Research Article

Global Bifurcation for Second-Order Neumann Problem with a Set-Valued Term

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We study the global bifurcation of the differential inclusion of the form $-(ku')' + g(\cdot, u) \in \mu F(\cdot, u)$, $u'(0) = 0 = u'(1)$, where F is a “set-valued representation” of a function with jump discontinuities along the line segment $[0, 1] \times \{0\}$. The proof relies on a Sturm-Liouville version of Rabinowitz’s bifurcation theorem and an approximation procedure.

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1. Introduction

We are concerned with the following differential inclusion which arises from a Budyko-North type energy balance climate models:

$$\begin{aligned} -(ku')'(x) + g(x, u(x)) &\in \mu F(x, u(x)), \quad x \in (0, 1) \text{ a.e.} \\ u'(0) &= 0, \quad u'(1) = 0; \end{aligned} \tag{1.1}$$

see [1–6] and the references therein. In particular, the set-valued right-hand side arises from a jump discontinuity of the albedo at the ice-edge in these models. By filling in such a gap, one arrives at the set-valued problem (1.1). As in [6], we are here interested in a considerably simplified version as compared to the situation from climate modeling; for example, a one-dimensional regular Sturm-Liouville differential operator substitutes for a two-dimensional Laplace-Beltrami operator or a singular Legendre-type operator, and the jump discontinuity is transformed to $u = 0$ in a way, which resembles only locally the climatological problem.

Assume that

$$(H1) \quad k \in C^1([0, 1]), \inf k > 0;$$

(H2) $g \in C([0, 1] \times \mathbb{R})$, $g(x, \cdot)$ strictly increasing for $x \in [0, 1]$,

$$g_1(x) := \lim_{|y| \rightarrow 0} \frac{g(x, y)}{y} \quad (1.2)$$

exists uniformly for $x \in [0, 1]$, and $g_1(x) > 0$ on $[0, 1]$,

(H2') g satisfies that

$$g_2(x) := \lim_{|y| \rightarrow \infty} \frac{g(x, y)}{y} \quad (1.3)$$

exists uniformly for $x \in [0, 1]$;

(H3) $f_+ \in C([0, 1] \times \mathbb{R}_+, (0, \infty))$, $\inf f_+ > 0$, $f_- \in C([0, 1] \times \mathbb{R}_-, (-\infty, 0))$, $\sup f_- < 0$.

Let F in (1.1) be given by

$$F(x, y) := \begin{cases} \{f_+(x, y)\}, & x \in [0, 1], y > 0, \\ [f_-(x, 0), f_+(x, 0)], & x \in [0, 1], \\ \{f_-(x, y)\} & x \in [0, 1], y < 0, \end{cases} \quad (1.4)$$

and set

$$\mathcal{S} := \{(\mu, w) \in \mathbb{R} \times C^1([0, 1]) \mid (\mu, w) \text{ solves (1.1)}\}. \quad (1.5)$$

Throughout $\mathcal{S}$ will be considered as subset of the Banach space $Y := \mathbb{R} \times C^1[0, 1]$ under the norm

$$\|(\mu, w)\|_Y := \max\{|\mu|, \|w\|_\infty, \|w'\|_\infty\}. \quad (1.6)$$

Let

$$\mathbb{Z}_+ := \{0, 1, 2, \dots\}. \quad (1.7)$$

Using a Sturm-Liouville version of Rabinowitz's bifurcation theorem and an approximation procedure, Hetzer [6] proved the following.

