Research Article

Study on the Accuracy Improvement of the Second-Kind Fredholm Integral Equations by Using the Buffa-Christiansen Functions with MLFMA

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Former works show that the accuracy of the second-kind integral equations can be improved dramatically by using the rotated Buffa-Christiansen (BC) functions as the testing functions, and sometimes their accuracy can be even better than the first-kind integral equations. When the rotated BC functions are used as the testing functions, the discretization error of the identity operators involved in the second-kind integral equations can be suppressed significantly. However, the sizes of spherical objects which were analyzed are relatively small. Numerical capability of the method of moments (MoM) for solving integral equations with the rotated BC functions is severely limited. Hence, the performance of BC functions for accuracy improvement of electrically large objects is not studied. In this paper, the multilevel fast multipole algorithm (MLFMA) is employed to accelerate iterative solution of the magnetic-field integral equation (MFIE). Then a series of numerical experiments are performed to study accuracy improvement of MFIE in perfect electric conductor (PEC) cases with the rotated BC as testing functions. Numerical results show that the effect of accuracy improvement by using the rotated BC as the testing functions is greatly different with curvilinear or plane triangular elements but falls off when the size of the object is large.

1. Introduction

Surface integral equations (SIEs) are widely used for computing electromagnetic scattering in real-life problems. Integral equations (IEs) are termed as the first-kind Fredholm IEs, such as the EFIE for PEC cases and the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) equations or the combined tangential formulation (CTF) for dielectric cases. The first-kind Fredholm IEs are well known to have good accuracy but bad condition number; besides this, IEs are termed as the second-kind Fredholm IEs, such as the MFIE for PEC cases and the combined normal formulation (CNF) or the normal Muller formulation (NMF) for dielectric cases. The second-kind Fredholm IEs are less accurate but significantly better convergence [1–7].

Application of the second-kind integral equations is limited especially when high accuracy is required in real-life

application. In recent years, much work has been done to improve the numerical accuracy of the second-kind IEs. Some of them focus on accurate calculation of the impedance elements by using the proper basis functions [8–12]. Besides these, other good methods are also mentioned by the way of improving the discretization schemes [13–16]. Careful examination shows that the second-kind IEs have identity operators, which will cause large discretization error [17, 18]. Recently, the rotated BC function is used as testing function in the second-kind IEs for both the PEC and dielectric cases to achieve a better accuracy. It is proved in [19, 20] that the discretization error caused by the identity operators can be greatly reduced by using the rotated BC function as testing functions. It is investigated that the second-kind IEs can achieve comparable or even better accuracy to the first-kind IEs while keeping their fast convergence speed [19]. The discretization procedure which uses the rotated BC as testing

function for the second-kind IEs is shown with spherical examples in [19]. A later work generalized this technique for arbitrary shaped objects by handling near singularities with singular extraction technique [20]. However, the size of the objects is severely limited with the conventional MoM which is implemented to solve the integral equation; numerical performance of using the rotated BC as the testing function for large objects is not well studied.

In this paper, the MLFMA is employed to accelerate matrix-vector multiplication in the iterative solution of the MFIE with the rotated BC as the testing function (NBC-MFIE). Since, for PEC cases, there is only the K operator which has the identity operator, only PEC cases are studied. Accuracy and convergence rate of the proposed NBC-MFIE is studied in the case of scattering by cube and spherical objects with different radius. It is demonstrated that the accuracy of the MFIE can be improved significantly by using this discretization scheme. For some cases, this proposed procedure even achieves better accuracy than the EFIE with Rao-Wilton-Glisson (RWG) [21] as the testing function (RWG-EFIE). Compared with the conventional MFIE (RWG-MFIE), better convergence in iterative solution can be achieved by NBC-MFIE. It should be noted that different effects of the geometry modeling with planar or curvilinear triangular patches are studied in this paper.

The rest of this paper is organized as follows: A brief description of the formulation of NBC-MFIE is presented in Section 2. Various discretization schemes of SIEs with PEC scattering objects are compared in Section 3. Based on the numerical results, a discussion is presented in Section 4. Finally, Section 5 gives our conclusions.

