Research Article

Cramér-Rao Bound Study of Multiple Scattering Effects in Target Separation Estimation

Edwin A. Marengo and Paul Berestesky

Department of Electrical and Computer Engineering, Northeastern University, Boston, MA 02115, USA

Correspondence should be addressed to Edwin A. Marengo; emarengo@ece.neu.edu

Received 16 January 2013; Accepted 2 April 2013

Academic Editor: Stefano Selleri

Copyright © 2013 E. A. Marengo and P. Berestesky. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The information about the distance of separation between two-point targets that is contained in scattering data is explored in the context of the scalar Helmholtz operator via the Fisher information and associated Cramér-Rao bound (CRB) relevant to unbiased target separation estimation. The CRB results are obtained for the exact multiple scattering model and, for reference, also for the single scattering or Born approximation model applicable to weak scatterers. The effects of the sensing configuration and the scattering parameters in target separation estimation are analyzed. Conditions under which the targets' separation cannot be estimated are discussed for both models. Conditions for multiple scattering to be useful or detrimental to target separation estimation are discussed and illustrated.

1. Introduction

An important question in imaging and inverse scattering is the quantification of theoretical limits in the information that can be extracted about parameters of a wave scatterer from given scattered field data. This question, with particular interest in the quantification of limits related to target localizability and resolution, has been tackled in a number of papers (see [1–3] and the references therein) via the statistical signal processing framework of the Fisher information and the associated Cramér-Rao bound (CRB) [4]. This approach quantifies the best precision with which scattering parameters can be estimated in the statistical framework of unbiased estimation under given signal corruption or noise models. Importantly, this quantification is algorithm-independent and showcases the role of both scattering parameters and imaging or sensing configuration. Furthermore, this theoretical approach holds under nonnegligible multiple scattering conditions where the mapping from object function to data is nonlinear, which prevents the direct application of the standard diffraction limits ($\lambda/2$ rule of thumb) of inverse scattering problems under the Born approximation as well as inverse source problems where the respective map is linear

and therefore tractable via band limitation considerations in spatial Fourier domain.

The present paper expands our research on CRB analysis of multiple scattering effects in the estimation of target parameters initiated in a previous paper [1] coauthored by the current authors. In that paper, we characterized analytically and computationally the roles of the sensing configuration and the scattering parameters in the task of localizing two point targets. Concrete conditions were derived under which localization is facilitated or obstructed. The separate roles of the sensing configuration and the scattering parameters were isolated and interpreted. The results were obtained within the exact scattering model including multiple scattering and, for reference purposes, also under the approximate first Born approximation scattering model so as to obtain insight on the role of multiple scattering in either facilitating or obstructing target localization relative to the baseline provided by the Born approximation. To ease mathematical tractability and insight, in that paper the focus was on the information about the targets' positions under the assumption that the target scattering strengths and the separation of the two targets are known. In the present paper we address the complementary question about the information on the targets' separation if

the position of one of the targets is known or if the position of the center of the two targets is known, which is related to the question of how well the two targets can be resolved from the scattering data.

The original contributions of the present paper can be summarized as follows. First, we provide the closed-form Fisher information and CRB expressions applicable to estimation problems relevant to target resolution and exploit the implications of the resulting developments with the aid of computer illustrations. Second, the present work expands the current understanding of the role of multiple scattering in imaging resolution by considering two different physical situations: one where prior knowledge of the position of one of the scatterers is available and another where prior knowledge of the center of the composite two-scatterer target is available. We derive for these two different scenarios the necessary and sufficient conditions under which the estimation of the targets' separation is impeded within the exact multiple scattering model. In addition, we derive analytically and illustrate computationally the conditions under which multiple scattering outperforms the Born approximation predictions and vice versa where the Born approximation predictions are unrealistically optimistic. As explained in [1], the estimation performance depends on both conditions intrinsic to the data, in particular the sensing configuration, as well as conditions that depend on the target, in the present case the scattering parameters of target strengths and positions. The adoption of the canonical system of two point scatterers allows us to gain insight into the role of both sensing configuration and scattering parameters and, in particular, the mathematical expressions for Fisher information and CRB derived in the paper demonstrate factors that depend only on configuration or on parameters as well as more complex factors that depend on both.

The remainder of the paper is organized as follows. Section 2 reviews the forward scattering model. Section 3 presents the CRB results for target separation estimation. Section 4 provides numerical illustrations of the derived theory. Section 5 provides concluding remarks. The appendix summarizes the basic Fisher information and CRB derivation.

2. Review of the Multiple Scattering Model

Following [1], we consider scattering within the framework of the scalar Helmholtz operator in three-dimensional free space, $\nabla^2 + k^2$, where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator and $k = 2\pi/\lambda$ is the wavenumber of the field corresponding to wavelength λ . For analytical tractability and insight, attention is restricted to a system of two-point scatterers having complex scattering strengths τ_1 and τ_2 and positions $\mathbf{R}_1 = (0, 0, d_1)$ and $\mathbf{R}_2 = (0, 0, d_2)$ in the z -axis as shown in Figure 1. We assume that $d_2 > d_1$ so that the targets' separation $d = d_2 - d_1 > 0$. Consider the incident plane waves $e^{i\mathbf{k}\cdot\mathbf{r}}$ traveling in the direction of the unit vector $\mathbf{s}_i$ corresponding to incidence polar angle α ($\cos \alpha = \mathbf{s}_i \cdot \hat{\mathbf{z}}$) and far-zone sensing in the direction $\mathbf{s}$ corresponding to scattering polar angle β ($\cos \beta = \mathbf{s} \cdot \hat{\mathbf{z}}$). These

FIGURE 1: Conceptual illustration of the scattering system.

