Research Article

A Solution of the Convective-Diffusion Equation for Solute Mass Transfer inside a Capillary Membrane Bioreactor

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Received 5 November 2009; Revised 23 March 2010; Accepted 27 April 2010

Academic Editor: Jose C. Merchuk

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This paper presents an analytical model of substrate mass transfer through the lumen of a membrane bioreactor. The model is a solution of the convective-diffusion equation in two dimensions using a regular perturbation technique. The analysis accounts for radial-convective flow as well as axial diffusion of the substrate specie. The model is applicable to the different modes of operation of membrane bioreactor (MBR) systems (e.g., dead-end, open-shell, or closed-shell mode), as well as the vertical or horizontal orientation. The first-order limit of the Michaelis-Menten equation for substrate consumption was used to test the developed model against available analytical results. The results obtained from the application of this model, along with a biofilm growth kinetic model, will be useful in the derivation of an efficiency expression for enzyme production in an MBR.

1. Introduction

Since the first uses of hollow-fiber membrane bioreactors (MBRs) to immobilize whole cells were reported in the early 1970s, this technology has been used in as wide ranging applications as enzyme production to bone tissue engineering. One of the current research areas of interest into biofilm-attached membrane bioreactors (MBRs) is the development of cost-effective and environmentally friendly methods of producing various primary and secondary metabolites from bacterial, fungal, and yeast cells. These include: manganese and lignin peroxidase, secreted by the fungus *Phanerochaete chrysosporium* [1, 2]; actinorhodin, a noncommercial antibiotic produced by the filamentous bacterium *Streptomyces coelicolor* [3]; glutamic acid, an ingredient in flavour enhancers of meats and vegetables, secreted by the bacterium *Corynebacterium glutamicum* [4]; ethanol, extracted from the yeast *Saccharomyces cerevisiae* [5]; and many others. With the exception of ethanol, these bioproducts are generally classified as products of intermediate value [6]. It has been reported that bioreactor productivity, in the production of these types of products, greatly impacts on the product cost [7].

The productivity of biofilm-attached MBRs is determined in large by the biomass growth, and one of the most important factors that influence biomass growth is the availability and transport of nutrients through the bioreactor [8, 9]. The momentum transfer of solutes through MBRs has been thoroughly studied, from a theoretical and experimental perspective, for a number of configurations [10–15]. Similarly, the mass transfer has received considerable attention [8, 9, 16–22]. With the exception of the models developed by Heath and Belfort [17]; Li and Tan [19]; and Willaert et al. [22], the mass transfer models were solved using numerical procedures such as finite difference schemes and control volumes. A difficulty in implementing such schemes is the choice of the appropriate technique for a specific MBR system [21], and these techniques are subject to discretization errors and stringent stability criterion. In the models presented by Heath and Belfort [17]; Willaert et al. [22]; and Li and Tan [19], the convective-diffusion equation governing mass transfer was solved analytically. These authors, however, neglected the effects of axial diffusion and radial convective flow in their models. Both these assumptions may not be justified in all cases. A number of theoretical and experimental investigations have

FIGURE 1: A schematic diagram of the single capillary membrane gradostat reactor (MGR).

demonstrated the significance of radial convective flows in improving MBR efficiencies [9, 15, 21]. In the dead-end ultrafiltration mode, particularly, the assumption of negligible radial convective flow is not justifiable. At axial Peclet numbers (Pe_u) smaller than unity large concentration gradients exist, and under these circumstances ignoring axial diffusion is also not justified.

The current study presents an analytical solution of the convective-diffusion equation, for solute transport through a single fiber isotropic capillary membrane, in two dimensions. This study will not include the growth kinetics of the microorganism, as conversion is assumed to take place in the shell side of the MBR; the current analysis is restricted to the lumen side. For comparison with literature models, however, the first-order limit of the Michaelis-Menten equation will be superimposed on the developed model in the results section.

2. Model Development

2.1. The Membrane Gradostat Reactor. The models developed in this study are applicable to a hollow-fiber MBR system, consisting of either a single fibre or a bundle of fibres; with nutrient flowing on the lumen side of the membrane and the micro-organism immobilised either on the lumen side or on the shell side. The notation used, however, is specifically for a single hollow fiber membrane gradostat reactor (MGR). The construction of the MGR, as patented by Edwards et al. [23], is illustrated schematically in Figure 1. It consists of a single hollow-fibre, made of surface modified polysulphone, encased in a glass bioreactor. The membranes are asymmetric and characterized by an internally skinned and externally unskinned region of microvoids; approximately 0.15 mm long and 0.015 mm thick. These membranes have inner and outer diameters of approximately 1.395 mm and 1.925 mm, respectively. The nutrient solution permeates from the lumen side to the shell side of the MGR due to the transmembrane pressure gradient. The micro-organism is immobilised on the shell side of the MGR. Humidified air

is supplied on the shell side, and two pressure transducers are fitted at the inlet and outlet of the MGR as shown in Figure 1.

2.2. Model Assumptions. The theoretical models to be developed will be based on the following conditions of operation and assumptions: (1) the system is isothermal, meaning the energy equation has been decoupled from the mass and momentum transfer; (2) the flow regime within the membrane lumen is fully developed, laminar, homogeneous, and at steady state; (3) the physical and transport parameters (e.g., density, viscosity, and diffusivity) are constant; (4) in the dense and spongy layers of the membrane matrix the flow is only one dimensional (i.e., there are no axial components of the velocity profiles in the membrane matrix); and (5) the aspect ratio of the membrane is much smaller than unity. The aspect ratio, ϕ , is the ratio of the membrane inner radius to the effective membrane length (i.e., R_L/L), and if it is much smaller than unity then normal stress effects are negligible in the momentum transfer analysis.