2. Outline of the MLFMA Enhanced the MFIE with the Rotated BC as the Testing Function

Consider the case of scattering by a given PEC object; based on the principle of equivalence, a set of integral equations in terms of equivalent electric current can be derived [22]: the EFIE and MFIE. The boundary S of the PEC body is taken as the equivalent surface, with the incident plane wave denoted as $(\mathbf{E}^i, \mathbf{H}^i)$. Hence,

$$\eta \mathbf{L}(\mathbf{J}) = -\hat{\mathbf{n}} \times \mathbf{E}^i, \quad (1)$$

$$\hat{\mathbf{n}} \times \left(\mathbf{K}_{\text{PV}}(\mathbf{J}) + \frac{\Omega}{4\pi} \mathbf{I}^{\times n}(\mathbf{J}) \right) = -\hat{\mathbf{n}} \times \mathbf{H}^i, \quad (2)$$

where η and Ω denote the free space wave impedance and solid angle ($0 \leq \Omega \leq 4\pi$). The identity operator is $\mathbf{I}^{\times n}(\mathbf{X}) = \hat{\mathbf{n}} \times \mathbf{X}$; operators L and K are defined as follows:

$$\begin{aligned} \mathbf{L}\{\mathbf{X}\}(\mathbf{r}) &= -jk\hat{\mathbf{n}} \\ &\times \int_{S'} \left[\mathbf{X}(\mathbf{r}') + k^{-2} \nabla' \cdot \mathbf{X}(\mathbf{r}') \nabla \right] G(\mathbf{r}, \mathbf{r}') d\mathbf{r}', \quad (3) \\ \mathbf{K}_{\text{PV}}\{\mathbf{X}\}(\mathbf{r}) &= \text{P.V.} \int_{S'} \nabla' G(\mathbf{r}, \mathbf{r}') \times \mathbf{X}(\mathbf{r}') d\mathbf{r}', \end{aligned}$$

where $j = \sqrt{-1}$ and

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (4)$$

which is free space Green's function.

As shown in Figure 1, by using the rotated BC as the testing function, an original large triangular patch is first divided into six small patches. Then the RWG basis function $\mathbf{f}^r$ and the BC function $\mathbf{f}^b$ can be expressed as the superposition of the RWG function defined on the barycentric refinement of the original triangular path:

$$\begin{aligned} \mathbf{f}^r &= \sum_{i=0}^{13} c_i \mathbf{g}_i, \\ \mathbf{f}^b &= d_0 \mathbf{g}_0 + \sum_{i=1}^{2N_d-1} d_i \mathbf{g}_i + \sum_{\tilde{i}=1}^{2\tilde{N}_d-1} d_{\tilde{i}} \mathbf{g}_{\tilde{i}} \end{aligned} \quad (5)$$

and definitions of the weighting factors c_i , d_i , $d_{\tilde{i}}$, N_d , and $\tilde{N}_d$ can be found in [23].

By expanding the electric current on the object surface using the RWG basis functions $\mathbf{f}^r$ and applying $\hat{\mathbf{n}} \times \mathbf{f}^b$ as the testing function, we can obtain the matrix form of (1):

$$[\mathbf{P}^T] [\mathbf{Z}] [\mathbf{R}] \mathbf{J} = [\mathbf{P}^T] \mathbf{f} \quad (6)$$

with

$$\begin{aligned} Z_{ij} &= \int_S d\mathbf{r} (\hat{\mathbf{n}} \times \mathbf{f}_i^b) \cdot \left(\hat{\mathbf{n}} \times \left(K_{\text{PV}}\{\mathbf{f}_j^r\} + \frac{\Omega}{4\pi} \mathbf{I}^{\times n}\{\mathbf{f}_j^r\} \right) \right) \\ &= - \int_{\tau} d\mathbf{r} \mathbf{f}_i^b \cdot K_{\text{PV}}(\mathbf{f}_j^r) \\ &\quad + \frac{1}{4\pi} \int_{\tau} d\mathbf{r} \Omega(\mathbf{r}) \mathbf{f}_i^b \cdot \mathbf{I}^{\times n}(\mathbf{f}_j^r) \\ &= - \int_{\tau} d\mathbf{r} \mathbf{f}_i^b \cdot \int_{\tau'} d\mathbf{r}' \mathbf{f}_j^r \cdot \nabla' G(\mathbf{r}, \mathbf{r}') \\ &\quad + \frac{1}{2} \int_{\tau} d\mathbf{r} \mathbf{f}_i^b \cdot (\hat{\mathbf{n}} \times \mathbf{f}_j^r), \\ \mathbf{f} &= \int_S \hat{\mathbf{n}} \times \mathbf{f}_i^b \cdot (-\hat{\mathbf{n}} \times \mathbf{H}^i) d\mathbf{r} = - \int_S \mathbf{f}_i^b \cdot \mathbf{H}^i d\mathbf{r}. \end{aligned} \quad (7)$$