(polar) angles lie in the range $[0, \pi]$. For this two-scatterer system under plane wave excitation, the scattering amplitude including multiple scattering is given by [1] as follows:

$$f(\alpha, \beta) = F(d, \tau_1, \tau_2) e^{ikd_1 g(\alpha, \beta)} \times [\tau_1 + \tau_2 e^{ikd g(\alpha, \beta)} + \tau_1 \tau_2 G(d) Q(d, \alpha, \beta)], \quad (1)$$

where

$$G(d) = -\frac{e^{ikd}}{4\pi d},$$

$$F(d, \tau_1, \tau_2) = [1 - \tau_1 \tau_2 G^2(d)]^{-1},$$

$$g(\alpha, \beta) = \cos \alpha - \cos \beta, \quad (2)$$

$$Q(d, \alpha, \beta) = e^{ikd \cos \alpha} + e^{-ikd \cos \beta}$$

$$= 2e^{i(kd/2)g(\alpha, \beta)} \cos\left(\frac{kd}{2} g'(\alpha, \beta)\right),$$

where

$$g'(\alpha, \beta) = \cos \alpha + \cos \beta. \quad (3)$$

Also, it is assumed that $\tau_1 \tau_2 G^2(d) \neq 1$ (nonresonance condition). For the special case of weak scatterers where $|\tau_m G(d)| \ll 1$, $m = 1, 2$, this takes the first Born approximation form $f(\alpha, \beta) \approx f_{\text{Born}}(\alpha, \beta)$, where

$$f_{\text{Born}}(\alpha, \beta) = e^{ikd_1 g(\alpha, \beta)} [\tau_1 + \tau_2 e^{ikd g(\alpha, \beta)}]. \quad (4)$$

3. Target Separation Estimation

This section describes the statistical information that is contained in scattering data about the separation distance d between two known point scatterers. In imaging systems, the ability to estimate the separation distance d of two-point targets is usually regarded as a metric for the resolution or ability of the imaging system to resolve target details. The smaller the separation distance d of two-point targets that can be properly estimated, the higher the resolution. The question is how the resolution is affected by the target parameters and the remote sensing configuration, particularly in the case when multiple scattering is significant. In the following, we study this question in the exact multiple scattering framework. For reference and to facilitate interpretation of the results, we

also derive the respective Born approximation results valid for weak scatterers. We derive and discuss the necessary and sufficient condition under which it is theoretically impossible to resolve the targets (“no-resolution condition”). This, of course, implies as a corollary the contrary “resolution condition” under which the targets can be resolved. We also comparatively examine the predictions of the exact and Born approximation models, paying particular attention to contrasting the degrees of freedom associated to the no-resolution conditions of the two models, and investigate the conditions where multiple scattering is beneficial or detrimental to resolution relative to the Born approximation model limits. In the following analysis, we consider two different scenarios associated to two different physical and informational situations. One is the case where the position of one of the targets, say target 1, is known (d_1 is known). The other case is where the position of the center of the two targets $d_c = (d_1 + d_2)/2$ is known. The results for the two cases are different but have the same general mathematical structure and, therefore, lead to similar general conclusions about the role of the scattering and configuration parameters in the task of resolving the targets and the differences between the exact and approximate models. A summary of the basic Fisher information and CRB results needed to carry out the computations of this section are given in the appendix.

3.1. Calculation for Known- d_1 Case. The following calculation assumes prior knowledge of the first scatterer’s position, d_1 , as well as of the scattering strengths, τ_1 and τ_2 . Adopting the Born approximation model (A.3), the Fisher information $I^{(n)}(d)$ of the n th scattering experiment corresponding to given pairs (α_n, β_n) of incidence and scattering angles $\alpha_n \in [0, \pi]$ and $\beta_n \in [0, \pi]$, respectively, is found from (A.7) with $\xi = d$ to be given by

$$I_{\text{Born}}^{(n)}(d) = 2\sigma^{-2}k^2|\tau_2|^2g^2(\alpha_n, \beta_n). \quad (5)$$

On the other hand, in the exact multiple scattering model, we obtain the following from (A.2) and (A.7):

$$I^{(n)}(d) = 2\sigma^{-2}|\tau_2|^2|F(d, \tau_1, \tau_2)|^2|C(\alpha_n, \beta_n)|^2, \quad (6)$$

where

$$\begin{aligned} C(\alpha_n, \beta_n) &= ikg(\alpha_n, \beta_n)e^{ikdg(\alpha_n, \beta_n)} + \tau_1 G(d) \\ &\times \left\{ \frac{\partial}{\partial d} Q(d, \alpha_n, \beta_n) + \left(ik - \frac{1}{d} \right) \right. \\ &\times \left. \left[Q(d, \alpha_n, \beta_n) + 2G(d)e^{-ikd_1g(\alpha_n, \beta_n)}K_n(d) \right] \right\}. \end{aligned} \quad (7)$$

As in the localization problem considered in [1], where $I^{(n)}(d_1) \propto g^2(\alpha_n, \beta_n)$ and $I_{\text{Born}}^{(n)}(d_1) \propto g^2(\alpha_n, \beta_n)$, the Fisher information $I_{\text{Born}}^{(n)}(d)$ in the Born approximation model (5) is also proportional to $g^2(\alpha_n, \beta_n)$, which implies that under the condition $g(\alpha_n, \beta_n) = 0$ (which is equivalent to the line-of-sight (LOS) condition, $\alpha_n = \beta_n$) no information is contained

in the data about the target separation d . However, this is not the case under the multiple scattering model, where, according to (6) and (7), $I^{(n)}(d)$ does not necessarily vanish if $g(\alpha_n, \beta_n) = 0$ or $\alpha_n = \beta_n$; that is, the respective LOS data may carry information about d . This issue is illustrated and discussed further as follows.

