Research Article

An Approximation Mathematical Formula of Pattern Analysis for Distorted Reflector Antennas considering Surface Normal Vector Variation

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An approximation mathematical formula to compute the distorted radiation pattern for reflector antennas is presented. In this approximation formula, besides the phase error caused by the structural deformation being added in the far field integral, the surface normal vector variation is also taken into consideration. The formula is derived by expanding the surface normal vector into a first-order Taylor series and the phase error into a second-order Taylor series. By assembling the integrals including the contributions of both surface normal vector variation and phase error, the far field electrical vector expressed as a function of structural nodal displacements is further obtained in a matrix form. Simulation results of a distorted reflector show the application of this formula.

1. Introduction

Reflector antennas are widely used in radar, telecommunication, radio astronomy, and other applications. Reflector surface distortions are inevitable in outer working conditions owing to thermal, gravitational, and dynamic effects and manufacturing tolerance [1]. With the increasing frequency, the stringent requirements on reflector surface accuracy become demanding [2]. Towards the evaluation of surface accuracy on electromagnetic performance, several tolerance analysis techniques have been proposed in scientific literature for random error [3–6] and systematic error [7–11] in centered reflectors [7–10], offset reflectors [11], and membrane reflectors [11]. By performing tolerance analysis, the reflector antenna can be designed and manufactured with considering the surface tolerance.

The process of antenna design belongs to the area of multidisciplinary design, which contains two main steps of electromagnetic design and structural design. In the previous antenna design, the structural design procedure was implemented by simply reducing the surface root-mean-square (RMS) error to fulfill the requirements of antenna electromagnetic performance, which usually could not obtain the

final satisfactory electromagnetic performance as observed in [10, 12]. Thus, the antenna design has been transformed from just reducing surface RMS distortion to an integrated structural-electromagnetic optimization design with a multidisciplinary optimization model [12–15]. The integrated structural-electromagnetic design concept is employed in various applications, which directly combines the electromagnetic performance gain or sidelobe levels with structural inputs other than simply reducing the surface RMS error. Some studies have been performed in the antenna design based on the integrated structural-electromagnetic concept, such as the shape control of antennas [12, 13], multidisciplinary optimization [14, 15], and electromechanical coupling analysis [9, 10]. In the integrated optimization design procedure, the antenna far field pattern will be calculated in the optimization iteration repeatedly, which requires an accurate and fast pattern analysis method for a given reflector with various deformations. An approximation radiation integral method for distorted reflector antennas was proposed in [16] using the decomposition orthogonal basis functions. The proposed method benefits the repeated calculations with allowable computation accuracy. Furthermore, referencing to the three-term Taylor series of phase error in [16], a matrix form

formula between the surface nodal displacements obtained by structural finite element method (FEM) and far field electrical vector was derived in [17]. The far field pattern can be directly and easily calculated by substituting the surface nodal displacements into the matrix form, which benefits the iterative pattern analysis procedure of antenna design.

As for the distorted reflector antennas, phase error caused by the surface nodal displacement is usually added in the integral to evaluate its effect on far field radiation pattern [9–11]. The phase error is introduced based on the assumption that the reflector is located at the far field zone of the primary feed, and the amplitude variation of incident magnetic field is neglected. Strictly speaking, the surface distortion introduces not only phase error but also surface normal vector variation. In the previous study, the surface normal variation is neglected, which is based on the fact that, for small-amplitude, smoothly varying errors, the surface normal vector will not deviate substantially from the surface of an undistorted reflector [16]. The questions appear as what percentage of the surface normal variation contributes in the far field integral for distorted reflectors so as to be neglected, and what is the final expression for the matrix form in [17] when considering both the surface normal variation and phase error. The aim of this study, which is a continuation of the work in [17], is to further develop the matrix form formula considering both the phase error and the surface normal vector variation caused by surface nodal displacements.

In this study, the reflector surface is divided into many small elements as in structural FEM analysis, and the far field pattern is obtained by superposition of each element's far field contribution. The surface normal vector variation caused by surface nodal displacement is expanded into a first-order Taylor series during the implementation, and the shape functions in [17] are also employed to express the integration point displacements as functions of nodal displacements. Both the phase error and the surface normal variation are included in these integrals, and the matrix form in [17] by assembling the integrals on elements is thus extended.