Theorem A (see [6, Theorem]). *Let (H1)–(H3) be fulfilled. Then there exist sequences $\{C_n^\pm\}_{n \in \mathbb{Z}_+}$ of unbounded, closed, connected subsets of $\mathcal{S}$ with $(0, 0) \in C_n^\pm$ and the property that u has exactly n zeroes, which are all simple, if $(\mu, u) \in C_n^\pm \setminus \{(0, 0)\}$. Moreover, u is positive (negative) on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0, 1]$, if $(\mu, u) \in C_n^+$ ($(\mu, u) \in C_n^-$) and $u \not\equiv 0$.*

It is easy to see from Theorem A that the effect of the discontinuity at zero is a solution branch which consists of infinitely many subbranches all meeting in $(0, 0)$. Two subbranches are distinguished by the number of zeroes of the respective solutions. However, Theorem A provides no any information about the asymptotic behavior of $C_n^\pm$ at infinity.

It is the purpose of this paper to study the asymptotic behavior of $C_n^\pm$ at infinity, and accordingly, to determine values of μ , for which there exist infinitely many *nodal solutions* of (1.1) (here and after, a function $u \in AC^1[0, 1]$ is a *nodal solution* of (1.1) if all of zeroes of u are simple). To wit, we have the following.

Theorem 1.1. *Let (H1)–(H3) and (H2') be fulfilled. Assume that*

(H4)

$$(f_+)_{\infty}(x) = (f_-)_{\infty}(x) =: b(x) \in C([0, 1], (0, \infty)), \quad (1.8)$$

where

$$(f_+)_{\infty}(x) := \lim_{s \rightarrow +\infty} \frac{f_+(x, s)}{s}, \quad (f_-)_{\infty}(x) := \lim_{s \rightarrow -\infty} \frac{f_-(x, s)}{s}. \quad (1.9)$$

Then for each $n \in \mathbb{Z}_+$, C_n^+ joins $(0, 0)$ with (η_n, ∞) , C_n^- joins $(0, 0)$ with (η_n, ∞) , where η_n , ($n \in \mathbb{Z}_+$), is the n -th eigenvalue of the linear problem:

$$\begin{aligned} -(ku')'(x) + g_2(x)u(x) &= \eta b(x)u(x), \quad x \in [0, 1], \\ u'(0) &= 0, \quad u'(1) = 0. \end{aligned} \quad (1.10)$$

Corollary 1.2. *Let (H1)–(H4) and (H2') be fulfilled. Let $k \in \mathbb{N}$ be fixed. Then*

(1) *for each $\mu \in [\eta_{k-1}, \eta_k)$, (1.1) has infinitely many solutions:*

$$u_j^{\nu}, \quad \nu \in \{+, -\}, j \in \{k, k+1, \dots\}, \quad (1.11)$$

which satisfies that u_j^+ has exactly j simple zeroes and u_j^+ is positive on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0, 1]$, u_j^- has exactly j simple zeroes and u_j^- is negative on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0, 1)$;

(2) *for each $\mu \in (0, \eta_0)$, (1.1) has infinitely many solutions:*

$$u_j^{\nu}, \quad \nu \in \{+, -\}, j \in \{0, 1, 2, \dots\} \quad (1.12)$$

which satisfies that u_j^+ has exactly j simple zeroes, and u_j^+ is positive on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0, 1]$, u_j^- has exactly j simple zeroes, and u_j^- is negative on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0, 1)$.

2. Notations and Preliminary Results

Recall Kuratowski's notion of lower and upper limits of sequences of sets.

Definition 2.1 (see [7]). Let X be a metric space and let $\{Z_l\}_{l \in \mathbb{N}}$ be a sequence of subsets of X . The set

$$\limsup_{l \rightarrow \infty} Z_l := \left\{ x \in X : \liminf_{l \rightarrow \infty} \text{dist}(x, Z_l) = 0 \right\} \quad (2.1)$$

is called the upper limit of the sequence $\{Z_l\}$, whereas

$$\liminf_{l \rightarrow \infty} Z_l := \left\{ x \in X : \lim_{l \rightarrow \infty} \text{dist}(x, Z_l) = 0 \right\} \quad (2.2)$$

is called the lower limit of the sequence $\{Z_l\}$.

Definition 2.2 (see [7]). A component of a set M is meant a maximal connected subset of M .