The two special cases of LOS data ($\alpha_n = \beta_n$) and of backscattering data ($\alpha_n = \pi - \beta_n$) allow further simplifications which facilitate visualization. Under the LOS condition, the Fisher information is given by (6) where $C(\alpha_n, \alpha_n)$ defined by (7) takes the particular form

$$\begin{aligned} C(\alpha_n, \alpha_n) &= \tau_1 G(d) \left\{ -2k \cos \alpha_n \sin(kd \cos \alpha_n) + \left(ik - \frac{1}{d} \right) \right. \\ &\times \left[2 \cos(kd \cos \alpha_n) + \frac{2G(d)}{1 - \tau_1 \tau_2 G^2(d)} \right. \\ &\times \left. \left. \left(\tau_1 + \tau_2 + 2\tau_1 \tau_2 G(d) \cos(kd \cos \alpha_n) \right) \right] \right\}. \end{aligned} \quad (8)$$

In this LOS condition, the Born approximation model applicable to weak targets does not permit the estimation of the targets’ separation. This is a very specific situation where categorically only nonweak scatterers exhibiting nonnegligible multiple scattering can reveal the targets’ separation information. On the other hand, for very specialized combinations of parameters, $C(\alpha_n, \alpha_n)$ in (8) vanishes so that the respective LOS Fisher information $I^{(n)}(d)$ vanishes as well. For example, if we let $d > 0$ be arbitrary while α_n is chosen such that $kd \cos \alpha_n$ takes one of the discrete values $\pm(2p+1)\pi/2$, $p = 0, 1, 2, \dots, p_{\max} = [(kd/\pi) - 1/2]$, then the choice of scattering strengths $\tau_1 = \tau = \tau_2$ where

$$\begin{aligned} \tau &= 4\pi(-1)^p e^{-ikd} \\ &\times \left[\left(id - \frac{1}{k} \right) \sec \alpha_n \mp \sqrt{\left(id - \frac{1}{k} \right)^2 \sec^2 \alpha_n + 1} \right] \end{aligned} \quad (9)$$

gives $I^{(n)}(d) = 0$.

For backscattering data, the Fisher information about d is given by (6) with $\beta_n = \pi - \alpha_n$ where the corresponding $C(\alpha_n, \pi - \alpha_n)$ is given by

$$\begin{aligned} C(\alpha_n, \pi - \alpha_n) &= 2ik \cos \alpha_n e^{2ikd \cos \alpha_n} + 2\tau_1 G(d) \\ &\times \left\{ ik \cos \alpha_n e^{ikd \cos \alpha_n} + \left(ik - \frac{1}{d} \right) \right. \\ &\times \left[e^{ikd \cos \alpha_n} + \frac{G(d)}{1 - \tau_1 \tau_2 G^2(d)} \right. \\ &\times \left. \left. \left(\tau_1 + \tau_2 e^{2ikd \cos \alpha_n} + 2\tau_1 \tau_2 G(d) e^{ikd \cos \alpha_n} \right) \right] \right\}. \end{aligned} \quad (10)$$

Note that, according to (5), within the Born approximation for weak scatterers, the Fisher information is proportional to $g^2(\alpha_n, \pi - \alpha_n) = 2 \cos \alpha_n$ so that it vanishes for $\alpha_n = \pi/2$. This situation contrasts with the exact model result (10) which gives

$$C\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 2\tau_1 G(d) \left(ik - \frac{1}{d}\right) \left\{1 + \frac{G(d) [\tau_1 + \tau_2 + 2\tau_1 \tau_2 G(d)]}{1 - \tau_1 \tau_2 G^2(d)}\right\} \quad (11)$$

which vanishes only for specialized values of the parameters d , τ_1 , and τ_2 .

3.2. Calculation for Known- d_c Case. Alternatively, we may consider another scenario where the center of the two targets, $d_c = (d_1 + d_2)/2$, is known. Physically and informationally, this is a different situation from the one of the preceding results and, therefore, gives different Fisher information and CRB results. In this case, the d -dependence of the data is conveniently highlighted by rewriting (A.2) and (A.3) as

$$\begin{aligned} K_n(d) &= F(d, \tau_1, \tau_2) e^{ik(d_c - d/2)g(\alpha_n, \beta_n)} \\ &\quad \times [\tau_1 + \tau_2 e^{ikdg(\alpha_n, \beta_n)} + \tau_1 \tau_2 G(d) Q(d, \alpha_n, \beta_n)], \\ K_n^{\text{Born}}(d) &= e^{ikd_c g(\alpha_n, \beta_n)} [\tau_1 e^{-i(kd/2)g(\alpha_n, \beta_n)} + \tau_2 e^{i(kd/2)g(\alpha_n, \beta_n)}] \\ &= e^{ik(d_c - d/2)g(\alpha_n, \beta_n)} [\tau_1 + \tau_2 e^{ikdg(\alpha_n, \beta_n)}]. \end{aligned} \quad (12)$$