The main difference between this formula and the one in [17] lies in the normal vector variation. The approximated formula in [17] just takes the phase error caused by surface nodal displacements into account, while it fails to consider the surface normal vector variation. The derived formula in this study will take both the phase error and normal vector variation into account theoretically, which can be considered as an extended formula for the one in [17].

The study is organized as follows: Section 2 outlines the derivation of the approximation mathematical formula considering both the surface normal variation and phase error, a distorted reflector is calculated to show the proportion of the surface normal variation contribution in Section 3, and the conclusion is drawn in Section 4.

2. The Approximation Mathematical Formula

The radiation integral for Physical Optics (PO) formula of reflector antennas is expressed as [16]

$$\vec{T}(\theta, \phi) = \int_{\sigma} 2\vec{N} \times \vec{H}(\vec{r}') e^{jk\vec{r}' \cdot \vec{r}} d\sigma, \quad (1)$$

FIGURE 1: A distorted reflector.

where $\vec{N}$ is the surface normal vector, $\vec{H}$ is the incident magnetic field, the vector $\vec{r}'$ locates the integration point, $j = \sqrt{-1}$ is the imaginary unit, $k = 2\pi/\lambda$ is the wavenumber, λ is the free-space wavelength at the working frequency, the unit vector $\vec{r}$ is in the observation direction, and σ is the projected aperture region of reflector antennas.

Provided that a nodal displacement from an ideal undistorted reflector is defined as $\vec{\epsilon}'_z = \Delta z \hat{z}$ in the z -direction shown in Figure 1, Δz is a scalar function that defines the nodal deformation. Strictly speaking, in the distorted antenna analysis the surface deformation introduces not only the phase error but also the surface normal vector variation. Thus, the far field integral becomes

$$\vec{T}(\theta, \phi) = \int_{\sigma} 2(\vec{N} + \Delta\vec{N}) \times \vec{H}(\vec{r}') e^{jk\vec{r}' \cdot \vec{r}} e^{jk\Delta z(\cos\theta_s + \cos\theta)} d\sigma, \quad (2)$$

where $\Delta\vec{N}$ is the normal vector variation, θ_s is the undistorted spherical component in the feed coordinate system as shown in Figure 1, and θ defines the observation direction. Both $\vec{N}$ and $\vec{H}$ are in the ideal undistorted nominal state. As described in Figure 1, the solid line represents the ideal surface and the dashed line is for the distorted surface. A surface nodal displacement is shown in Figure 1. The global coordinate system (x, y, z) is defined as the reflector unprimed coordinate system and (x_s, y_s, z_s) is for the feed coordinate system. Shown in Figure 1, $\vec{r}$ denotes the observation point and $\vec{r}'$ is the surface point vector in global coordinate system.

The surface nodal displacements shown in Figure 1 will introduce phase error and surface normal vector variation in far field radiation integral as illustrated in (2). In [16], a three-term Taylor series approximation about phase error is adopted to express the phase error as a second-order polynomial expansion of nodal displacements, and its accuracy can be sufficient enough for errors profiles up to about 0.1λ RMS amplitude [16]:

$$e^{jk\Delta z(\cos\theta_s + \cos\theta)} \approx \sum_{i=0}^2 \frac{1}{i!} [jk\Delta z(\cos\theta_s + \cos\theta)]^i. \quad (3)$$

In this error analysis, the reflector surface is divided into a series of small triangular elements, and the far field pattern

FIGURE 2: Two equilateral triangles in aperture plane.

is thus calculated by superposition of each element's far field vector contribution. In this derivation, the small elements are required to be equilateral triangles projected in the aperture plane, such that the element normal vector variation can be written as a linear combination of triangular nodal displacements in a first-order Taylor series, and the coefficients of these series depend only on the triangular side length and the nodal position with respect to the element center point. The normal vector variation in (2) can be expressed as

$$\Delta \vec{N} = \sum_{i=1}^3 \frac{\partial \vec{N}}{\partial z_i} \Delta z_i, \quad (4)$$

where Δz_i is the surface nodal z -displacement of the i th node and $\partial \vec{N} / \partial z_i$ is the gradient of normal vector with respect to the i th nodal displacement.