Lemma 2.3 (see [7]). Suppose that Y is a compact metric space, A and B are nonintersecting closed subsets of Y , and no component of Y intersects both A and B . Then there exist two disjoint compact subsets Y_A and Y_B , such that $Y = Y_A \cup Y_B$, $A \subset Y_A$, $B \subset Y_B$.

Using the above Whyburn Lemma, Ma and An [8] proved the following.

Lemma 2.4 (see [8, Lemma 2.1]). Let Z be a Banach space and let $\{A_n\}$ be a family of closed connected subsets of Z . Assume that

- (i) there exist $z_n \in A_n$, $n = 1, 2, \dots$, and $z^* \in Z$, such that $z_n \rightarrow z^*$;
- (ii) $r_n = \sup\{\|x\| \mid x \in A_n\} = \infty$;
- (iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty} A_n) \cap B_R$ is a relatively compact set of Z , where

$$B_R = \{x \in Z \mid \|x\| \leq R\}. \quad (2.3)$$

Then there exists an unbounded component C in $\limsup_{l \rightarrow \infty} A_l$ and $z^* \in C$.

Remark 2.5. The limiting processes for sets go back at least to the work of Kuratowski [9]. Lemma 2.4 will play an important role in the proof of Theorem 1.1. It is a slight generalization of the following well-known results due to Whyburn [7].

Proposition 2.6 (Whyburn [7, page 12]). Let Z be a Banach space and let $\{A_n\}$ be a family of closed connected subsets of Z . Let $\liminf_{l \rightarrow \infty} A_l \neq \emptyset$ and $\bigcup_{l \in \mathbb{N}} A_l$ is relatively compact. Then $\limsup_{l \rightarrow \infty} A_l$ is nonempty, compact, and connected.

Lemma 2.7. Let $q \in C([0, 1], (0, \infty))$. Let $p_m \in C([0, 1], (0, \infty))$ be such that

$$p_m(t) \geq \rho, \quad t \in [0, 1] \quad (2.4)$$

for some $\rho > 0$. Suppose that the sequence $\{(\mu_m, y_m)\}$ satisfies

$$-(ky'_m)' + q(t)y_m = \mu_m p_m(t)y_m, \quad y'_m(0) = y'_m(1) = 0 \quad (2.5)$$

with either

$$(y_m|_I)(t) > 0 \quad \forall m \text{ sufficiently large} \quad (2.6)$$

or

$$(y_m|_I)(t) < 0 \quad \forall m \text{ sufficiently large,} \quad (2.7)$$

where $I := [\alpha, \beta]$ with $\alpha < \beta$ being a given closed subinterval of $(0, 1)$. Then

$$|\mu_m| \leq M_0 \quad (2.8)$$

for some positive constant M_0 .

Proof. We only deal with the case that $(y_m|_I)(t) > 0$ for all m sufficiently large. The other case can be treated by the similar way. We may assume that $(y_m|_I)(t) > 0$ for all $m \in \mathbb{N}$.

We divide the proof into three cases.

Case 1. Let (α_m, β_m) be a subinterval of $[0, 1]$ satisfying

- (i) $I \subset (\alpha_m, \beta_m)$;
- (ii) $y_m(\alpha_m) = y_m(\beta_m) = 0$;
- (iii) $y_m(t) > 0$ for all $t \in (\alpha_m, \beta_m)$.

Let $\psi_m(t)$ and $\varphi_m(t)$ be the unique solution of the problems:

$$\begin{aligned} -(ky')' + q(t)y &= 0, \quad t \in (\alpha_m, \beta_m), \\ y(\alpha_m) &= 0, \quad y'(\alpha_m) = 1, \\ -(ky')' + q(t)y &= 0, \quad t \in (\alpha_m, \beta_m), \\ y(\beta_m) &= 0, \quad y'(\beta_m) = -1, \end{aligned} \quad (2.9)$$