The corresponding Fisher information, to be denoted as $I_{\text{Born}}^{(n)}(d)$ and $I^{(n)}(d)$, respectively, is found from (12) and (A.7) to be given by

$$I_{\text{Born}}^{(n)}(d) = \frac{k^2}{2} g^2(\alpha_n, \beta_n) \text{DNR}_{\text{Born}}(\alpha_n, \beta_n), \quad (13)$$

where

$$\text{DNR}_{\text{Born}}(\alpha, \beta) = \frac{1}{\sigma^2} |\tau_1 - \tau_2 e^{ikdg(\alpha, \beta)}|^2, \quad (14)$$

$$I^{(n)}(d) = \frac{2}{\sigma^2} |F(d, \alpha_n, \beta_n)|^2 |C'(d, \alpha_n, \beta_n)|^2, \quad (15)$$

where

$$\begin{aligned} C'(d, \alpha_n, \beta_n) &= -\frac{i}{2} kg(\alpha_n, \beta_n) [\tau_1 - \tau_2 e^{ikdg(\alpha_n, \beta_n)}] \\ &\quad + \tau_1 \tau_2 G(d) \left[\left(ik - \frac{1}{d}\right) Q(d, \alpha_n, \beta_n) \right. \\ &\quad \left. + \frac{i}{2} kg'(\alpha_n, \beta_n) Q'(d, \alpha_n, \beta_n) + 2G(d) \right. \\ &\quad \left. \times \left(ik - \frac{1}{d}\right) e^{-ik(d_c - d/2)g(\alpha_n, \beta_n)} K_n(d) \right], \end{aligned} \quad (16)$$

$$\begin{aligned} Q'(d, \alpha, \beta) &= e^{ikd \cos \alpha} - e^{-ikd \cos \beta} \\ &= 2ie^{i(kd/2)g(\alpha, \beta)} \sin\left(\frac{kd}{2} g'(\alpha, \beta)\right). \end{aligned} \quad (17)$$

The behavior of $I_{\text{Born}}^{(n)}(d)$ in (13) applicable to the known- d_c case is similar to that in the counterpart result (5) for known d_1 in that the Fisher information is proportional to the quantity $g^2(\alpha_n, \beta_n)$. Again, the Fisher information vanishes for $g(\alpha_n, \beta_n) = 0$, that is, $\alpha_n = \beta_n$, so that under this LOS condition d cannot be estimated. However, instead of being proportional to $|\tau_2|^2$ as in (5), the Fisher information in (13) is proportional to DNR_{Born} which is a function of both scattering strengths and the incidence and sensing angles. Previously, because d_1 was known, an estimation of d was effectively an estimation of d_2 and, therefore, the problem was essentially of locating the second scatterer. Hence, a stronger second scatterer strength leads directly to better-resolution information. In contrast, in the known- d_c case, because it is the center point that is known, an estimation of d is effectively the estimation of both d_1 and d_2 where it is known that the scatterers are equidistant from the known position d_c on the z -axis. Thus, within the Born approximation, the two estimation problems (known d_1 and known d_c) have a fundamental difference in that, while for known d_1 the no-resolution condition is the LOS condition, $\alpha_n = \beta_n$, for known d_c , it is the LOS condition $\alpha_n = \beta_n$ or $\text{DNR}_{\text{Born}}(\alpha_n, \beta_n) = 0$. Thus, unlike in the known- d_1 case, in the known- d_c case, zero resolution information is possible for certain values of the scattering parameters at non-LOS (NLOS) angles $\alpha_n \neq \beta_n$. On the other hand, the exact multiple scattering expression (15) resembles its known- d_1 counterpart (6). In this case, the angular dependence is more complex and $g(\alpha_n, \beta_n) = 0$ does not imply zero information about d . In the following, we elaborate the no-resolution conditions for both the exact and approximate models (whose disobedience defines, of course, the contrary “resolution conditions” under which it is theoretically possible to resolve the targets).

3.3. No-Resolution Condition: Known- d_1 Case

3.3.1. Born Approximation. As explained in the preceding paragraph, it follows from (5) that, under the Born approximation, the no-resolution condition for the known- d_1 case

is simply the LOS requirement, $\alpha_n = \beta_n$. Note that this condition depends only on the sensing angles and hence applies to any value of the scattering parameters d , τ_1 and τ_2 . In particular, this no-resolution condition defines a plane which has 6 degrees of freedom in the 7-dimensional parameter space associated to $(d, \tau_1, \tau_2, \alpha_n, \text{ and } \beta_n)$.

