Application Article

Using Truncated Data Sets in Spherical-Scanning Antenna Measurements

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We discuss the mitigation of truncation errors in spherical-scanning measurements by use of a constrained least-squares estimation method. The main emphasis is the spherical harmonic representation of probe transmitting and receiving functions; however, our method is applicable to near-field measurement of electrically small antennas for which full-sphere data are either unreliable or unavailable.

1. Introduction

The transmitting function of an electrically small probe tends to be very broad, so that full-sphere data are needed to compute the spherical-harmonic representation of the pattern by use of standard methods [1]. Unfortunately, backward-hemisphere data are often unavailable or at best unreliable due to support-structure blockage. Simply setting the backward-hemisphere data to zero leads to ringing effects that degrade the accuracy of the probe-pattern representation in the forward hemisphere. In this paper, we approximate the spherical-harmonic expansion of transmit/receive functions using only forward hemisphere data. The solution presented is a least-squares fit with energy constraints (to limit backward-hemisphere radiation) and proper weighting of measurements (to compensate for clustering near the poles). The energy constraint is specified using an estimated value of directivity. Results are given that compare pattern representations obtained using both full-sphere and half-sphere data.

Others have also considered truncation error mitigation in various near-field measurement contexts [2–5].

2. Theory

We wish to approximate a physical quantity $\mathbf{b}$ by a spherical-harmonic expansion

$$\mathbf{b}(\theta, \varphi) = b^\theta(\theta, \varphi)\hat{\boldsymbol{\theta}} + b^\varphi(\theta, \varphi)\hat{\boldsymbol{\varphi}} \approx \hat{\mathbf{b}}(\theta, \varphi), \quad (1)$$

$$\hat{\mathbf{b}}(\theta, \varphi) = \sum_{\mu=-M}^M \sum_{\nu=n(\mu)}^N \left[a_{\nu\mu}^1 \mathbf{X}_{\nu\mu}(\theta, \varphi) + a_{\nu\mu}^2 \mathbf{Y}_{\nu\mu}(\theta, \varphi) \right], \quad (2)$$

$$n(\mu) = \max(1, |\mu|), \quad |M| \leq N.$$

Here, $\mathbf{X}_{nm}$ and $\mathbf{Y}_{nm} = i\hat{\mathbf{r}} \times \mathbf{X}_{nm}$ are vector spherical harmonics [6, chapter 16]. For example, $\mathbf{b}(\theta, \varphi)$ might be the measured transmitting or receiving pattern of a probe, or it might be the “measurement vector” $\mathbf{w}(\theta, \varphi)$ for a spherical near-field scanning measurement [1, 7–9].

When full-sphere data are available, the orthogonality of vector spherical harmonics can be used to determine the coefficients $a_{\nu\mu}^1$ and $a_{\nu\mu}^2$ from discrete measurements of $\mathbf{b}(\theta, \varphi)$ gathered on a uniform grid in θ and φ . When so determined, $\hat{\mathbf{b}}(\theta, \varphi)$ is optimal in the sense that

$$\int_0^\pi \sin \theta d\theta \int_0^{2\pi} \left| \mathbf{b}(\theta, \varphi) - \hat{\mathbf{b}}(\theta, \varphi) \right|^2 d\varphi \quad (3)$$

is minimal.

In this paper, we assume that data are available only in the forward hemisphere $\theta \leq \pi/2$ on the grid

$$\begin{aligned}\theta_n &= \frac{\pi}{2P'}n, \quad 0 \leq n \leq P', \\ \varphi_m &= \frac{2\pi}{Q}m, \quad 0 \leq m < Q,\end{aligned}\quad (4)$$

with

$$\begin{aligned}P' &> N, \\ Q &> 2M.\end{aligned}\quad (5)$$

Requirement (5) preserves the standard number of measurement points by halving the maximum sample interval in θ . Ideally, P' and Q are products of small primes to allow efficient application of the fast Fourier transform (FFT) algorithm; however, factorization may not be important when antennas are “electrically small.” Our goal is to minimize the discrepancy