respectively. Then it is easy to check $\psi_m(\cdot)$ is nondecreasing on (α_m, β_m) , $\varphi_m(\cdot)$ is nonincreasing on (α_m, β_m) , and that Green's function $G_m(t, s)$ of

$$\begin{aligned} -(ky')' + q(t)y &= 0, \quad t \in (\alpha_m, \beta_m), \\ y(\alpha_m) &= y(\beta_m) = 0 \end{aligned} \quad (2.10)$$

is explicitly given by

$$G_m(t, s) = \frac{1}{\varphi_m(\alpha_m)} \begin{cases} \psi_m(t)\varphi_m(s), & \alpha_m \leq t \leq s \leq \beta_m, \\ \varphi_m(t)\psi_m(s), & \alpha_m \leq s \leq t \leq \beta_m. \end{cases} \quad (2.11)$$

Let $\Psi(t)$ and $\Phi(t)$ be the unique solution of the problems:

$$\begin{aligned} -(ky')' + q(t)y &= 0, & t \in (0, 1), \\ y(0) &= 0, & y'(0) = 1, \\ -(ky')' + q(t)y &= 0, & t \in (0, 1), \\ y(1) &= 0, & y'(1) = -1, \end{aligned} \quad (2.12)$$

respectively. Then it is easy to check that $\Psi(\cdot)$ is nondecreasing on $(0, 1)$ and $\Phi(\cdot)$ is nonincreasing on $(0, 1)$, and

$$\Phi(0) \geq \varphi_m(\alpha_m), \quad \Psi(1) \geq \varphi_m(\beta_m). \quad (2.13)$$

Let $\varphi_I(t)$ and $\psi_I(t)$ be the unique solution of the problems

$$\begin{aligned} -(ky')' + q(t)y &= 0, & t \in (\alpha, \beta), \\ y(\alpha) &= 0, & y'(\alpha) = 1, \\ -(ky')' + q(t)y &= 0, & t \in (\alpha, \beta), \\ y(\beta) &= 0, & y'(\beta) = -1, \end{aligned} \quad (2.14)$$

respectively. Then, for $(t, s) \in [\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4] \times [\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]$,

$$G_m(t, s) \geq \frac{1}{\Phi(0)} \varphi_I\left(\alpha + \frac{\beta - \alpha}{4}\right) \varphi_I\left(\beta - \frac{\beta - \alpha}{4}\right). \quad (2.15)$$

Since

$$\begin{aligned}
 \frac{G_m(t, s)}{G_m(s, s)} &\geq \begin{cases} \frac{\psi_m(t)}{\psi_m(s)}, & \alpha_m \leq t \leq s \leq \beta_m, \\ \frac{\varphi_m(t)}{\varphi_m(s)}, & \alpha_m \leq s \leq t \leq \beta_m, \end{cases} \\
 &\geq \begin{cases} \frac{\psi_m(t)}{\Psi(1)}, & \alpha_m \leq t \leq s \leq \beta_m, \\ \frac{\varphi_m(t)}{\Phi(0)}, & \alpha_m \leq s \leq t \leq \beta_m, \end{cases} \\
 &\geq \min \left\{ \frac{\psi_m(t)}{\Psi(1)}, \frac{\varphi_m(t)}{\Phi(0)} \right\} =: \delta_m(t),
 \end{aligned} \tag{2.16}$$

it follows that for $t \in [\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]$,

$$\begin{aligned}
 y_m(t) &= \mu_m \int_{\alpha_m}^{\beta_m} G_m(t, s) p_m(s) y_m(s) ds \\
 &\geq \delta_m(t) \mu_m \int_{\alpha_m}^{\beta_m} G_m(s, s) p_m(s) y_m(s) ds \\
 &\geq \delta_m(t) \left\| (y_m|_{[\alpha_m, \beta_m]}) \right\|_{\infty} \\
 &\geq \delta_m(t) \left\| (y_m|_{[\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]}) \right\|_{\infty} \\
 &\geq \delta_I(t) \left\| (y_m|_{[\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]}) \right\|_{\infty},
 \end{aligned} \tag{2.17}$$

where

$$\delta_I(t) := \min \left\{ \frac{\psi_I(t)}{\Psi(1)}, \frac{\varphi_I(t)}{\Phi(0)} \right\}. \tag{2.18}$$