3.3.2. Exact Model. Note that $F(d, \tau_1, \tau_2) \neq 0$ for finite parameters d , τ_1 , and τ_2 . It follows from (6) and (7) that $I^{(n)}(d) = 0$ if and only if $C(\alpha_n, \beta_n) = 0$, in particular,

$$\begin{aligned} & ikg(\alpha_n, \beta_n) e^{ikdg(\alpha_n, \beta_n)} + \tau_1 G(d) \\ & \times \left\{ \frac{\partial}{\partial d} Q(d, \alpha_n, \beta_n) + \left(ik - \frac{1}{d} \right) \right. \\ & \left. \times [Q(d, \alpha_n, \beta_n) + 2G(d) e^{-ikd_1 g(\alpha_n, \beta_n)} K_n(d)] \right\} = 0. \end{aligned} \quad (18)$$

By using expression (A.2) for $K_n(d)$ in (18) and manipulating the resulting equation for $(d, \tau_1, \tau_2, \alpha_n, \beta_n)$, it is not hard to show that this equation cannot be obeyed for arbitrary values of the scattering strengths τ_1 and τ_2 in clear contrast with the corresponding Born approximation no-resolution condition (the LOS condition), which holds for any τ_1 and τ_2 . The Born approximation condition corresponds to 6 degrees of freedom which is dimensionally less restrictive than the exact multiple scattering condition, suggesting that as the targets scatter more it also becomes less likely for the scattering and configuration parameters to be such that zero information is available about the targets' separation d . In particular, it follows from (A.2) that if the values of d , α_n , β_n , and τ_1 (or τ_2) are arbitrarily fixed, condition (18) reduces to a quadratic equation in τ_2 (or τ_1) which gives either two different or one (double-root) solution for τ_2 (or τ_1). The number of degrees of freedom of this no-resolution condition is 5 in the 7-dimensional parameter space of $(d, \tau_1, \tau_2, \alpha_n, \beta_n)$. On the other hand, while in the Born approximation the angles for no-resolution are restricted by $\alpha_n = \beta_n$, under exact multiple scattering, the angles for no-resolution can take any value as long as the other parameters are such that condition (18) holds.

3.4. No-Resolution Condition: Known- d_c Case

3.4.1. Born Approximation. As outlined previously, it follows from (13) that, for the Born approximation, the no-resolution condition for the known- d_c case is

$$\alpha_n = \beta_n, \quad \text{or} \quad \text{DNR}_{\text{Born}}(\alpha_n, \beta_n) = 0. \quad (19)$$

Thus, there are two alternative ways of having zero resolution information. One is the LOS condition, which involves 6 degrees of freedom in the 7-dimensional parameter space.

Another is the zero DNR_{Born} condition, which can be put explicitly as

$$\begin{aligned} |\tau_1| &= |\tau_2|, \\ kd g(\alpha_n, \beta_n) &= kd (\cos \alpha_n - \cos \beta_n) = \theta_1 - \theta_2 + l\pi, \end{aligned} \quad (20)$$

$$l = \pm 2p, \quad p = 0, 1, 2, \dots, p_{\max},$$

where $p_{\max}$ is constrained by

$$4p_{\max}^2 \pm \frac{4}{\pi} p_{\max} (\theta_1 - \theta_2) + \frac{1}{\pi^2} (\theta_1 - \theta_2)^2 \leq 16 \frac{d^2}{\lambda^2}. \quad (21)$$

The condition involves 5 degrees of freedom in 7-dimensional space. This can be readily visualized by noting that the first condition in (20) defines two planes ($\tau_1 = \pm \tau_2$) in the parameter space, reducing the dimensionality from 7 to 6, while the second condition defines for any $(\theta_1, \theta_2, \alpha_n, \beta_n)$ a countably infinite set of values of d . In particular, the values $d_p^\pm > 0$, $p = 0, 1, 2, \dots, \infty$, obeying

$$d_p^\pm = \left[\frac{(\theta_1 - \theta_2)}{2\pi} \pm p \right] \frac{\lambda}{\cos \alpha_n - \cos \beta_n}, \quad p = 0, 1, 2, \dots, \infty, \quad (22)$$

implying the loss of another dimension for a total of 5 degrees of freedom. The $\text{DNR}_{\text{Born}} = 0$ alternative has less degrees of freedom than the LOS condition. However, it offers greater flexibility to the no-resolution angles which are not required to be the same under this other condition.

3.4.2. Exact Model. It follows from (15) and (16) that, within the exact model, the no-resolution condition is $C'(\alpha_n, \beta_n) = 0$, that is,

$$\begin{aligned} & -\frac{i}{2} kg(\alpha, \beta) [\tau_1 - \tau_2 e^{ikdg(\alpha, \beta)}] + \tau_1 \tau_2 G(d) \\ & \times \left[\left(ik - \frac{1}{d} \right) Q(d, \alpha, \beta) + \frac{i}{2} kg'(\alpha, \beta) Q'(d, \alpha, \beta) \right. \\ & \left. + 2G(d) \left(ik - \frac{1}{d} \right) e^{-ik(d_c - d/2)g(\alpha, \beta)} K_n(d) \right] = 0. \end{aligned} \quad (23)$$

This result has the same general form as the known- d_1 counterpart (18) and yields, through the methodology outlined above, the same general conclusions. For instance, it is a constraint of 5 degrees of freedom. The comparative discussion of the exact and approximate models given for the known- d_1 case still applies. However, there is a minor difference which is that, in the known- d_c case, the Born approximation model allows situations where estimation of the targets' separation is impossible at arbitrary sensing angles unlike in the known- d_1 case where the LOS condition is the only way of impeding estimation of this distance.

4. Numerical Illustrations

4.1. Single Observation. Next we discuss a selection of single-observation experiments which illustrate how variations in

the system's parameters (scatterer separation d , scatterer strengths τ_1 and τ_2) and observer configuration (incident and observation angles α_n and β_n) affect the estimation of the targets' separation. In generating the following plots, we use $\sigma^2 = 1$ so that the plotted CRB results are normalized by the noise variance σ^2 . In some of the plots, the CRB of the targets' separation d , $\text{CRB}(d)$, is additionally normalized by d^2 (equivalently $\sqrt{\text{CRB}(d)}$ is normalized by σd). This highlights the estimation error relative to d which facilitates interpretation. Also, we consider unit value wavelength $\lambda = 1$ so that the wavenumber $k = 2\pi/\lambda = 2\pi$ and all distances (e.g., d) can be given in the plots in terms of the wavelength.