$$\Delta = \sum_{n=0}^{P'} w_n \int_0^{2\pi} |\mathbf{b}(\theta_n, \varphi) - \hat{\mathbf{b}}(\theta_n, \varphi)|^2 d\varphi, \quad (6)$$

where the w_n are positive weights. Equation (6) may be viewed as an adaptation of (3). Because the sampling theorem has been satisfied, the φ integral can be readily evaluated analytically. The θ integral, however, has been discretized. (We generally set $w_n = \sin \theta_n$).

The discrepancy Δ is minimized subject to the “energy” constraint

$$E = \sum_{\gamma\mu} \left(|a_{\gamma\mu}^1|^2 + |a_{\gamma\mu}^2|^2 \right), \quad (7)$$

in order to limit the power radiated into directions where there are no measurements. When $\mathbf{b}(\hat{\mathbf{r}})$ is proportional to the transmitting pattern of the test antenna, we have

$$E = \frac{4\pi |\mathbf{b}(\hat{\mathbf{r}})|^2}{D(\hat{\mathbf{r}})}, \quad (8)$$

so that E may be chosen using an estimate of the directivity D in some measurement direction $\hat{\mathbf{r}}$. (Directivity and gain are approximately equal when ohmic losses are small).

Avoiding details for the moment, we write

$$\mathbf{b} \approx \hat{\mathbf{b}} = \mathbf{A}\mathbf{x}, \quad (9)$$

where $\mathbf{b}$ and $\hat{\mathbf{b}}$ are vectors of measurements and predicted values, $\mathbf{x}$ is a vector of coefficients to be determined, and the matrix $\mathbf{A}$ represents a known linear relationship. We choose $\mathbf{x}$ to minimize $\Delta = (\mathbf{b} - \mathbf{A}\mathbf{x})^* \boldsymbol{\kappa} (\mathbf{b} - \mathbf{A}\mathbf{x})$ subject to the constraint $E = \|\mathbf{x}\|^2$. Here, $\boldsymbol{\kappa}$ is a diagonal matrix of positive weights, and an asterisk implies Hermitian transpose. It is easy to show that this $\mathbf{x}$ is determined by the equations

$$\mathbf{A}^* \boldsymbol{\kappa} \mathbf{A} \mathbf{x} - \lambda \mathbf{x} = \mathbf{A}^* \boldsymbol{\kappa} \mathbf{b}, \quad (10)$$

$$\mathbf{x}^* \mathbf{x} = E \quad (11)$$

The significance of the Lagrange multiplier λ is seen in the relation

$$|\lambda| = \frac{\|\mathbf{A}^* \boldsymbol{\kappa} \mathbf{A} \mathbf{x} - \mathbf{A}^* \boldsymbol{\kappa} \mathbf{b}\|}{\sqrt{E}} \quad (12)$$

that is, $|\lambda|$ is a measure of the residual of the fit. In the absence of the energy constraint (7) of course, we can choose $\mathbf{x}$ to be the solution of the normal equations $\mathbf{A}^* \boldsymbol{\kappa} \mathbf{A} \mathbf{x} = \mathbf{A}^* \boldsymbol{\kappa} \mathbf{b}$.

Since $\mathbf{A}^* \boldsymbol{\kappa} \mathbf{A}$ is Hermitian nonnegative definite, we have the singular-value decomposition (SVD)

$$\mathbf{A}^* \boldsymbol{\kappa} \mathbf{A} = \mathbf{S} \mathbf{D} \mathbf{S}^*, \quad (13)$$

where $\mathbf{S}$ is unitary, $D_{ij} = d_i \delta_{ij}$, and $d_i \geq 0$. From (13), (10), and (11), we are led to