3. Mathematical Formulation

The starting point of the analysis is the convective-diffusion equation [24]:

$$\frac{Dc}{Dt} = D_{AB} \nabla^2 c + r_A, \quad (1)$$

where c is the local substrate concentration; t is time; D_{AB} is the substrate diffusivity, assumed to be constant; and r_A , the rate of substrate production (or consumption), is a function of the local biofilm density, and time. Equation (1), for steady state, two-dimensional flow, without reaction, in cylindrical coordinates may be written as

$$u \frac{\partial c}{\partial z} - D_{AB} \frac{\partial^2 c}{\partial z^2} = \frac{D_{AB}}{r} \left(\frac{\partial c}{\partial r} + r \frac{\partial^2 c}{\partial r^2} \right) - v \frac{\partial c}{\partial r}. \quad (2)$$

It is convenient to express this equation in dimensionless form by introducing the following dimensionless variables:

$$\begin{aligned} U &= \frac{u}{u_0}, & V &= \frac{v}{v_0}, & C &= \frac{c}{c_0}, \\ Z &= \frac{z}{L}, & R &= \frac{r}{R_L}, & \phi &= \frac{R_L}{L}. \end{aligned} \quad (3)$$

The expressions of U and V in (3) are obtained from the momentum transfer analysis given in Appendix A. Substituting the dimensionless variables in (3) into (2) results in

$$\phi Pe_u U \frac{\partial C}{\partial Z} - \phi^2 \frac{\partial^2 C}{\partial Z^2} = \frac{1}{R} \left(\frac{\partial C}{\partial R} + R \frac{\partial^2 C}{\partial R^2} \right) - Pe_v V \frac{\partial C}{\partial R}, \quad (4)$$

where the axial and radial Peclet numbers ($Pe_{u,v}$) are defined as

$$Pe_u = \frac{u_0 R_L}{D_{AB}}, \quad Pe_v = \frac{v_0 R_L}{D_{AB}}. \quad (5)$$

The boundary conditions which match the imposed operating conditions of the MBR system are presented in Table 1.

FIGURE 2: A cross section of the single fibre MGR.

Boundary condition 1 (B.C.1) corresponds to a uniform inlet substrate concentration; B.C.2 and B.C.8 corresponds to cylindrical symmetry at the centre of the membrane lumen; B.C.3 indicates that the substrate concentration is only a function of the axial spatial coordinate at the membrane wall; B.C.4 corresponds to continuity of the substrate flux at the lumen-matrix interface; B.C.5 indicates a distance along the length of the membrane above which the axial concentration gradient becomes zero. The boundary conditions B.C.7–11 are employed in the solution of the velocity profiles given in Appendix A. A cross section of the MGR illustrating the three regions of the reactor, with the notation used in the model development, is shown in Figure 2.

If U in (4) is radially averaged, to become U_{av} , then the left-hand side (LHS) of (4) is only a function of Z and the right-hand side (RHS) only a function of R . This can only be true if both the LHS and RHS are independent of the variables R and Z . Equation (4) may therefore be solved by separation of variables to give a solution of the form

$$C = F(Z)T(R). \quad (6)$$

Substituting (6) into (4) gives

$$\begin{aligned} \phi \frac{Pe_u U_{av}}{F} \frac{dF}{dZ} - \frac{\phi^2}{F} \frac{d^2 F}{dZ^2} \\ = \frac{1}{RT} \left(\frac{dT}{dR} + R \frac{d^2 T}{dR^2} \right) - \frac{Pe_v V}{T} \frac{dT}{dR} = -\lambda^2. \end{aligned} \quad (7)$$

The equating of the two ordinary differential equations (ODEs) to the arbitrary constant $-\lambda^2$ in (7) is due to the fact that the two ODEs are independent of the variables R and Z .

3.1. Solution of the Axial Concentration Function $F(Z)$. To solve for the axial function $F(Z)$ of the substrate concentration profile, the ODE on the LHS of (7) is considered

$$\frac{d^2 F}{dZ^2} - \frac{Pe_u U_{av}}{\phi} \frac{dF}{dZ} - \frac{\lambda^2}{\phi^2} F = 0. \quad (8)$$

The radially-averaged axial velocity U_{av} in (8) is defined as

$$U_{av} = 2 \int_0^1 UR dR = -\frac{1}{8} \left(\frac{dP}{dZ} - \frac{Re}{Fr} \right), \quad (9)$$

where Fr and Re are the Froude and Reynolds number, respectively, given by

$$\begin{aligned} Fr &= \frac{u_0^2}{gR_L}, \\ Re &= \frac{\rho u_0 R_L}{\mu}, \end{aligned} \quad (10)$$

where g is the gravitational acceleration; ρ is the solution density; and μ is the solution dynamic viscosity. The solution of the axial velocity U in (9) is given in Appendix A as

$$U = -\frac{1}{4} (1 - R^2) \left(\frac{dP}{dZ} - \frac{Re}{Fr} \right), \quad (11)$$

with the axial pressure gradient given by

$$\frac{dP}{dZ} = 4\sqrt{\varphi^{-1}\kappa}\beta \sinh(4\sqrt{\varphi^{-1}\kappa})Z + \phi a \cosh(4\sqrt{\varphi^{-1}\kappa})Z, \quad (12)$$

where P is the dimensionless hydrostatic pressure; β is the dimensionless transmembrane pressure; a is the dimensionless entrance pressure drop; and κ is the dimensionless membrane hydraulic permeability. The entrance pressure drop a in (12) is given by

$$a = \frac{4\sqrt{\varphi^{-1}\kappa}\beta \sinh(4\sqrt{\varphi^{-1}\kappa}) - Re Fr^{-1}(1-f)}{\phi [f - \cosh(4\sqrt{\varphi^{-1}\kappa})]}, \quad (13)$$

where f is the fraction retentate ($f = 0$ for the dead-end mode and $f = 1$ for the closed-shell mode). The membrane hydraulic permeability κ in (12) is much smaller than unity ($\kappa \ll 1$), therefore, this equation can be approximated by the following expression:

$$\frac{dP}{dZ} \approx \phi a + 16\beta\varphi^{-1}\kappa Z. \quad (14)$$

This approximation makes (8) a confluent hypergeometric type differential equation. This is more evident if the following sequential substitutions are made.

Substitution 1.

$$\xi = \begin{cases} -\frac{Pe_u}{8\phi} \left(a\varphi + \frac{16\beta\kappa Z}{\phi} - \frac{Re}{Fr} \right), & 0 \leq Z < Z_0, \\ 0, & Z_0 \leq Z \leq 1. \end{cases} \quad (15)$$

TABLE 1: The boundary conditions of the MBR.