Figure 2 shows plots of $\text{CRB}(d)$ for $\tau_1 = 1 = \tau_2$ as a function of the observation angle β while the incidence angle is held constant at $\alpha = 0$. Both models (exact and approximate), as well as the two formulation cases (known- d_1 and known- d_c cases), are plotted together for comparison. Two values of the separation d are considered as follows: $d = \lambda/4$ in Figure 2(a) and $d = \lambda/2$ in Figure 2(b). In both the known- d_1 case and the known- d_c case, the Fisher information under the Born approximation is proportional to g^2 (see (5), (13)) so that (as shown in the plots) for $\beta = 0$, $\text{CRB}(d) = \infty$, since then $g = 0$. Also, in the known- d_c case, the Fisher information under the Born approximation is proportional to DNR_{Born} (see (13)) which vanishes for ($d = \lambda/2$, $\beta = \pi$) so that $\text{CRB}(d) = \infty$ in that case, as is shown in the respective plot in Figure 2(b). On the other hand, the exact CRB is finite in all these cases: the information about the targets' separation contained in the data is thus greater than the Born approximation value. As a general trend, the exact CRB is seen to decrease with the observation angle, reaching its lowest values for the near-backscattering angles, except for the known- d_c case and $d = \lambda/2$ where the forward scattering and backscattering values are comparable. This can be understood intuitively from the corresponding Born approximation result where the CRB goes to infinity for both $\beta = 0$ and $\beta = \pi$. In addition to lacking blind spots, the exact CRB varies only smoothly with limited regions of raised bound or peaks such as the ones for the known- d_1 formulation near ($d = \lambda/2$, $\beta = 0.2\pi$) and for the known- d_c formulation near ($d = \lambda/4$, $\beta = 0.45\pi$).

Figures 3 and 4 show plots of $\text{CRB}(d)$ for backscattering experiments as a function of (d, α) where $d \in [0.1\lambda, 10\lambda]$ and $\alpha \in [0, \pi]$. The results are for an equal scatterer system having $\tau_1 = 1 = \tau_2$. Figure 3 shows the results for the known- d_1 case while Figure 4 shows the results for the known- d_c case. The dark areas highlight regions where the CRB is high. These do not explicitly correspond to blind conditions but the black regions do follow loci of infinite CRB conditions in general. The simplest plot is the Born approximation known- d_1 case result shown in Figure 3(b) which shows a single high CRB ridge along $\alpha = \pi/2$ where $g = 0$ so that, according to (5), the CRB is infinite. The Born approximation known- d_c case plot shown in Figure 4(b) contains the same ridge with the addition of a widening region of increasing bound in the vicinity of $\alpha = \pi/2$ for small separations d as well as a number of blind zones per given value of d which increases as d grows larger. All of this is in agreement with the analysis

FIGURE 2: $\text{CRB}(d)$ as the observation angle β varies, normalized by $(d\sigma)^2$, for target separations $d = \lambda/4$ and $d = \lambda/2$. The incidence angle is $\alpha = 0$ and the scatterer strengths are $\tau_1 = 1 = \tau_2$.

presented in (13), (20), and (21). The exact CRB results shown in these figures exhibit similar general trends with differences between the exact and approximate results becoming more prominent for smaller d . This is also as expected since this is where the multiple scattering becomes stronger. These effects appear to be favorable toward facilitating estimation of d as the blind conditions predicted by the Born approximation break up into sparser individual peaks or disappear entirely as d decreases.

Figures 5 and 6 illustrate further the results in Figures 3 and 4, showing the relation between the Fisher information for the multiple scattering and Born approximation models. The most noticeable difference between the known- d_1 and known- d_c cases is the shape of the alternating beneficial and harmful influence regions of the multiple scattering. In the known- d_1 case, the pattern resembles stripes rather than a checker-board as in the known- d_c case. As in the localization case studied in [1], the alternating pattern ends below some

(a) Multiple Scattering Model

(b) Born Approximation Model

FIGURE 3: Plots of $\text{CRB}(d)$ as a function of incidence angle α and scatterer separation d , normalized by σ^2 , for the case where d_1 is known. The observation angle $\beta = \pi - \alpha$ and $\tau_1 = 1 = \tau_2$.

value of d ($d \leq 0.05\lambda$ for known- d_1 and $d \leq 0.08\lambda$ for known- d_c). Notably, the behavior in this small d region is opposite from the localization case in that here $\text{CRB}(d)$ of the multiple scatter is consistently higher. This result is also consistent with the multiple observation results discussed later and illustrated in Figure 8(a) where the error bounds of the multiple scattering and Born approximation models averaged over all observation and incidence angles converge for higher values of d but, for lower values of d , the CRB of the multiple scattering model is consistently lower.

We consider next the LOS or forward scattering configuration, $\alpha = \beta$, for $\tau_1 = 1 = \tau_2$. In this configuration, the exact expressions for $\text{CRB}(d)$ in terms of d_1 and d_c become equal. On the other hand, the Born approximation predicts $\text{CRB}(d) = \infty$ due to the vanishing of g for this configuration. Figure 7 shows a plot of the CRB for the multiple scattering model. As expected, the CRB increases as d grows because the multiple scattering becomes less pronounced with greater scatter separation and the results approach the Born approximation prediction. High bound ridges appear for separations above one wavelength and increase in number as d increases above this size. This is as expected from the discussion in (20) and (21).