$$E = \sum_i \frac{|f_i|^2}{(d_i - \lambda)^2}, \quad (14)$$

$$\mathbf{f} \equiv \mathbf{S}^* \mathbf{A}^* \boldsymbol{\kappa} \mathbf{b}.$$

The right side of (14) tends to 0 as $|\lambda| \rightarrow \infty$, and because of poles at the singular values d_i , there may be numerous solutions for λ . According to (12), the best choice corresponds to the unique solution for which $\lambda < d_1$, where d_1 is the smallest singular value. In general, $\lambda < 0$, since the constrained solution is expected to have lower energy than the unconstrained solution. Finally, given λ , we may compute $\mathbf{x}$,

$$x_i = \sum_j \frac{S_{ij} f_j}{d_j - \lambda}. \quad (15)$$

Numerical methods used in this paper are discussed in greater detail in [10].

At this point, we have discussed the main ideas of this paper. What follows are some rather unpleasant details that we summarize dutifully and concisely. To begin with, (1) can be written as a Fourier series

$$\mathbf{b}(\theta, \varphi) = \sum_{\mu=-M}^M \left[b_\mu^\theta(\theta) \hat{\boldsymbol{\theta}} + b_\mu^\varphi(\theta) \hat{\boldsymbol{\varphi}} \right] \exp(i\mu\varphi), \quad (16)$$

where the coefficients may be computed from the data with a discrete Fourier transform

$$b_\mu^{\theta, \varphi}(\theta) = \frac{1}{Q} \sum_{m=0}^{Q-1} b^{\theta, \varphi}(\theta, \varphi_m) \exp\left(-i \frac{2\pi}{Q} m\mu\right). \quad (17)$$

Similarly, for (2),

$$\hat{\mathbf{b}}(\theta, \varphi) = \sum_{\mu=-M}^M \left[\hat{b}_\mu^\theta(\theta) \hat{\boldsymbol{\theta}} + \hat{b}_\mu^\varphi(\theta) \hat{\boldsymbol{\varphi}} \right] \exp(i\mu\varphi), \quad (18)$$

$$\hat{b}_\mu^{\theta, \varphi}(\theta) = \sum_{\nu=n(\mu)}^N \left[a_{\gamma\mu}^1 X_{\gamma\mu}^{\theta, \varphi}(\theta, 0) + a_{\gamma\mu}^2 Y_{\gamma\mu}^{\theta, \varphi}(\theta, 0) \right],$$

$$\mathbf{X}_{\gamma\mu} = X_{\gamma\mu}^\theta \hat{\boldsymbol{\theta}} + X_{\gamma\mu}^\varphi \hat{\boldsymbol{\varphi}}, \quad (19)$$

$$\mathbf{Y}_{\gamma\mu} = Y_{\gamma\mu}^\theta \hat{\boldsymbol{\theta}} + Y_{\gamma\mu}^\varphi \hat{\boldsymbol{\varphi}},$$

$$Y_{\gamma\mu}^\theta = -iX_{\gamma\mu}^\varphi, \quad Y_{\gamma\mu}^\varphi = iX_{\gamma\mu}^\theta.$$

With substitution of (16) and (18), the discrepancy (6) becomes

$$\Delta = 2\pi \sum_{n=0}^{P'} w_n \sum_{\mu=-M}^M \left[\left| b_{\mu}^{\theta}(\theta_n) - \hat{b}_{\mu}^{\theta}(\theta_n) \right|^2 + \left| b_{\mu}^{\varphi}(\theta_n) - \hat{b}_{\mu}^{\varphi}(\theta_n) \right|^2 \right]. \quad (20)$$

This may be rewritten in the form

$$\Delta = 2\pi \sum_{\mu=-M}^M (\mathbf{b}_{\mu} - \mathbf{A}_{\mu} \mathbf{x}_{\mu})^* \boldsymbol{\kappa}_{\mu} (\mathbf{b}_{\mu} - \mathbf{A}_{\mu} \mathbf{x}_{\mu}). \quad (21)$$