B.C.	R, Z	C, U, V, P_L	Range	Equation
B.C.1	$Z = 0$	$C = 1$	$0 \leq R \leq 1$	(6a)
B.C. 2	$R = 0$	$\frac{\partial C}{\partial R} = 0$	$0 \leq Z \leq 1$	(6b)
B.C. 3	$R = 1$	$C = C(Z)$	$0 \leq Z \leq 1$	(6c)
B.C. 4	$R = 1$	$-\frac{\partial C}{\partial R} + Pe_v VC = Sh[C(Z, 1) - C(Z, R_2)]$	$0 \leq Z \leq 1$	(6d)
B.C. 5	$Z \geq \frac{\varphi}{16\beta\kappa} \left(\frac{Re}{Fr} - a\varphi \right)$	$C(Z) = \text{constant}$	$0 \leq R \leq 1$	(6e)
B.C. 6	$R = 0$	$C = \text{finite}$	$0 \leq Z \leq 1$	(6f)
B.C. 7	$R = 1$	$U = 0$	$0 \leq Z \leq 1$	(6g)
B.C. 8	$R = 0$	$\frac{\partial U}{\partial R} = 0$	$0 \leq Z \leq 1$	(6h)
B.C. 9	$R = 0$	$V = 0$	$0 \leq Z \leq 1$	(6i)
B.C.10	$R = 1$	$V = V_M$	$0 \leq Z \leq 1$	(6j)
B.C.11	$Z = 0$	$P = P_0; P' = \varphi a$	$0 \leq R \leq 1$	(6k)

The substitution ξ in (15) represents the axial gradient (or driving force) of the substrate concentration profile. The axial distance along the membrane length at which this gradient is zero represents the point at which the axial function of the concentration profile, $F(Z)$, is constant. This axial distance, Z_0 , is obtained by equating (15) to zero as follows:

$$\therefore \xi = 0 \quad \text{at } Z_0 = \frac{\varphi}{16\beta\kappa} \left(\frac{Re}{Fr} - \varphi a \right). \quad (16)$$

The substitution in (15) transforms (8) to

$$\theta \frac{d^2 F}{d\xi^2} - A\xi \frac{dF}{d\xi} - \frac{A^2 \lambda^2}{\varphi^2} F = 0, \quad (17)$$

where

$$A = -\varphi^2 (2Pe_u \beta \kappa)^{-1}. \quad (18)$$

Substitution 2.

$$\theta = \frac{1}{2} A \xi^2. \quad (19)$$

This substitution transforms (17) to

$$\theta \frac{d^2 F}{d\theta^2} + \left(\frac{1}{2} - \theta \right) \frac{dF}{d\theta} - \frac{A\lambda^2}{2\varphi^2} F = 0. \quad (20)$$

Equation (20) is the standard Kummer hypergeometric equation and has two solutions, the Kummer function of the first kind $M(\alpha, \gamma, \theta)$ and the Tricomi function $\Phi(\alpha, \gamma, \theta)$, respectively [25]:

$$M(\alpha, \gamma, \theta) = 1 + \frac{\alpha}{\gamma} \theta + \frac{(\alpha)_2}{(\gamma)_2 2!} \theta^2 + \cdots + \frac{(\alpha)_n}{(\gamma)_n n!} \theta^n, \quad (21)$$

where

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2) + \cdots (\alpha+n-1), \quad (\alpha)_0 = 1,$$

$$\Phi(\alpha, \gamma, \theta) = \frac{\pi}{\sin \pi \gamma} \left\{ \frac{M(\alpha, \gamma, \theta)}{\Gamma(1+\alpha-\gamma)\Gamma(\gamma)} - \theta^{1-\gamma} \frac{M(1+\alpha-\gamma, 2-\gamma, \theta)}{\Gamma(\alpha)\Gamma(2-\gamma)} \right\}, \quad (22)$$

where $\Gamma(n)$ is the gamma function. Therefore, the solution of (20) becomes

$$F(\theta) = F_0 M\left(\frac{A\lambda^2}{2\varphi^2}, \frac{1}{2}, \theta\right) + F_1 \Phi\left(\frac{A\lambda^2}{2\varphi^2}, \frac{1}{2}, \theta\right), \quad (23)$$

where F_0 and F_1 are coefficients obtained from the inlet condition ((6a) in Table 1). The Tricomi function approaches infinity as values of θ approach zero [26], therefore, the coefficient F_1 in (23) must be zero for this equation to satisfy ((6a) in Table 1). The coefficient F_0 is obtained from the inlet condition ((6a) in Table 1):

$$C = F_0 M\left(\frac{A\lambda^2}{2\varphi^2}, \frac{1}{2}, \theta_0\right) T(R) = 1 \quad \text{at } Z = 0, \quad (24)$$

where

$$\theta_0 = -\frac{Pe_u}{256\beta\kappa} \left(a\varphi - \frac{Re}{Fr} \right)^2. \quad (25)$$

From the definition in (21), the Kummer function $M(\alpha, \gamma, 0)$ is equal to 1 for all real values of α and γ (where $\gamma \neq 0$). The two piecewise solutions of the axial function of the dimensionless concentration profile, $F(\theta)$, are therefore

$$F(\theta) = \begin{cases} F_0 M\left(\frac{A\lambda^2}{2\varphi^2}, \frac{1}{2}, \theta\right), & 0 \leq Z < Z_0, \\ F_0, & Z_0 \leq Z \leq 1. \end{cases} \quad (26)$$

3.2. Solution of the Radial Concentration Function $T(R)$

3.2.1. Zero-Order Approximation. To solve for the radial function $T(R)$ of the substrate concentration profile, the ODE on the RHS of (7) is solved

$$\frac{d^2 T}{dR^2} + \left(\frac{1}{R} - Pe_v V \right) \frac{dT}{dR} + \lambda^2 T = 0. \quad (27)$$

The radial velocity V in (27) is given in Appendix A as

$$V = \varphi \left(\frac{u_0}{v_0} \right) \left[\frac{R}{8} \left(1 - \frac{R^2}{2} \right) \right] \frac{d^2 P}{dZ^2}. \quad (28)$$

From the approximation in (14)

$$\frac{d^2 P}{dZ^2} \approx 16\beta\varphi^{-1}\kappa, \quad \kappa \ll 1. \quad (29)$$

Equation (27) is solved by a regular perturbation technique:

$$T(R) = \sum_{n=0}^{\infty} \kappa^n T_n. \quad (30)$$

The magnitude of the membrane hydraulic permeability κ is very small; hence, validity of the perturbation method is assured. The equations to solve for the zero-order approximation, T_0 , of (27) are

$$\frac{d^2 T_0}{dR^2} + \frac{1}{R} \frac{dT_0}{dR} + \lambda^2 T_0 = 0; \quad (31)$$

$$T_0(0) - B_1 = 0; \quad \frac{dT_0(0)}{dR} - B_2 = 0 \quad (32)$$

Equation (31) is Bessel's differential equation and has a standard solution of the form

$$T_0 = B_1 J_0(\lambda R) + B_2 Y_0(\lambda R). \quad (33)$$

As R approaches zero in (33), the function Y_0 tends to minus infinity; and therefore, B_2 must be zero for the equation to satisfy B.C.6 at R is equal to zero.

$$\therefore T_0 = B_1 J_0(\lambda R). \quad (34)$$

Imposing B.C.3, the dimensionless concentration C is only a function of the axial coordinate Z at $R = 1$; therefore, the function T_0 should be equal to the constant $B_1 J_0(\lambda)$ at $R = 1$, but $J_0(\lambda)$ is an oscillating function that can have a number of roots that satisfy the condition of B.C.3:

$$\therefore T_0 = \sum_{m=1}^{\infty} B_{1m} J_0(\lambda_m R). \quad (35)$$

The solution for the coefficient B_{1m} in (35) is given in Appendix B.1; the eigenvalues λ_m are derived from B.C.4 and are given in Appendix B.2.