(a) Multiple Scattering Model

(b) Born Approximation Model

FIGURE 4: Plots of $\text{CRB}(d)$ as a function of incidence angle α and scatterer separation d , normalized by σ^2 , for the case where d_c is known. The observation angle $\beta = \pi - \alpha$ and $\tau_1 = 1 = \tau_2$.

FIGURE 5: Map of the regions where multiple scattering effects do not aid the estimation of target separation in relation to blind conditions of the Born approximation model. Black lines mark conditions where $I_{\text{Born}}(d) = 0$. Gray areas mark regions where $I(d) \leq I_{\text{Born}}(d)$. The same parameters are used here as in Figure 3 ($\tau_1 = \tau_2 = 1$, $\beta = \pi - \alpha$, d_1 is known).

FIGURE 6: Map of the regions where multiple scattering effects do not aid the estimation of target separation in relation to blind conditions of the Born approximation model. Black lines mark conditions where $I'_{\text{Born}}(d) = 0$. Gray areas mark regions where $I'(d) \leq I'_{\text{Born}}(d)$. The same parameters are used here as in Figure 4 ($\tau_1 = \tau_2 = 1$, $\beta = \pi - \alpha$, d_c is known).

FIGURE 7: Plot of $\text{CRB}(d)$ as a function of incidence angle α and scatterer separation d , normalized by σ^2 , for the multiple scattering model. The results apply for $\tau_1 = 1 = \tau_2$ in the LOS or forward scattering configuration, $\alpha = \beta$.

4.2. Multiple Observations. We consider next the case of multiple observations data. The corresponding calculations provide values of the CRB averaged over the sensing configuration parameters which effectively reduces the parameter dimensionality and allows us to decipher general patterns that are not obvious from the single observation analysis and associated computer results.

Figure 8 shows the behavior of the average $\text{CRB}(d)$ normalized by $(d\sigma)^2$. It is calculated from the average Fisher information corresponding to 10^4 scattering experiments involving 100 uniformly spaced values of incidence and scattering angles in the full interval $[0, \pi]$. Two scatterer strength cases are shown as follows: (a) identical scatterers, $\tau_1 = 1 = \tau_2$, and (b) unequal scatterers, $\tau_1 = 1$, $\tau_2 = 1/4$. It is worth noting that the exact formulations in terms of d_1

FIGURE 8: $\text{CRB}(d)$ as the scatterer separation d varies, normalized by $(d\sigma)^2$. The Fisher information on which the bound is based is averaged over all combinations of 100 samples each of incidence angles $\alpha_n \in [0, \pi]$ and observation angles $\beta_n \in [0, \pi]$ taken at regular intervals. Two variations are shown: (a) equal ($\tau_1 = 1 = \tau_2$) and (b) unequal ($\tau_1 = 1$, $\tau_2 = 1/4$) scatterer strengths.

and d_c approach each other below $d = \lambda/10$. In addition, the relation between the exact and Born approximation models is similar to the one in the localization context considered in [1] in that the two models match reasonably above $d = \lambda/2$ but diverge below this separation. However, the small d behavior is the opposite of what we obtained for the localization problem in [1]. Thus, unlike in the localization problem, where the Born approximation is unrealistically optimistic for small $d \leq \lambda/10$, in the target separation problem, the Born approximation CRB is significantly higher than the exact CRB for $d \leq \lambda/10$ so that the approximate predictions are in reality quite pessimistic. It is important to note that, in clear contrast with the target localization problem where the strong multiple scattering present for small d has a destructive effect in SNR which in turn reduces target localizability for small d

as we discussed earlier, in the related problem of estimating the target separation, the strong multiple scattering for small d plays a beneficial role since the information about d does not rely on SNR. In particular, in the targets' separation problem, d is the sought-after parameter and estimation is based on the rate at which the data signal changes due to variations in d which is significant for small values of d due to multiple scattering. Therefore, the associated information that can be extracted about d is also significant for small values of d , as is shown in these plots.

5. Conclusions

This paper has expanded the research on CRB for scattering parameter estimation initiated in the previous paper [1], coauthored by the current authors, where we studied, via the CRB approach, the effects of multiple scattering in the localization of a two-target scattering system. In the present paper, we have addressed the pending question of the effects of multiple scattering in the estimation of the separation distance between two targets which is related to the resolvability of the two targets from scattering data. As in [1], we have assumed that the target strengths are known which simplifies the formulation and the interpretation of the derived findings. Two possible situations were considered: one where the position of one of the scatterers is known, which essentially models localization of the other target in a (nonfree-space) medium including the other target which acts as a multipathing agent, and another where the center of the two targets system is known but the targets' separation is unknown, which essentially simulates estimation of the size of the total two-target scatterer. The presented results complete the discussion initiated in [1], clarifying further the role of multiple scattering in the localization and resolution of targets.

