; \zeta). \quad (\text{B.4})$$

C. Formal Quantum Theory of Light Rays

In the quantum theory of light rays, the coordinates x, y retain their meaning as numbers, but the canonically conjugate variables p, q (the momenta), (52), and the Hamiltonian H , (51), become differential operators

$$p = -i \frac{\lambda_0}{2\pi} \frac{\partial}{\partial x}, \quad (\text{C.1a})$$

$$q = -i \frac{\lambda_0}{2\pi} \frac{\partial}{\partial y}, \quad (\text{C.1b})$$

$$H = i \frac{\lambda_0}{2\pi} \frac{\partial}{\partial z}, \quad (\text{C.1c})$$

where λ_0 is the free-space wavelength. Thus, the operator equation of the quantum theory of light rays can be obtained from the optical Hamiltonian (25) and operator equation (C.1c) by replacing momenta p and q with the momentum operators (C.1a) and (C.1b)

$$-\sqrt{n^2(x, y; z) + \left(\frac{\lambda_0}{2\pi}\right)^2 \frac{\partial^2}{\partial x^2} + \left(\frac{\lambda_0}{2\pi}\right)^2 \frac{\partial^2}{\partial y^2}} = i \frac{\lambda_0}{2\pi} \frac{\partial}{\partial z}. \quad (\text{C.2})$$

A wave function $U(x, y; z)$ of the quantum theory of light rays is the usual distribution of the scalar wave field over the plane z . Applying the operator equation (C.2) to the wave function $U(x, y; z)$ results in the equation

$$i \frac{\partial U}{\partial z} = -\sqrt{\left(\frac{2\pi}{\lambda_0}\right)^2 n^2(x, y; z) + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}} \cdot U \quad (\text{C.3})$$

that takes into account the symmetry of the optical medium relative to the $0z$ axis. Equation (C.3) can be interpreted as the symbolic form of the Helmholtz equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \left(\frac{2\pi}{\lambda_0}\right)^2 n^2(x, y; z) U = 0. \quad (\text{C.4})$$

D.

One has

$$\begin{aligned}
 & (a) \\
 & \frac{\partial f}{\partial(x + \Delta x/2)} \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{f(x + \Delta x/2 + \varepsilon) - f(x + \Delta x/2)}{\varepsilon} \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{f\left(x + \frac{\varepsilon}{2} + (\Delta x + \varepsilon)/2\right) - f(x + \Delta x/2)}{\varepsilon} \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon/2 + (\Delta x + \varepsilon)/2) - f(x + (\Delta x + \varepsilon)/2)}{\varepsilon} \\
 &\quad + \lim_{\varepsilon \rightarrow 0} \frac{f(x + (\Delta x + \varepsilon)/2) - f(x + \Delta x/2)}{\varepsilon} \\
 &= \frac{1}{2} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial \Delta x}.
 \end{aligned} \tag{D.1}$$

One has the following:

$$\begin{aligned}
 & (b) \\
 & \frac{\partial f}{\partial(x - \Delta x/2)} \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{f(x - \Delta x/2 + \varepsilon) - f(x - \Delta x/2)}{\varepsilon} \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon/2 - (\Delta x - \varepsilon)/2) - f(x - \Delta x/2)}{\varepsilon} \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon/2 - (\Delta x - \varepsilon)/2) - f(x - (\Delta x - \varepsilon)/2)}{\varepsilon} \\
 &\quad + \lim_{\varepsilon \rightarrow 0} \frac{f(x - (\Delta x - \varepsilon)/2) - f(x - \Delta x/2)}{\varepsilon} \\
 &= \frac{1}{2} \frac{\partial f}{\partial x} - \frac{\partial f}{\partial \Delta x}.
 \end{aligned} \tag{D.2}$$

Acknowledgments

It is pleasure to thank Dr. Wilhelm Becker and Dr. Mikhail Kalashnikov for helpful discussions and conversations.

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