3.3. First-Order and Second-Order Approximations. The equations to solve for the first-order approximation, T_1 , of (27) are

$$\frac{d^2 T_1}{dR^2} + \frac{1}{R} \frac{dT_1}{dR} + \lambda_m^2 T_1 = \delta \left[R \left(1 - \frac{R^2}{2} \right) \right] \frac{dJ_0(\lambda_m R)}{dR}, \quad (36)$$

$$T_1(0) = 0, \quad \frac{dT_1(0)}{dR} = 0, \quad (37)$$

where

$$\delta = 2Pe_v \beta \left(\frac{u_0}{v_0} \right) \sum_{m=1}^{\infty} B_{1m}. \quad (38)$$

Equation (36) is evaluated by making use of the following identity of Bessel functions [25]:

$$\frac{dJ_0(\lambda_m R)}{dR} = -\lambda_m J_1(\lambda_m R). \quad (39)$$

Substituting (39) into (36) results in the following inhomogeneous O.D.E:

$$\frac{d^2 T_1}{dR^2} + \frac{1}{R} \frac{dT_1}{dR} + \lambda_m^2 T_1 = -\delta \lambda_m \left[R \left(1 - \frac{R^2}{2} \right) \right] J_1(\lambda_m R). \quad (40)$$

Equation (40) is further simplified by making use of the following substitution:

$$x = \lambda_m R. \quad (41)$$

This substitution simplifies (40) to

$$x \frac{d^2 T_1}{dx^2} + \frac{dT_1}{dx} + x T_1 = -\frac{\delta x^2}{\lambda_m^2} \left(1 - \frac{x^2}{2\lambda_m^2} \right) J_1(x). \quad (42)$$

Some mathematical architecture is required to solve (42) and this is described in Appendix C.1. The solution of this equation is given in Appendix C.2 as

$$T_1(x) = i_3 \left[\frac{x^2 J_2(x)}{3!!} + i_1 \frac{x^3 J_3(x)}{5!!} + i_2 \frac{x^4 J_4(x)}{7!!} \right], \quad (43)$$

where

$$i_1 = -\frac{20}{3\lambda_m^2}, \quad i_2 = \frac{35}{4\lambda_m^2}, \quad i_3 = -\frac{3\delta}{4\lambda_m^2}. \quad (44)$$

The differential equations to solve for the second-order approximation, T_2 , of (27) are

$$x \frac{d^2 T_2}{dx^2} + \frac{dT_2}{dx} + x T_2 = -\frac{\delta x^2}{\lambda_m^2} \left(1 - \frac{x^2}{2\lambda_m^2} \right) T_1'(x), \quad (45)$$

$$T_2(0) = 0, \quad \frac{dT_2(0)}{dx} = 0. \quad (46)$$

The mathematical architecture required for the solution of these equations is also given in Appendix C.1, and the solution is given in Appendix C.3:

$$\begin{aligned}
 T_2(x) &= \left(\frac{3\delta^2}{2\lambda_m^6} \right) \left\{ \frac{20}{3} \lambda_m^2 \frac{x^3 J_3(x)}{5!!} \right. \\
 &\quad - \left[105 - \frac{7}{3} \lambda_m^2 (6i_1 - 5) \right] \frac{x^4 J_4(x)}{7!!} \\
 &\quad + \frac{9}{10} [7(60 - 48i_1) + 2\lambda_m^2 (8i_2 - 7i_1)] \frac{x^5 J_5(x)}{9!!} \\
 &\quad - \frac{99}{12} [(35 - 112i_1 + 80i_2) + 2\lambda_m^2 i_2] \frac{x^6 J_6(x)}{11!!} \\
 &\quad - \frac{9 \cdot 11 \cdot 13}{14} (7i_1 - 20i_2) \frac{x^7 J_7(x)}{13!!} \\
 &\quad \left. - \frac{9 \cdot 11 \cdot 13 \cdot 15}{16} \cdot i_2 \frac{x^8 J_8(x)}{15!!} \right\}. \quad (47)
 \end{aligned}$$

Substituting (26) and (30) into (6), the solution of the dimensionless luminal concentration profile is therefore

$$C(\theta, x) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} F_m(\theta) T_{mn}(x) \kappa^n. \quad (48)$$

The perturbation solution is only extended up to second-order approximation; therefore, (48) reduces to

$$C(\theta, x) = \sum_{m=1}^{\infty} F_m(\theta) [T_{m0}(x) \kappa^0 + T_{m1}(x) \kappa^1 + T_{m2}(x) \kappa^2]. \quad (49)$$

4. Results

4.1. Model Validation. The small diameters of the capillary membranes used in the construction of the MBRs render it a very difficult task to validate the accuracy of the developed models experimentally, and thus the predictive power of the models will be compared to currently available results for similar MBR systems; in particular the model developed by Heath and Belfort [17]. The approach used by these authors in solving the convective-diffusion equation was to assume a constant cross-sectional concentration profile in the membrane lumen and matrix regions, and to solve the simplified form of the equation. The lumen side axial concentration profile was then obtained from a mass flux balance. This approach is equivalent to assuming that the substrate consumption takes place in the lumen side of the MBR; in such case the convective-diffusion equation given in (1) becomes

$$u \frac{\partial c}{\partial z} - D_{AB} \frac{\partial^2 c}{\partial z^2} = \frac{D_{AB}}{r} \left(\frac{\partial c}{\partial r} + r \frac{\partial^2 c}{\partial r^2} \right) - v \frac{\partial c}{\partial r} - \frac{V_m c}{K_m + c}, \quad (50)$$

where V_m is the maximum rate of reaction and K_m is the Michaelis constant. This equation is made dimensionless by introducing the variables in (3) and (5), and two additional variables the Thiele modulus ϕ and the dimensionless Michaelis constant K_m^* , respectively:

$$K_m^* = \frac{K_m}{c_0}, \quad \phi = \sqrt{\frac{V_m R_L^2}{c_0 D_{AB}}}. \quad (51)$$

Equation (50) then becomes

$$\begin{aligned}
 \phi Pe_u U \frac{\partial C}{\partial Z} - \phi^2 \frac{\partial^2 C}{\partial Z^2} \\
 = \frac{1}{R} \left(\frac{\partial C}{\partial R} + R \frac{\partial^2 C}{\partial R^2} \right) - Pe_v V \frac{\partial C}{\partial R} - \frac{\phi^2 C}{K_m^* + C}. \quad (52)
 \end{aligned}$$