$[\mathbf{R}]$ and $[\mathbf{P}]$ are sparse transformation matrices mapping the RWG function $\mathbf{f}^r$ and the BC function $\mathbf{f}^b$ defined on the original mesh to the RWG function $\mathbf{g}$ on the refined mesh, and their columns are composed by c and d in (5). For the conventional MFIE with the RWG basis function as its testing function, there is an identity operator in the final impedance matrix:

$$\begin{aligned} \mathbf{I}^{\times n} &= \int (\hat{\mathbf{n}} \times \mathbf{g}_i(\mathbf{r})) \cdot (\hat{\mathbf{n}} \times \mathbf{g}_j(\mathbf{r}')) d\mathbf{r} \\ &= \int \mathbf{g}_i(\mathbf{r}) \cdot \int \delta(\mathbf{r}, \mathbf{r}') \mathbf{g}_j(\mathbf{r}') d\mathbf{r}' d\mathbf{r} \end{aligned} \quad (9)$$

FIGURE 1: The relationship between the RWGs defined on the barycentric mesh and the RWG/BC defined on the original mesh.

which causes large discretization error and makes the MFIE less accurate. From (7) we can know that when the rotated BC is used as the testing function, there also exists an identity operator $\mathbf{I}$ but has the form of

$$\mathbf{I} = \int \mathbf{g}_i(\mathbf{r}) \cdot (\hat{\mathbf{n}} \times \mathbf{g}_j(\mathbf{r}')) d\mathbf{r}. \quad (10)$$

Compared with the identity operator in (9), (10) is weakly tested and propagates its inaccuracy less in final results.

Due to the computational and storage complexity of $O(N^2)$, MoM is conventionally limited and difficult to be used for dealing with electrically large-scale problems. Various acceleration methods including MLFMA and fast Fourier transform (FFT) are developed for matrix-vector multiplication during iterative solution steps [21, 24]. In this paper, MLFMA is employed to accelerate the computing and reduce the memory requirements for the impedance matrix $\mathbf{Z}$ in (7).

When MLFMA is employed, near interaction matrix is computed and stored explicitly in set up step. For far interactions between the basis and testing elements, MLFMA calculates them in a group-by-group manner consisting of three stages called aggregation, translation, and disaggregation [24]. In MLFMA, far interactions can be factorized as

$$\begin{aligned} & \int_S \mathbf{g}_i \cdot \mathbf{K}(\mathbf{g}_j) d\mathbf{r} \\ &= k^{-2} (4\pi)^{-2} \int \mathbf{V}_i(\hat{\mathbf{k}}) \cdot T(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \mathbf{V}_j(\hat{\mathbf{k}}) d^2\hat{\mathbf{k}}, \end{aligned} \quad (11)$$

where the disaggregation term $\mathbf{V}_j$, the aggregation terms $\mathbf{V}_i$, and the translation term T are explicitly expressed as follows:

$$\begin{aligned} \mathbf{V}_i &= \int_S e^{j\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_C)} (\hat{\mathbf{k}} \times \mathbf{g}_i) d\mathbf{r}, \\ \mathbf{V}_j &= \int_{S'} e^{j\mathbf{k} \cdot (\mathbf{r}' - \mathbf{r}_{C'})} \mathbf{g}_j d\mathbf{r}', \\ T &= \sum_{m=0}^L (-j)^m (2m+1) h_m^{(2)}(kr_{CC'}) P_m(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}), \end{aligned} \quad (12)$$

where $\mathbf{k} = k\hat{\mathbf{k}}$ and $\mathbf{g}_i$ and $\mathbf{g}_j$ are the basis functions residing in groups C or C' centered at $\mathbf{r}_C$ and $\mathbf{r}_{C'}$ respectively. $h_m^{(2)}$ denotes the spherical Hankel function of the second kind, P_m is the Legendre polynomial of degree m , and L is the number of multipole expansion terms; here it is determined by

$$L = kd + 2 \ln(kd + \pi) \quad (13)$$

with d being diagonal length of the cubical box. And then construct the MLFMA tree in a multilevel way.

3. Numerical Results

To demonstrate the accuracy, efficiency, and capability of the presented algorithm, a series of numerical experiments are performed in this section. All the computations are performed on a computer platform Liuhui-II at the Center for Electromagnetic Simulation, Beijing Institute of Technology. It has 2 Intel X5650 2.66 GHz CPUs with 12 cores for each CPU, 96 GB memory. The GMRES solver is employed and the convergence criterion is set to 0.001.