Concrete examples of conditions under which multiple scattering aids or impedes target separation estimation relative to the baseline of the Born approximation have been derived and illustrated. We have derived for both the exact and approximate scattering models the conditions under which the targets' separation cannot be estimated and concluded that these conditions are generally more restrictive under the Born approximation than for the exact model. Thus, in general, multiple scattering enhances target resolvability. The provided numerical results of CRB(d) corresponding to multiple observations show that, for $d \geq \lambda/2$, the Born approximation gives results which are, on average, very similar to those of the exact model. On the other hand, when considering single observations, specific cases exist where one model or the other exhibits a larger CRB. The multiple observation results for the two models diverge for smaller values of d ($d \leq \lambda/2$). The boundary depends on whether the center of the two targets or the position of one of the targets is known. However, in contrast to the localization problem of the previous paper [1], the error bound of the multiple scattering model is consistently smaller than the error bound of the Born approximation model and, for

very small target separations, multiple scattering significantly enhances the estimability of the two targets' separation.

Appendix

CRB Derivation

Following [1], we express the signal model as

$$\tilde{\mathbf{K}}(\xi) = \mathbf{K}(\xi) + \mathbf{W}, \quad (\text{A.1})$$

where $\mathbf{K}$ is the noise-free data vector, $\tilde{\mathbf{K}}$ is the collected noisy data vector, ξ is the parameter vector to be estimated and $\mathbf{W}$ is complex Gaussian noise with known variance σ^2 . In this paper, $\xi = d$, assuming prior knowledge of the strengths and a reference position such as the position of the first target, d_1 , or the position of the center of the two targets ($d_c = (d_1 + d_2)/2$).

The data entries K_n , $n = 1, 2, \dots, N$, of the $N \times 1$ data vector $\mathbf{K}$ are the values of the scattering amplitudes $f(\alpha_n, \beta_n)$ measurable in scattering experiments corresponding to given pairs (α_n, β_n) of incidence and scattering angles $\alpha_n \in [0, \pi]$ and $\beta_n \in [0, \pi]$, respectively. From (1), the entries K_n are given by

$$K_n(\xi) = F(d, \tau_1, \tau_2) e^{ikd_1 g(\alpha_n, \beta_n)} \times [\tau_1 + \tau_2 e^{ikd g(\alpha_n, \beta_n)} + \tau_1 \tau_2 G(d) Q(d, \alpha_n, \beta_n)]. \quad (\text{A.2})$$

The corresponding Born approximation is given from (4) by $K_n(\xi) \approx K_n^{\text{Born}}(\xi)$, where

$$K_n^{\text{Born}}(\xi) = e^{ikd_1 g(\alpha_n, \beta_n)} [\tau_1 + \tau_2 e^{ikd g(\alpha_n, \beta_n)}]. \quad (\text{A.3})$$

The CRB of the parameter ξ_i , CRB(ξ_i), constitutes a lower bound, achievable under mild conditions, for the variance $\text{var}(\hat{\xi}_i) = E[(\xi_i - \hat{\xi}_i)^2]$ (where E denotes the expected value) of any unbiased estimate $\hat{\xi}_i$ of the parameter ξ_i . It is given by the diagonal elements of the Fisher information matrix (FIM) [4, equation (3.20)]. In particular,

$$\text{var}(\hat{\xi}_i) \geq [I^{-1}(\xi)]_{i,i} = \text{CRB}(\xi_i), \quad (\text{A.4})$$

where the FIM, $I(\xi)$, is given by [4, equation (15.52)] as follows:

$$I(\xi)_{i,j} = 2\Re \left[\frac{\partial \mathbf{K}^H(\xi)}{\partial \xi_i} C_{\tilde{\mathbf{K}}}^{-1}(\xi) \frac{\partial \mathbf{K}(\xi)}{\partial \xi_j} \right], \quad (\text{A.5})$$

where $(\cdot)^H$ denotes the conjugate transpose and $C_{\tilde{\mathbf{K}}}$ is the covariance matrix which, in our case, is $C_{\tilde{\mathbf{K}}} = \sigma^2 \mathbf{I}$. $\mathbf{I}$ denotes the $N \times N$ identity matrix. Therefore, (A.5) reduces to

$$I(\xi)_{i,j} = \sum_{n=1}^N I_{i,j}^{(n)}(\xi), \quad (\text{A.6})$$

where the entry $I_{i,j}^{(n)}$ of the FIM of the n th scattering experiment, $I^{(n)}(\xi)$, is given by

$$I_{i,j}^{(n)}(\xi) = 2\sigma^{-2} \Re \left[\frac{\partial K_n^*(\xi)}{\partial \xi_i} \frac{\partial K_n(\xi)}{\partial \xi_j} \right]. \quad (\text{A.7})$$

Acknowledgment

This research was supported by the Air Force Office of Scientific Research under Grant FA9550-12-1-0285.

References

- [1] E. A. Marengo, M. Zambrano-Nunez, and P. Berestesky, "Cramér-Rao bound study of multiple scattering effects in target localization," *International Journal of Antennas and Propagation*, vol. 2012, Article ID 390312, 13 pages, 2012.
- [2] A. Sentenac, C. A. Guérin, P. C. Chaumet et al., "Influence of multiple scattering on the resolution of an imaging system: a Cramér-Rao analysis," *Optics Express*, vol. 15, no. 3, pp. 1340–1347, 2007.
- [3] F. Simonetti, M. Fleming, and E. A. Marengo, "Illustration of the role of multiple scattering in subwavelength imaging from far-field measurements," *Journal of the Optical Society of America A*, vol. 25, no. 2, pp. 292–303, 2008.
- [4] S. Kay, *Fundamentals of Signal Processing: Estimation Theory*, Prentice Hall, Englewood Cliffs, NJ, USA, 1993.