Assuming the first-order limit of the Michaelis-Menten equation (i.e., $K_m^* \gg C$) the solution of (52) takes the same general approach as that of (4). The solution of this equation is identical to (49) with an adjustment of the axial function $F_m(\theta)$ and the coefficient of the perturbation solution B_{1m} to account for the reaction rate:

$$C(\theta, x) = \sum_{m=1}^{\infty} \tilde{F}_m(\theta) [T_{m0}(x) \kappa^0 + T_{m1}(x) \kappa^1 + T_{m2}(x) \kappa^2], \quad (53)$$

where

$$\begin{aligned}
 \tilde{F}_m(\theta) &= F_0 M \left(\frac{A(\lambda_m^2 K_m^* + \phi^2)}{2\phi^2 K_m^*}, \frac{1}{2}, \theta \right), \\
 \tilde{B}_{1m} &= \frac{2}{\lambda_m F_0 M (A(\lambda_m^2 K_m^* + \phi^2)/2\phi^2 K_m^*, 1/2, \theta_0)} \\
 &\quad \times \left[\frac{J_1(\lambda_m)}{J_0^2(\lambda_m) + J_1^2(\lambda_m)} \right]. \quad (54)
 \end{aligned}$$

The input variables used in validating (53) are obtained from Heath and Belfort [17] and are given in Table 2. The values of the membrane hydraulic permeability (k_m), the fraction retentate (f), the lumen-side inlet hydrostatic pressure (p_0), the shell-side hydrostatic pressure (p_s), the solution density (ρ), and viscosity (μ) are not specified by these authors; therefore, typical operational values of these parameters will be used. A transmembrane (TMP) pressure of 5 kPa is assumed across the MBR, and the solution properties are assumed to be those of water at 30°C.

In Figure 3, it can be seen that (53) compares satisfactorily with the model of Heath and Belfort [17] for the parameter values listed in Table 2. At the centre of the MBR ($R = 0$) Equation (53) predicts that the concentration decreases axially to 34% of its original value when the applied TMP is 5 kPa, and at the membrane wall ($R = 1$) the concentration decreases up to 24% of its original value as shown in Figure 4. The model of Heath and Belfort [17] predicts a 44% decrease for all values of R for the corresponding conditions. It is important to note, however, that only the resulting trends from the two models can be

TABLE 2: Parameter values used to determine the concentration profile [17].

Model parameter	Symbol	Unit	Basic measured value
Membrane hydraulic permeability ^a	k_m	m/Pas	3.82×10^{-11}
Fraction retentate ^a	F		0.70
Membrane inner radius	R_L	m	1.30×10^{-4}
Annulus radius	r_3	m	4.08×10^{-4}
Effective membrane length	L	m	5.7×10^{-2}
Lumen-side entrance axial velocity	u_0	ms^{-1}	1.67×10^{-3}
Permeation velocity	v_0	ms^{-1}	3.82×10^{-7}
Number of fibres	N		150
Lumen-side inlet hydrostatic pressure ^a	p_0	Pa	106 325
Shell-side hydrostatic pressure ^a	p_s	Pa	101 325
Glucose diffusivity	D_{AB}	m^2s	1.0×10^{-10}
solution density ^a	ρ	kg m^{-3}	998
solution viscosity ^a	μ	Pas	9.7×10^{-4}
Glucose inlet concentration	c_0	g dm^{-3}	2.00
Kinetic constants	V_m/K_m	s^{-1}	1.00

^aTypical operational values are used.

FIGURE 3: A comparison of the concentration profiles resulting from (53) with Heath and Belfort [17] assuming a first-order limit for substrate consumption (at $R = 0$).

compared, since (53) requires a more detailed description of the MBR system than that required for the model of Heath and Belfort [17].

5. Conclusion

A rigour analytical mathematical model for substrate concentration profiles in the lumen of a hollow fiber MBR was developed. The model was based on the solution of the convective-diffusion equation in dimensionless form. The model allows evaluation of the influence of the general

FIGURE 4: Substrate concentration profiles assuming a first-order limit for substrate consumption at different radial positions.

operating parameters of a MBR on the concentration profiles. These parameters are the fraction retentate (f); the membrane hydraulic permeability (κ); the axial and radial Peclet numbers ($Pe_{u,v}$); the Thiele modulus (ϕ); the fluid properties; and the dimensions of the MBR. The developed model can be further used to evaluate reactor performance from basic principles, since it allows analytical evaluation of performance parameters (e.g., the performance index and the effectiveness factor). The current paper is a precursor to a paper by the same authors on bioreactor performance. The models developed in the current paper will be used to develop expressions for the effectiveness factor and

performance index (PI) of a capillary membrane gradostat reactor.

Appendices

A. Momentum Transfer Analysis

A.1. Momentum Transfer Analysis. The z -component of the nonconservative form of the Navier-Stokes equation in cylindrical coordinates is made dimensionless by introducing the variables in (4) and the following additional variables:

$$P = \frac{pR_L^2}{\mu u_0 L}, \quad \tau = \frac{tu_0}{R_L}, \quad \text{Fr} = \frac{u_0^2}{gR_L}; \quad (\text{A.1})$$

where p and P are the hydrostatic and dimensionless hydrostatic pressures, respectively; t and τ are the time and dimensionless time, respectively; μ is the solution dynamic viscosity; and Fr is the Froude number. The dimensionless form of the Navier-Stokes thus becomes

$$\frac{\partial U}{\partial \tau} = \frac{1}{\text{Re}} \left[\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial U}{\partial R} \right) + \varphi^2 \frac{\partial^2 U}{\partial Z^2} - \frac{dP}{dZ} \right] + \frac{1}{\text{Fr}}, \quad (\text{A.2})$$

and the continuity equation

$$\frac{1}{R} \frac{\partial(RV)}{\partial R} = -\varphi \left(\frac{u_0}{v_0} \right) \frac{\partial U}{\partial Z}. \quad (\text{A.3})$$

Equation (A.2) is solved following the procedure of Godongwana et al. [13], and the boundary conditions listed in Table 1 (B.C.7 and B.C.8). Ignoring normal stresses $\partial^2 U / \partial Z^2$ and considering steady-state conditions, the solution of (A.2) is given by

$$U = -\frac{1}{4}(1 - R^2) \left(\frac{dP}{dZ} - \frac{\text{Re}}{\text{Fr}} \right). \quad (\text{A.4})$$