Here

$$\begin{aligned} \mathbf{x}_{\mu} &= \begin{pmatrix} \mathbf{x}_{\mu}^1 \\ \mathbf{x}_{\mu}^2 \end{pmatrix}, \quad \mathbf{b}_{\mu} = \begin{pmatrix} \mathbf{b}_{\mu}^{\theta} \\ \mathbf{b}_{\mu}^{\varphi} \end{pmatrix}, \\ \mathbf{x}_{\mu}^{1,2} &= \begin{pmatrix} a_{n(\mu),\mu}^{1,2} \\ \vdots \\ a_{N_{\mu}}^{1,2} \end{pmatrix}, \quad \mathbf{b}_{\mu}^{\theta,\varphi} = \begin{pmatrix} b_{\mu}^{\theta,\varphi}(\theta_0) \\ \vdots \\ b_{\mu}^{\theta,\varphi}(\theta_{P'}) \end{pmatrix}, \\ \mathbf{A}_{\mu} &= \begin{pmatrix} \boldsymbol{\alpha}_{\mu} & \boldsymbol{\beta}_{\mu} \\ i\boldsymbol{\beta}_{\mu} & i\boldsymbol{\alpha}_{\mu} \end{pmatrix}, \\ \boldsymbol{\alpha}_{\mu} &= \begin{pmatrix} X_{n(\mu),\mu}^{\theta}(\theta_0, 0) & \cdots & X_{N_{\mu}}^{\theta}(\theta_0, 0) \\ \vdots & \ddots & \vdots \\ X_{n(\mu),\mu}^{\theta}(\theta_{P'}, 0) & \cdots & X_{N_{\mu}}^{\theta}(\theta_{P'}, 0) \end{pmatrix}, \\ \boldsymbol{\beta}_{\mu} &= -i \begin{pmatrix} X_{n(\mu),\mu}^{\varphi}(\theta_0, 0) & \cdots & X_{N_{\mu}}^{\varphi}(\theta_0, 0) \\ \vdots & \ddots & \vdots \\ X_{n(\mu),\mu}^{\varphi}(\theta_{P'}, 0) & \cdots & X_{N_{\mu}}^{\varphi}(\theta_{P'}, 0) \end{pmatrix}, \\ \boldsymbol{\kappa}_{\mu} &= \begin{pmatrix} \mathbf{w} & \mathbf{0} \\ \mathbf{0} & \mathbf{w} \end{pmatrix}, \end{aligned} \quad (22)$$

$$\mathbf{w} = \begin{pmatrix} w_0 & 0 & \cdots & 0 \\ 0 & w_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & w_{P'} \end{pmatrix}.$$

Finally, with the definitions

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{-M} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{-M+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_M \end{pmatrix},$$

$$\boldsymbol{\kappa} = \begin{pmatrix} \boldsymbol{\kappa}_{-M} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\kappa}_{-M+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\kappa}_M \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_{-M} \\ \vdots \\ \mathbf{x}_M \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{b}_{-M} \\ \vdots \\ \mathbf{b}_M \end{pmatrix}, \quad (23)$$

we have

$$\begin{aligned} \mathbf{b} &\approx \hat{\mathbf{b}} = \mathbf{A} \mathbf{x}, \\ E &= \|\mathbf{x}\|^2, \end{aligned} \quad (24)$$

$$\Delta = (\mathbf{A} \mathbf{x} - \mathbf{b})^* \boldsymbol{\kappa} (\mathbf{A} \mathbf{x} - \mathbf{b}).$$