Because of the similarity of the equilateral triangles, two triangles ABC and $A'BC$ are employed in Figure 2 to show the gradient of surface normal vector with respect to nodal displacement, where the symbols A , B , C , and A' denote the triangular nodes. The gradients of triangular normal vector with respect to each nodal displacement are listed in Table 1, where l is the triangular side length.

The same as in [17], the triangular element ABC in aperture plane shown in Figure 3 is chosen in the analysis, where the symbols A , B , and C represent the triangle nodes and P is an arbitrary point inside the element. Supposing that the nodal displacements in z -direction are labeled as Δz_a , Δz_b , and Δz_c for nodes A , B , and C , respectively, the displacement of an integration point P in (3) within the element can be expressed as a function of nodal displacements multiplied by the element shape functions, and

$$\Delta z = \{\xi_a, \xi_b, \xi_c\} \begin{pmatrix} \Delta z_a \\ \Delta z_b \\ \Delta z_c \end{pmatrix}, \quad (5)$$

where ξ_a , ξ_b , and ξ_c are the element shape functions with respect to the nodes A , B , and C , respectively.

FIGURE 3: A triangular element ABC in FEM analysis [17].

Substituting (3), (4), and (5) into (2) and neglecting the high-order terms, the integral on the triangular element ABC (labeled with e as superscript) can be expanded as

$$\vec{T}^e = \vec{T}_0^e + \vec{T}_1^e + \vec{T}_2^e + \vec{T}_3^e + \vec{T}_4^e, \quad (6)$$

where the superscript e designates the element ABC , $\vec{T}_0^e$ is the undistorted contribution in the integral, and $\vec{T}_1^e$ and $\vec{T}_2^e$ are the linear and square terms in [17]; the first three terms in the right side are not repeated here for brevity, and

$$\begin{aligned} \vec{T}_3^e &= \sum_i \Delta z_i \vec{T}_{3,i}^e, \\ \vec{T}_4^e &= \sum_{u,v} \Delta z_u \Delta z_v \vec{T}_{4,uv}^e, \end{aligned} \quad (7)$$

where Δz_i , Δz_u , and Δz_v are the nodal displacements of nodes i , u , and v , where $i = a, b, c$ and $u, v = a, b, c$, respectively, and

$$\vec{T}_{3,i}^e = \int_{\sigma_{ABC}} 2 \frac{\partial \vec{N}}{\partial z_i} \times \vec{H}(\vec{r}') e^{jk\vec{r}' \cdot \vec{r}} d\sigma_{ABC}, \quad (8)$$

$$\begin{aligned} \vec{T}_{4,uv}^e &= \int_{\sigma_{ABC}} 2 \frac{\partial \vec{N}}{\partial z_u} \\ &\times \vec{H}(\vec{r}') e^{jk\vec{r}' \cdot \vec{r}} jk\xi_v (\cos \theta_s + \cos \theta) d\sigma_{ABC}, \end{aligned} \quad (9)$$

where σ_{ABC} is the integral region of element ABC , $\partial \vec{N} / \partial z_u$ is the gradient of surface normal vector with respect to u th nodal displacement, ξ_v are the element shape function with respect to v th node in element ABC . From its expression, it can be seen that $\vec{T}_{4,uv}^e \neq \vec{T}_{4,vu}^e$ because of the different sequences of normal vector gradient and shape function in (9).

The same as in [17], the integral in (6) can be rewritten as a matrix form:

$$\begin{aligned} \Delta \vec{T}^e &= (h_1^e + h_3^e) \cdot \Delta z^e + (h_2^e + h_4^e) \\ &\cdot \text{vec}(\Delta z^e \cdot (\Delta z^e)^T), \end{aligned} \quad (10)$$

TABLE 1: Gradients of normal vector with respect to triangular nodal displacement.