The dimensionless radial velocity profile V is obtained by substituting (A.4) into (A.3) and imposing B.C.9:

$$V = \varphi \left(\frac{u_0}{v_0} \right) \left[\frac{R}{8} \left(1 - \frac{R^2}{2} \right) \right] \frac{d^2 P}{dZ^2}. \quad (\text{A.5})$$

The dimensionless pressure profile P is obtained by imposing B.C.10, where the matrix velocity V_M is governed by Darcy's law:

$$V_M = - \left(\frac{u_0}{v_0} \right) \kappa (P_S - P_0), \quad (\text{A.6})$$

where the dimensionless membrane hydraulic permeability κ is given by

$$\kappa = \frac{\mu k_m L}{R_L^2}. \quad (\text{A.7})$$

Substituting (A.5) and (A.6) into (6j) results in

$$\frac{\varphi}{16} \frac{d^2 P}{dZ^2} = -\kappa (P_S - P_0). \quad (\text{A.8})$$

Equation (A.8) is solved by applying B.C.11 to give:

$$P = \beta \cosh \left(4\sqrt{\varphi^{-1}\kappa} \right) Z + \frac{\varphi a}{4\sqrt{\varphi^{-1}\kappa}} \sinh \left(4\sqrt{\varphi^{-1}\kappa} \right) Z + P_S, \quad (\text{A.9})$$

where the dimensionless entrance pressure drop a in (A.9) is obtained by defining a fraction retentate, f , as the ratio of the exit to the inlet velocity, that is, $f = U_1/U_0$:

$$a = \frac{4\sqrt{\varphi^{-1}\kappa} \beta \sinh \left(4\sqrt{\varphi^{-1}\kappa} \right) - \text{Re Fr}^{-1} (1 - f)}{\varphi \left[f - \cosh \left(4\sqrt{\varphi^{-1}\kappa} \right) \right]}. \quad (\text{A.10})$$

B. Solution of Constants

B.1. Solution of the Coefficient B_{1m} . Imposing the inlet condition (6a) into (48) gives

$$C(\theta_0, R) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} F_m(\theta_0) T_{mn}(\lambda_m R) \kappa^n = 1, \quad \text{at } Z = 0. \quad (\text{B.1})$$

For the zero-order approximation of the radial function $T(R)$, (B.1) becomes

$$F_0 \sum_{m=1}^{\infty} M \left(\frac{A\lambda_m^2}{2\varphi^2}, \frac{1}{2}, \theta_0 \right) B_{1m} J_0(\lambda_m R) = 1, \quad \text{at } Z = 0. \quad (\text{B.2})$$

To solve for B_{1n} in (B.2) both the LHS and the RHS are multiplied by $J_0(\lambda_{mx} R) R$ and integrated with respect to R over the interval 0-1,

$$\begin{aligned} \int_0^1 J_0(\lambda_{mx} R) R \left\{ \sum_{m=1}^{\infty} M \left(\frac{A\lambda_m^2}{2\varphi^2}, \frac{1}{2}, \theta_0 \right) B_{1m} J_0(\lambda_m R) \right\} dR \\ = \frac{1}{F_0} \int_0^1 J_0(\lambda_{mx} R) R dR. \end{aligned} \quad (\text{B.3})$$

The RHS of (B.3) is evaluated by making use of the following property of Bessel functions:

$$\int_0^z r^v J_{v-1}(r) dr = z^v J_v(z). \quad (\text{B.4})$$

The R.H.S of (B.3) then becomes

$$\text{R.H.S} = \frac{J_1(\lambda_{mx})}{\lambda_{mx} F_0}. \quad (\text{B.5})$$

The L.H.S of (B.3) may be evaluated by making use of Lommel integrals:

$$\begin{aligned} \int_0^x J_n(\alpha_k r) J_n(\alpha_m r) r dr &= 0 \quad (k \neq m), \\ \int_0^x r J_n^2(\alpha_m r) dr &= \frac{x^2}{2} \left[J_n'(\alpha_m x)^2 + \left(1 - \frac{n^2}{\alpha_m^2 x^2} \right) J_n^2(\alpha_m x) \right]. \end{aligned} \quad (\text{B.6})$$

TABLE 3: Positive roots of (B.10).

m	λ_m
1	4.0×10^{-08}
2	3.701
3	6.945
4	10.125
5	13.286
6	16.440
7	19.591
8	22.738
9	25.885
10	29.030

To give the following equation:

$$\text{LHS} = \frac{B_{1m}}{2} M \left(\frac{A\lambda_m^2}{2\varphi^2}, \frac{1}{2}, \theta_0 \right) [J_0^2(\lambda_m) + J_1^2(\lambda_m)]. \quad (\text{B.7})$$

Substituting (B.5) and (B.7) back into the RHS and LHS of (B.3), respectively

$$B_{1m} = \frac{2}{\lambda_m F_0 M \left(\frac{A\lambda_m^2}{2\varphi^2}, \frac{1}{2}, \theta_0 \right)} \left[\frac{J_1(\lambda_m)}{J_0^2(\lambda_m) + J_1^2(\lambda_m)} \right]. \quad (\text{B.8})$$

B.2. Solution of the Eigenvalues λ_m . The solution of the eigenvalues is obtained from B.C.4 in Table 1. When the enzyme is immobilized on the lumen side of the membrane this equation reduces to

$$\frac{\partial C}{\partial R} - Pe_v VC = 0, \quad \text{at } R = 1. \quad (\text{B.9})$$

Substituting the expression for C , (49) into (B.9) and limiting the perturbation to first-order approximation:

$$\left(Pe_v VT_0 - \frac{dT_0}{dR} \right) \kappa^0 - \left(Pe_v VT_1 - \frac{dT_1}{dR} \right) \kappa^1 = 0, \quad \text{at } R = 1. \quad (\text{B.10})$$

The eigenvalues of (B.8) are values of λ_m that satisfy (B.10). The first 10 of these values are listed in Table 3 for the parameter values in Table 2.