We consider various discretization schemes for SIEs. Table 1 summarizes 6 discretization schemes, where the

FIGURE 2: The normalized RMS of RCS calculated by the RWG-MFIE, RWG-EFIE, NBC-MFIE, CRWG-MFIE, CRWG-EFIE, and NCBC-MFIE as a function of number of unknowns for a PEC sphere with radius of 1 m at 75 MHz.

TABLE 1: Discretization schemes for the SIEs.

Basis function	Testing function	Equation	Procedure
RWG	RWG	EFIE	RWG-EFIE
CRWG	CRWG	EFIE	CRWG-EFIE
RWG	RWG	MFIE	RWG-MFIE
CRWG	CRWG	MFIE	CRWG-MFIE
RWG	$\hat{n} \times \text{BC}$	MFIE	NBC-MFIE
CRWG	$\hat{n} \times \text{CBC}$	MFIE	NCBC-MFIE

first and second columns show the choices of the basis functions and the testing functions; the third column shows the equations; the last column shows the abbreviations for different discretization schemes. For example, the abbreviation “NCBC-MFIE” means we choose the CRWG basis functions to expand the current densities and the $\hat{n} \times \text{CBC}$ to test MFIE. CBC means the BC basis function which is defined on curvilinear patch. Similarly, CRWG represents the curvilinear RWG.

Firstly, we consider plane wave scattering by a PEC sphere with a radius of 1 m. We compute the normalized root mean square (RMS) error of the bistatic radar cross section (RCS) for this sphere problem; the reference value is obtained with the Mie series. The normalized RMS error is defined as

$$\Delta = \frac{\text{RMS} \{ \sigma_{\text{ref}} - \sigma_{\text{cal}} \}}{\text{RMS} \{ \sigma_{\text{ref}} \}}. \quad (14)$$

The frequency of the incident plane wave is 75 MHz and 150 MHz, respectively. These two examples are exactly the same as Figure 4 in [19]; what is more, different effects by using the curvilinear triangular patches and planar triangular patches are studied. Figures 2(a) and 2(b) show the normalized RMS error as a function of the number

FIGURE 3: Iteration number for the RWG-MFIE, RWG-EFIE, NBC-MFIE, CRWG-MFIE, CRWG-EFIE, and NCBC-MFIE as a function of number of unknowns for a PEC sphere with radius of 1 m at 75 MHz.

of unknowns for various approaches with the planar and curvilinear patches, respectively. In Figure 3, faster iterative convergence is investigated for the NCBC-MFIE. Then we increase the frequency of incident wave to 150 MHz; the same results can be obtained obviously from Figure 4 as the frequency of the incident wave is 75 MHz. We can observe from these figures the following:

- (1) The accuracy of the EFIE (RWG-EFIE and CRWG-EFIE) is always better than the MFIE (RWG-MFIE

FIGURE 4: The normalized RMS of RCS calculated by the RWG-MFIE, RWG-EFIE, NBC-MFIE, CRWG-MFIE, CRWG-EFIE, and NCBC-MFIE as a function of number of unknowns for a PEC sphere with radius of 1 m at 150 MHz.

and CRWG-MFIE), which is the same as known to all. However, when the rotated BC function is used as the testing function, the MFIE (NBC-MFIE and NCBC-MFIE) can achieve comparable or even better accuracy than that of EFIE.

- (2) When the curvilinear patches are employed, all CRWG-MFIE, CRWG-EFIE, and NCBC-MFIE can achieve better accuracy than those with the planar triangular patches. The NCBC-MFIE with curvilinear has the best accuracy, even better than that of the CRWG-EFIE. On one hand, the discretization error of the identity operator is suppressed when the rotated BC is used as the testing function; on the other hand, when the size of the objects is relatively small, the curvilinear patches will affect the accuracy of the final results greatly.
- (3) The numbers of iterations for solving final matrix equation system for the NCBC-MFIE/NBC-MFIE and CRWG-MFIE/RWG-MFIE keep constant as a function of number of unknowns, but for CRWG-EFIE/RWG-EFIE, the number of iterations increases fast. The NBC-MFIE converges even faster than that of RWG-MFIE. We can conclude that NBC-MFIE can achieve better accuracy than both RWG-EFIE and RWG-MFIE and maintain fast convergence of the second-kind integral equation at the same time.