Thus, the measurement truncation problem has been reduced, explicitly, to the constrained least-square optimization problem discussed earlier. In particular, (10) and (11) can be written as

$$\mathbf{A}_{\mu}^* \boldsymbol{\kappa}_{\mu} \mathbf{A}_{\mu} \mathbf{x}_{\mu} - \lambda \mathbf{x}_{\mu} = \mathbf{A}_{\mu}^* \boldsymbol{\kappa}_{\mu} \mathbf{b}_{\mu}, \quad -M \leq \mu \leq M, \quad (25)$$

$$E = \sum_{\mu=-M}^M \|\mathbf{x}_{\mu}\|^2. \quad (26)$$

Because of the block-diagonal structure, the overall SVD can be broken into smaller parts, one μ at a time, resulting in an algorithm with computational complexity $\mathcal{O}(N^4)$. Comparing unfavorably with the $\mathcal{O}(N^3)$ complexity for the full-sphere case, our method may be less useful for larger antennas. The order of $\mathbf{A}_{\mu}$ is $2(P' + 1) \times 2(N - n(\mu) + 1)$, so there will be more equations than unknowns in every block $|\mu| \leq M$ as long as $P' \geq N$. Actually, there is always a unique solution to our constrained optimization problem; however, the quality of this solution is observed to degrade dramatically as P' is decreased below the threshold $P' = N$.

3. Experimental and Simulated Results

To investigate the utility of our least-squares technique for reducing the truncation error that results from zero filling in the rear hemisphere, we examined a number of typical probes for which full-sphere far-field patterns are available. In each case, we computed the far-field pattern using the spherical mode expansion obtained (a) from the standard algorithm with *full-sphere* data, (b) from the standard algorithm with *zero-fill* in the backward-hemisphere, and finally, (c) from the constrained *least-squares* fit to forward-hemisphere measurements described previously. Ideally, the far-field pattern calculated from the spherical-mode expansion should agree with the original pattern. For the *full-sphere* modal calculation, this is of course the case if the sampling theorem is satisfied. We show E-plane results only,

since truncation effects are more important because the patterns tend to be broader. For all cases, the sampling is 2° in θ and 5° in ϕ .

3.1. High-Gain Symmetric Probe. In this case, the input data were simulated using a far-field pattern calculated from a specified set of ($m = \pm 1$) modal coefficients. For this antenna, we used $N = 10$, $M = 5$. The directivity of 15.44 dB was used in (8) to determine the energy constraint. Figures 1 and 2 indicate excellent agreement between the *full-sphere* and *least-squares* techniques in the forward hemisphere.

In this example, the *constrained least-squares* technique also performs well in the backward-hemisphere if the correct energy is specified. This is not necessarily true in the presence of noise. As a test, we added random errors with an RMS value of 0.1% relative to the noise-free RMS pattern level. An *unconstrained* fit resulted in an on-axis directivity of -32 dB. In other words, the back hemisphere overwhelmed the forward hemisphere. On the other hand, a *constrained* fit forced the back hemisphere pattern to remain at a reasonable level even though the details were incorrect. Although an *unconstrained* fit may be somewhat better in the forward directions, a *constrained* fit is undoubtedly more reasonable given our knowledge of the directivity of the test antenna.

In probe-corrected spherical near-field measurements, for example, the complete probe pattern is required. We often argue that the forward hemisphere pattern is most important, but all bets are off when the back hemisphere pattern dwarfs the forward hemisphere pattern.

3.2. NIST Circularly Cylindrical Waveguide Probe. This cylindrical waveguide probe was designed and built for spherical near-field measurements at 3.3 GHz. The far-field pattern was measured over the entire sphere, although support-structure blockage limits the value of backward-hemisphere information. In this case, we used $N = 17$, $M = 17$. A directivity of 7.87 dB was calculated using the *full-sphere* data. Deviations (Figure 3) between the *full-sphere* and *zero-fill* methods will lead to significant errors when the *zero-fill* results are used in spherical scanning measurements. Within the range $\pm 75^\circ$, the maximum error in the *zero-fill* result is about 0.6 dB, while the *least-squares* technique has a maximum error of 0.08 dB. Figure 4 shows how the *full-sphere*, *zero-fill*, and *least-squares* results compare in the backward-hemisphere. In this case, the truncation level is as high as -12 dB, which causes a greater discrepancy between the *zero-fill* and *full-sphere* patterns than observed in the previous example.