C. Laplace Transform Solutions

C.1. Mathematical Architecture for the Solution of $T(x)$. We define a parameter p in Laplace space as

$$p = \frac{1}{\sqrt{1+s^2}}. \quad (\text{C.1})$$

The following properties of p are all noteworthy, differentiation:

$$p' = -sp^3. \quad (\text{C.2})$$

Integration:

$$\int p^n s ds = \frac{p^{n-2}}{2-n}, \quad n \neq 2. \quad (\text{C.3})$$

Inverse Laplace Transform:

$$\begin{aligned} \mathcal{L}^{-1}\{p\} &= J_0(x), \\ \mathcal{L}^{-1}\{p^{2n+1}\} &= \frac{x^n J_n(x)}{(2n-1)!!}. \end{aligned} \quad (\text{C.4})$$

Laplace Transforms involving the First-Order Bessel function:

$$\mathcal{L}\{J_1(x)\} = 1 - sp. \quad (\text{C.5})$$

More relations involving p :

$$s^2 = p^{-2} - 1, \quad (\text{C.6})$$

$$\frac{d}{ds}[1 - sp] = -p^3,$$

$$\frac{d^2}{ds^2}[1 - sp] = 3sp^5, \quad (\text{C.7})$$

$$\frac{d^3}{ds^3}[1 - sp] = 15p^7 - 12p^5,$$

$$\frac{d^4}{ds^4}[1 - sp] = 60sp^7 - 105sp^9.$$

We also define the following polynomial in p , which is significant for the first- and second-order approximation of $T(x)$ in Sections C.2 and C.3, respectively,

$$u(p) = p^5 + i_1 p^7 + i_2 p^9. \quad (\text{C.8})$$

Then it is easy to show that the second derivative of the product $su(p)$ with respect to the Laplace variable s is

$$\begin{aligned} \frac{d^2}{ds^2}[su(p)] &= s[20p^7 + 7(6i_1 - 5)p^9 + 9(8i_2 - 7i_1)p^{11} - 99i_2p^{13}]. \end{aligned} \quad (\text{C.9})$$

The fourth derivative of the product $su(p)$ with respect to the Laplace variable s is

$$\begin{aligned} \frac{d^4}{ds^4}[su(p)] &= s[840p^9 - 63(60 - 48i_1)p^{11} \\ &\quad + 99(35 - 112i_1 + 80i_2)p^{13} \\ &\quad - 9 \cdot 11 \cdot 13(20i_2 - 7i_1)p^{15} \\ &\quad + 9 \cdot 11 \cdot 13 \cdot 15 \cdot i_2 p^{17}]. \end{aligned} \quad (\text{C.10})$$

Both (C.9) and (C.10) will become important in Appendix C.3.

C.2. Solution of the First-Order Approximation Function, $T_1(x)$, in (42). If the function $g(s)$ is taken as the Laplace transform of the function $T_1(x)$, that is, $\mathcal{L}\{T_1(x)\} = g(s)$, then the Laplace transform of (42) yields

$$\begin{aligned} -s^2 \frac{d}{ds} g(s) - sg(s) - \frac{d}{ds} g(s) \\ = \frac{\delta}{2\lambda_m^4} \{ \mathcal{L}[x^4 J_1(x)] - 2\lambda_m^2 \mathcal{L}[x^2 J_1(x)] \}. \end{aligned} \quad (C.11)$$

In terms of the parameter p , defined in Appendix C.1, this equation may be written as

$$\begin{aligned} p^{-2} \frac{d}{ds} g(s) + sg(s) \\ = -\frac{\delta}{2\lambda_m^4} \left\{ \frac{d^4}{ds^4} (1-sp) - 2\lambda_m^2 \frac{d^2}{ds^2} (1-sp) \right\}. \end{aligned} \quad (C.12)$$

The right-hand side of (C.12) is evaluated from the relations involving p in Appendix C.1 (C.7). Multiplying through by p (C.12) becomes

$$\frac{d}{ds} [p^{-1} g(s)] = -\frac{\delta}{2\lambda_m^4} \{-60sp^8 + 105s^2p^{10} + 6\lambda_m^2 sp^6\}. \quad (C.13)$$

Integrating (C.13) with respect to s , to find the Laplace space solution:

$$g(s) = -\frac{3\delta}{4\lambda_m^2} \left(p^5 - \frac{20}{3\lambda_m^2} p^7 + \frac{35}{4\lambda_m^2} p^9 \right). \quad (C.14)$$

For convenience of expressing the solutions of the first- and second-order approximations, $T_1(x)$ and $T_2(x)$, respectively, the following constants and function are defined:

$$i_1 = -\frac{20}{3\lambda_m^2}, \quad (C.15a)$$

$$i_2 = \frac{35}{4\lambda_m^2}, \quad (C.15b)$$

$$g(s) = i_3 u(p), \quad (C.15c)$$

where

$$i_3 = -\frac{3\delta}{4\lambda_m^2}. \quad (C.16)$$

The polynomial $u(p)$ in (C.15c) was defined in (C.8). The inverse Laplace transform of the function $g(s)$ in (C.14) is the solution of $T_1(x)$:

$$\begin{aligned} T_1(x) &= \mathcal{L}^{-1}\{g(s)\} \\ &= i_3 \mathcal{L}^{-1}\{p^5 + i_1 p^7 + i_2 p^9\} \\ &= i_3 \left[\frac{x^2 J_2(x)}{3!!} + i_1 \frac{x^3 J_3(x)}{5!!} + i_2 \frac{x^4 J_4(x)}{7!!} \right]. \end{aligned} \quad (C.17)$$

C.3. Solution of the Second-Order Approximation Function, $T_2(x)$, in (45). Similar to the first-order approximation, the Laplace transform of the second-order approximation, for $\mathcal{L}\{T_2(x)\} = h(s)$, yields

$$\begin{aligned} -s^2 \frac{d}{ds} h(s) - sh(s) - \frac{d}{ds} h(s) \\ = \frac{\delta}{2\lambda_m^4} \{ \mathcal{L}[x^4 T_1'(x)] - 2\lambda_m^2 \mathcal{L}[x^2 T_1'(x)] \}. \end{aligned} \quad (C.18)$$

In terms of the parameter p , defined in Appendix C.1, this equation may be written as

$$p^{-2} \frac{d}{ds} h(s) + sh(s) = -\frac{2i_3^2}{3\lambda_m^2} \left\{ \frac{d^4}{ds^4} su - 2\lambda_m^2 \frac{d^2}{ds^2} su \right\}. \quad (C.19)$$

The expressions for the fourth derivative and second derivative of $su(p)$ are given in Section C.1, (C.9)-(C.10). The first-order differential equation in $h(s)$ has an integrating factor p^{-1} , so again (C.19) is multiplied by p to obtain

$$\begin{aligned} \frac{d}{ds} [p^{-1} h(s)] &= \frac{2i_3^2 s}{3\lambda_m^2} \{ [840p^{10} - 63(60 - 48i_1)p^{12} \\ &\quad + 99(35 - 112i_1 + 80i_2)p^{14} \\ &\quad + 9 \cdot 11 \cdot 13(7i_1 - 20i_2)p^{16} \\ &\quad + 9 \cdot 11 \cdot 13 \cdot 15 \cdot i_2 p^{18}] \\ &\quad - 2\lambda_m^2 [20p^8 + 7(6i_1 - 5)p^{10} \\ &\quad + 9(8i_2 - 7i_1)p^{12} - 99i_2 p^{14}] \}. \end{aligned} \quad (C.20)$$