Element	Node	Gradient with respect to nodal displacement	Value
ABC	A	$\frac{\partial \vec{N}}{\partial z_A}$	$\frac{(0, -2/\sqrt{3}, 0)}{l}$
	B	$\frac{\partial \vec{N}}{\partial z_B}$	$\frac{(1, 1/\sqrt{3}, 0)}{l}$
	C	$\frac{\partial \vec{N}}{\partial z_C}$	$\frac{(-1, 1/\sqrt{3}, 0)}{l}$
A'BC	A'	$\frac{\partial \vec{N}}{\partial z_{A'}}$	$\frac{(0, 2/\sqrt{3}, 0)}{l}$
	B	$\frac{\partial \vec{N}}{\partial z_B}$	$\frac{(1, -1/\sqrt{3}, 0)}{l}$
	C	$\frac{\partial \vec{N}}{\partial z_C}$	$\frac{(-1, -1/\sqrt{3}, 0)}{l}$

where $\Delta z^e = \{\Delta z_a, \Delta z_b, \Delta z_c\}^T$ is the nodal displacements vector in the element ABC, the operator $\text{vec}(\cdot)$ converts a matrix into a vector, h_1^e and h_2^e are the same coefficients as in [17], and

$$h_3^e = \{\vec{T}_{3,a}^e, \vec{T}_{3,b}^e, \vec{T}_{3,c}^e\},$$

$$h_4^e = \{\vec{T}_{4,aa}^e, \vec{T}_{4,ab}^e, \vec{T}_{4,ac}^e, \vec{T}_{4,ba}^e, \vec{T}_{4,bb}^e, \vec{T}_{4,bc}^e, \vec{T}_{4,ca}^e, \vec{T}_{4,cb}^e, \vec{T}_{4,cc}^e\}. \quad (11)$$

Assuming that there are m elements in the FEM analysis and n observation points in the far field requirements, the extended final matrix form can be expressed by assembling the whole elements integrals as

$$\Delta \vec{E} = -jk\eta \frac{e^{-jkr}}{4\pi r} \left(\vec{I} - \hat{r}\hat{r} \right) \cdot \left((\mathbf{H}_1 + \mathbf{H}_3) \Delta \mathbf{z} + (\mathbf{H}_2 + \mathbf{H}_4) \Delta \mathbf{z}^2 \right), \quad (12)$$

where $\Delta \mathbf{z}$ is the z -direction displacements of all nodes, $\Delta \mathbf{z}^2$ is the product term between each two nodal displacements in one element including square terms of each nodal displacement, and

$$\Delta \vec{E} = \{\Delta \vec{E}(\theta_1, \phi_1), \Delta \vec{E}(\theta_2, \phi_2), \dots, \Delta \vec{E}(\theta_n, \phi_n)\}^T,$$

$$\mathbf{H}_1 = \mathbf{A} \mathbf{h}_1^e,$$

$$\mathbf{H}_2 = \mathbf{A} \mathbf{h}_2^e,$$

$$\mathbf{H}_3 = \mathbf{A} \mathbf{h}_3^e,$$

$$\mathbf{H}_4 = \mathbf{A} \mathbf{h}_4^e,$$

$$\Delta \mathbf{z} = \mathbf{A} \Delta z^e,$$

$$\Delta \mathbf{z}^2 = \mathbf{A} \text{vec}(\Delta z^e \cdot (\Delta z^e)^T), \quad (13)$$

where $\Delta \vec{E} = \vec{E} - \vec{E}_0$, $\vec{E}_0$ is the undistorted electrical far field vector, and $\mathbf{A}_{e=1}^m$ is the finite element assembly operator [17, 18].

The matrix form in [17] is thus extended in (12) considering both the phase error and the surface normal vector variation. The same as the properties of the matrix form in [17], the extended formula can also benefit the iterative optimization procedure, and once the matrices $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, and $\mathbf{H}_4$ and the undistorted electrical far field $\vec{E}_0$ are determined, the distorted radiation pattern will be easily obtained by a simple matrix operation with the stored data.

The extended formula can also be employed in the application of pattern analysis for different surface-error profiles as addressed in [17]. For example, in the integrated structural-electromagnetic optimization design [14, 15], the antenna surface is determined through different various loads and structural design variables, which makes the repeated pattern analysis required for a given reflector with varied error profiles. It is also needed to analyze the electromagnetic performance of a reflector with different surface-error profiles as in the contoured beam pattern synthesis. Besides, this formula can also be applied into the uncertainty bounds analysis of reflector antennas with surface random errors. The main application of this approximated formula lies in the requirement of repeated pattern analysis, namely, pattern analysis for a given reflector with various surface-error distributions. Once the prestored data is computed, the antenna pattern can be easily calculated by recalling the stored matrices and performing a simple matrix operation for a given reflector.