Then we increase the frequency of the incident plane wave to 0.6 GHz and 1.2 GHz. The computed normalized RMS error as a function of unknowns is plotted in Figures 5 and 6, respectively. Since the size of the PEC sphere is relatively large now, the MLFMA is employed to speed matrix-vector

FIGURE 5: The normalized RMS of RCS calculated by the NBC-MFIE, CRWG-MFIE, CRWG-EFIE, and NCBC-MFIE as a function of number of unknowns for a PEC sphere with radius of 1 m at 600 MHz.

FIGURE 6: The normalized RMS of RCS calculated by the NBC-MFIE, CRWG-MFIE, CRWG-EFIE, and NCBC-MFIE as a function of number of unknowns for a PEC sphere with radius of 1 m at 1.2 GHz.

multiplication in these iterative solution steps. From these two figures we can conclude the following:

- (1) As observed in Figures 5 and 6, when we increase the frequency of the incident plane wave, the accuracy of the RWG-EFIE is still better than the RWG-MFIE. What is different is that the BC-MFIE can achieve better accuracy than the RWG-MFIE, but less accurate than that of the RWG-EFIE. With the size of the object increasing, the discretization error of the identity operator can be improved in the MFIE by using the BC function as the testing function.

FIGURE 7: The normalized RMS of RCS calculated by the RWG-MFIE, RWG-EFIE, and NBC-MFIE as a function of number of unknowns for a PEC cube with 2 m edges at 0.3 GHz.

However, as shown in (10), the identity operator does not vanish but still exists in MFIE. With the increment of the size, the discretization error of the identity operator in (10) increases. Hence the BC-MFIE is less accurate than the RWG-EFIE.

- (2) With the increment of the number of unknowns (increase in mesh density), the iteration numbers of NBC-MFIE with curvilinear and planar patches are almost the same.

To demonstrate various discretization schemes further, the last case is a PEC cube with 2 m edges; the frequency of the incident wave is 300 MHz. This problem is discretized with various mesh sizes and solved with the RWG-MFIE, RWG-EFIE, and NBC-MFIE. Figure 7 presents the normalized RMS error of RCS with various discretization schemes for this cube problem. We use the result of the RWG-EFIE with $1/50\lambda$ discretization mesh size as the reference result. Similar to the previous large spherical examples, the NBC-MFIE improves the accuracy of the MFIE by using the BC functions but gets less accurate result than the result of the RWG-EFIE.

4. Discussion

As was shown in the section of numerical results, the accuracy of the MFIE is highly improved by using the $\hat{n} \times \text{BC}$ as the testing function while maintaining high convergence rate. Especially when the object size is small, we can get the same conclusion as shown in [19]; the NCBC-MFIE can get more accurate result than the CRWG-EFIE. When the size of object becomes larger, the accuracy of the MFIE can be improved, but less accurate than EFIE, because there are two aspects which lead to high accuracy: the BC function and the curvilinear patch. The effect of the curvilinear patch gets

weaker with the size increasing, so the accuracy of the NCBC-MFIE is almost the same as that of the NBC-MFIE when the incident frequency is high. Cube case shows the same conclusion. From our analysis, there are two benefits for using the rotated BC functions:

- (1) During the process of calculating the impedance elements, the identity operator can be more accurately calculated which is shown in (10).
- (2) During the process of solving the matrix equation, the matrices P and R map the relationship between the test functions and the basis functions from the space of the barycentric meshes to the space of the original meshes, which leads to well conditioned matrix. That is why this discretization scheme can achieve good convergence rate. In contrast, using the rotated RWG functions as the testing functions can fulfill benefit (1) but also makes the matrices ill-conditioned.

5. Conclusion

In this paper, numerical performance of the MFIE with the rotated BC function as the testing function is studied and the MLFMA is employed to accelerate iterative solution for the electrical large objects. A series of numerical experiments on PEC cube and spheres with different radius show that, compared with using the RWG as the testing functions, the discretization error of the identity operator can indeed be suppressed by using the rotated BC as testing functions. When the size of the sphere is not very large, the NCBC-MFIE can achieve accurate numerical solutions that are comparable to (or even better than) the existing solutions of CRWG-EFIE, because the discretization error for the identity operator is sharply decreased and the geometry modeling is more accurate. However, the effect of the accuracy improvement by using the rotated BC function as testing function for the MFIE falls off when the radius of the sphere becomes large. When the rotated BC function is used instead of the conventional RWG function, the discretization error of the identity operator although being improved still exists.

Competing Interests

The authors declare that they have no competing interests.

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