The integral of (C.20), after simplifying by grouping terms with like powers of p , is

$$\begin{aligned} p^{-1} h(s) &= \frac{2i_3^2}{3\lambda_m^2} \left\{ \frac{20}{3} \lambda_m^2 p^6 - \left[105 - \frac{7}{3} \lambda_m^2 (6i_1 - 5) \right] p^8 \right. \\ &\quad + \frac{9}{10} [7(60 - 48i_1) + 2\lambda_m^2 (8i_2 - 7i_1)] p^{10} \\ &\quad - \frac{99}{12} [(35 - 112i_1 + 80i_2) + 2\lambda_m^2 i_2] p^{12} \\ &\quad - \frac{9 \cdot 11 \cdot 13}{14} (7i_1 - 20i_2) p^{14} \\ &\quad \left. - \frac{9 \cdot 11 \cdot 13 \cdot 15}{16} \cdot i_2 p^{16} \right\}. \end{aligned} \quad (C.21)$$

Recalling that $\mathcal{L}\{T_2(x)\} = h(s)$, the second-order approximation of (27) is simply the inverse Laplace transform of (C.21):

$$\begin{aligned}
 T_2(x) &= \left(\frac{3\delta^2}{2\lambda_m^6} \right) \left\{ \frac{20}{3} \lambda_m^2 \frac{x^3 J_3(x)}{5!!} \right. \\
 &\quad - \left[105 - \frac{7}{3} \lambda_m^2 (6i_1 - 5) \right] \frac{x^4 J_4(x)}{7!!} \\
 &\quad + \frac{9}{10} [7(60 - 48i_1) + 2\lambda_m^2 (8i_2 - 7i_1)] \frac{x^5 J_5(x)}{9!!} \\
 &\quad - \frac{99}{12} [(35 - 112i_1 + 80i_2) + 2\lambda_m^2 i_2] \frac{x^6 J_6(x)}{11!!} \\
 &\quad - \frac{9 \cdot 11 \cdot 13}{14} (7i_1 - 20i_2) \frac{x^7 J_7(x)}{13!!} \\
 &\quad \left. - \frac{9 \cdot 11 \cdot 13 \cdot 15}{16} \cdot i_2 \frac{x^8 J_8(x)}{15!!} \right\}. \quad (C.22)
 \end{aligned}$$

Nomenclature

a :	Dimensionless entrance pressure drop
B_n :	Constants of integration of Bessel's equation, $n = 1, 2$
c :	Substrate concentration (g dm^{-3})
c_0 :	Substrate feed concentration (g dm^{-3})
$C = c/c_0$:	Dimensionless substrate concentration
D_{AB} :	Substrate diffusivity ($\text{m}^2 \text{s}^{-1}$)
$f = u_1/u_0$:	Fraction retentate
F_n :	Coefficients of the solution of Kummer's confluent hypergeometric equation, $n = 1, 2$
$\text{Fr} = u_0^2/(gR_L)$:	Froude number
$F(Z)$:	Dimensionless axial concentration function
g :	Gravitational acceleration (m s^{-2})
$g(s)$:	Laplace transform of the first-order approximation of the function $T(x)$
$h(s)$:	Laplace transform of the second-order approximation of the function $T(x)$
i_n :	Constants in the first and second-order approximations of the function $T(x)$, $n = 1, 2, 3$
$J_n(\lambda)$:	Bessel function of order n of the first kind
k_a :	Mass transfer coefficient (m s^{-1})
k_m :	Membrane hydraulic permeability ($\text{m Pa}^{-1} \text{s}^{-1}$)
K_m :	Michaelis constant (g dm^{-3})
$K_m^* = K_m/c_0$:	Dimensionless Michaelis constant
L :	Membrane effective length (m)
$M(\alpha, \gamma, \theta)$:	Kummer function of the first kind
p :	Hydrostatic pressure (Pa)

$P = p/(\rho u_0^2)$:	Dimensionless hydrostatic pressure
$Pe_u = u_0 R_L/D_{AB}$:	Axial Peclet number
$Pe_v = v_0 R_L/D_{AB}$:	Radial Peclet number
r_A :	Rate of substrate production/consumption ($\text{g dm}^{-3} \text{s}^{-1}$)
r :	Radial spatial coordinate (m)
$R = r/R_L$:	Dimensionless radial spatial coordinate
R_L :	Membrane lumen radius (m)
$\text{Re} = \rho u_0 R_L/\mu$:	Reynolds number
$\text{Sh} = k_a R_L/D_{AB}$:	Sherwood number
t :	Time (s)
$T(R)$:	Dimensionless radial concentration function
u :	Axial velocity (m s^{-1})
u_0 :	Feed axial velocity (m s^{-1})
$U = u/u_0$:	Dimensionless axial velocity
v :	Radial velocity (m s^{-1})
$v_0 = k_m(p_0 - p_s)$:	Permeation velocity (m s^{-1})
$V = v/v_0$:	Dimensionless radial velocity
V_m :	Maximum rate of reaction ($\text{g dm}^{-3} \text{s}^{-1}$)
$x = \lambda_m R$:	Substitution variable
$Y_n(\lambda)$:	Bessel function of order n of the second kind
z :	Axial spatial coordinate (m)
$Z = z/L$:	Dimensionless axial spatial coordinate
Z_0 :	Dimensionless axial distance at which the concentration gradient is zero.

Greek Letters

α :	First parameter in the Kummer functions of the first and second kind
$\beta = P - P_S$:	Dimensionless transmembrane pressure
δ :	Lumped parameter in (38)
ε :	Membrane porosity
ϕ :	Thiele modulus
$\Phi(\alpha, \gamma, \theta)$:	Tricomi function/Kummer function of the second kind
γ :	Second parameter in the Kummer functions of the first and second kind
$\Gamma(n)$:	Gamma function, $n = 1, 2, \dots$
$\varphi = R_L/L$:	Aspect ratio
$\kappa = \mu k_m L/R_L^2$:	Dimensionless membrane hydraulic permeability
λ_m :	Eigen values, $m = 1, 2, \dots$
μ :	Solution dynamic viscosity (Pa s)
θ :	Substitution variable
ρ :	Solution density (kg m^{-3})
$\tau = tu_0/R_L$:	Dimensionless time
ξ :	Axial gradient/driving force of substrate concentration profile.

Subscripts

0:	Membrane entrance
1:	Membrane exit
S:	Shell-side of the MBR.

Acknowledgments

The authors would like to thank the Deutscher Akademischer Austausch Dienst (DAAD) for financial support, Dr. Aletta van der Merwe for her contribution in validating some of the mathematical principles.

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