3. Numerical Assessments

The approximation method in [16] expanded the phase error caused by the surface displacements into three-term Taylor series, and the second-order approximation showed very good agreement with the exact computation. To find out what percentage of surface normal vector variation is taken in the far field integral for error profiles up to about 0.1λ RMS amplitude, a simulation is performed in this study. The contributions of the matrices $\mathbf{H}_3$ and $\mathbf{H}_4$ in (12) are computed

TABLE 2: Maximum directivity of each pattern.

Unit: dB	Deformation (14)	Deformation (15)	Deformation (16)
First-order	49.072	49.023	49.023
Extended first-order	49.071	49.026	49.023
Second-order	48.273	48.269	48.214
Extended second-order	48.272	48.271	48.214

FIGURE 4: Far field pattern comparisons of the extended formula with the previous one under deformation (14).

FIGURE 5: Far field pattern comparisons of the extended formula with the previous one under deformation (15).

with the former matrices $\mathbf{H}_1$ and $\mathbf{H}_2$. For the distorted reflector in [16, 17] of which the diameter $D = 100\lambda$, the focal length $F = 100\lambda$ is adopted in this simulation. The feed parameters are also the same as the ones in [16, 17] with $Q_x = Q_y = 8$ to radiate a linearly polarized wave. The equilateral triangular side length projected in the aperture is λ . The three deformations in [16, 17] are also employed in this simulation, and their functions are

$$\Delta z(\rho', \phi') = 0.05\lambda \sin\left(\frac{2\pi\rho'^2}{(50\lambda)^2}\right), \quad (14)$$

$$\Delta z(\rho', \phi') = 0.05\lambda \cos\left(\frac{10\pi\rho'^2}{(50\lambda)^2}\right), \quad (15)$$

$$\Delta z(\rho', \phi') = 0.05\lambda \cos(5\phi'), \quad (16)$$

where the variables ρ' and ϕ' are the same as in [17] which describes the polar coordinate components in the projected aperture plane of the reflector.

Figures 4–6 show the far field patterns under three different deformations for the distorted reflector, respectively. Each figure has the patterns for the first-order approximation in [17] (dashed line), the extended first-order approximation (solid line), the second-order approximation in [17] (dot line), and the extended second-order approximation (the marked line) in xz plane. Table 2 lists the maximum directivities of the patterns. The simulation results of the extended first-order and second-order approximations show very close agreement with the pervious study in [17], and the maximum

FIGURE 6: Far field pattern comparisons of the extended formula with the previous one under deformation (16).

difference is in the order of 0.001 dB, which indicates that the surface normal variation takes little percentage in the distorted reflector pattern analysis and it can be neglected in the approximation for errors profiles up to about 0.1λ RMS amplitude. It can be understood from this that, in the equivalent induced current density ($\vec{J}_{\text{eff}} = 2\vec{N} \times \vec{H}$) of the cross product of surface normal vector and the incident magnetic field, the phase of induced current density is same as the incident magnetic field while its amplitude variation can be neglected when distortion happens.

4. Conclusion

An extended approximation mathematical formula to compute the radiation pattern for distorted reflector antennas was proposed considering both the phase error and surface normal vector variation. The surface normal vector variation was added in the integral by a first-order Taylor series. The matrix form combining the surface nodal displacements and far field electrical vector was thus extended by assembling the integrals on structural elements. The extended formula also has the property of the pervious one that once the matrices are precomputed, the distorted pattern can be easily obtained by recalling the stored matrices and performing a simple matrix operation for a given reflector. Simulation results showed that the surface normal vector variation takes little percentage in the distorted reflector pattern analysis and its contribution is in the order of 0.001 dB which can be neglected in the approximation for errors profiles up to about 0.1λ RMS amplitude.

Competing Interests

The authors declare that they have no competing interests.

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