3.3. Rectangular Waveguide Probe. This probe is a section of WR-284 rectangular waveguide. Far-field patterns were obtained over the entire sphere at 3.3 GHz. In this case, we used $N = 17$, $M = 17$. Strictly speaking, rectangular waveguide probes do not have the correct symmetry for spherical near-field scanning applications; however, they often perform satisfactorily when the scan radius is more than about an antenna diameter. Thus, we include results for this type of probe. A directivity of 7.12 dB, calculated from the *full-sphere* modal expansion, was used to determine the

FIGURE 1: Simulated probe: deviations from the *full-sphere* pattern (errors) are shown for the *zero-fill* (long dash) and *least-squares* (short dash) techniques. The *full-sphere* pattern is shown as a solid line.

FIGURE 2: Simulated probe: comparison of the *full-sphere* E-plane far-field patterns obtained using *zero-fill* (long dash) and *least-squares* (short dash) techniques. The *full-sphere* pattern is shown as a solid line. The *least-squares* and *full-sphere* patterns are indistinguishable at this scale.

energy constraint. Measured gain was 6.7 dB. The results are similar to those for the cylindrical waveguide case. As evident in Figure 5, within the angular range $\pm 75^\circ$, the *zero-fill* error is greatest at about 40° , where it is 0.66 dB. The error is only about 0.11 dB for the *least-squares* technique at this angle. In the forward hemisphere, the *zero-fill* technique exhibits large errors, while the *least-squares* technique errors remain

FIGURE 3: Cylindrical waveguide probe: deviations from the *full-sphere* pattern (errors) are shown for the *zero-fill* (long dash) and *least-squares* (short dash) techniques. The *full-sphere* pattern is shown as a solid line.

FIGURE 5: Rectangular waveguide probe: deviations from the *full-sphere* pattern (errors) are shown for the *zero-fill* (long dash) and *least-squares* (short dash) techniques. The *full-sphere* pattern is shown as a solid line.

FIGURE 4: Cylindrical waveguide probe: comparison of the full-sphere E-plane far-field patterns obtained using the *zero-fill* (long dash) and the *least-squares* (short dash) techniques. The *full-sphere* pattern is shown as a solid line.

FIGURE 6: Rectangular waveguide probe: comparison of the full-sphere E-plane far-field patterns obtained using the *zero-fill* (long dash) and the *least-squares* (short dash) techniques. The *full-sphere* pattern is shown as a solid line.

relatively small. Figure 6 provides a full-sphere comparison of the three techniques.

3.4. Choice of E . Several tests indicate that the quality of the *least-squares* fit in the forward hemisphere is relatively insensitive to the choice of E . In a typical case with $0.9E_0 \leq E \leq 1.1E_0$, the errors observed in the results were generally much less than 0.15 dB within 20° of the on-axis direction. (Here, E_0 is the correct value). When gain is used to

approximate directivity, experience indicates that the error in E usually will be less than 10%.

4. Summary and Future Work

We have demonstrated a constrained *least-squares* technique for calculating the spherical mode coefficients of a small antenna from forward-hemisphere far-field data. This technique can significantly reduce truncation errors that arise

when the standard algorithm is used with *zero-fill* in the backward-hemisphere. While the *least-squares* algorithm is not especially efficient, computational times are still acceptable for small antennas.

Our technique can also be used to process near-field data when reliable full-sphere measurements are not available. The method may be adapted to serve when measurements are made in the range $0 \leq \theta \leq \theta_0$, where θ_0 is not restricted to 90° . We are considering an iterative algorithm to find both $\mathbf{x}$ and λ simultaneously. For well-conditioned systems, this could effectively reduce the computational complexity to $O(N^3)$. By beginning with a spherical-wave expansion and projecting onto a scanning surface, our method should be useful for mitigation of truncation error in planar and cylindrical scanning measurements, as well.

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