Set

$$\delta_0 := \min \left\{ \delta_I(t) \mid t \in \left[\alpha + \frac{\beta - \alpha}{4}, \beta - \frac{\beta - \alpha}{4} \right] \right\}. \tag{2.19}$$

Then

$$\min_{t \in [\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]} y_m(t) \geq \delta_0 \left\| (y_m|_{[\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]}) \right\|_{\infty}. \tag{2.20}$$

By (2.5), we have that

$$y_m(t) = \mu_m \int_{\alpha_m}^{\beta_m} G_m(t, s) p_m(s) y_m(s) ds, \quad (2.21)$$

which together with (2.15) and (2.20) imply that for $t \in [\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]$,

$$\begin{aligned} & y_m(t) \\ & \geq \mu_m \int_I G_m(t, s) \rho y_m(s) ds \\ & \geq \mu_m \int_{\alpha + (\beta - \alpha)/4}^{\beta - (\beta - \alpha)/4} G_m(t, s) \rho y_m(s) ds \\ & \geq \delta_0 \mu_m \int_{\alpha + (\beta - \alpha)/4}^{\beta - (\beta - \alpha)/4} G_m(t, s) \rho ds \cdot \left\| (y_m|_{[\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]}) \right\|_{\infty} \\ & \geq \delta_0 \frac{\mu_m}{\Phi(0)} \varphi_I \left(\alpha + \frac{\beta - \alpha}{4} \right) \varphi_I \left(\beta - \frac{\beta - \alpha}{4} \right) \rho \int_{\alpha + (\beta - \alpha)/4}^{\beta - (\beta - \alpha)/4} ds \cdot \left\| (y_m|_{[\alpha + (\beta - \alpha)/4, \beta - (\beta - \alpha)/4]}) \right\|_{\infty}. \end{aligned} \quad (2.22)$$

Therefore

$$|\mu_m| \leq \left(\frac{\delta_0 \rho}{\Phi(0)} \varphi_I \left(\alpha + \frac{\beta - \alpha}{4} \right) \varphi_I \left(\beta - \frac{\beta - \alpha}{4} \right) \cdot \frac{\beta - \alpha}{2} \right)^{-1}. \quad (2.23)$$

Case 2. Let $(0, \beta_m)$ be a subinterval of $[0, 1]$ satisfying

- (i) $I \subset (0, \beta_m)$;
- (ii) $y'_m(0) = 0$, $y_m(\beta_m) = 0$;
- (iii) $y_m(t) > 0$ for all $t \in (0, \beta_m)$.

Let $\bar{\psi}_m(t)$ and $\bar{\varphi}_m(t)$ be the unique solution of the problems:

$$\begin{aligned} & -(ky')' + q(t)y = 0, \quad t \in (0, \beta_m), \\ & y'(0) = 0, \quad y(\beta_m) = 1, \\ & -(ky')' + q(t)y = 0, \quad t \in (0, \beta_m), \\ & y(\beta_m) = 0, \quad y'(\beta_m) = -1, \end{aligned} \quad (2.24)$$

respectively. Then it is easy to check that $\bar{\psi}_m(\cdot)$ is nondecreasing on $(0, \beta_m)$, $\bar{\varphi}_m(\cdot)$ is nonincreasing on $(0, \beta_m)$, and Green's function $G^*(t, s)$ of

$$\begin{aligned} & -(ky')' + q(t)y = 0, \quad t \in (0, \beta_m), \\ & y'(0) = y(\beta_m) = 0 \end{aligned} \quad (2.25)$$

is explicitly given by

$$G^*(t, s) = \frac{1}{\bar{\varphi}_m(0)} \begin{cases} \bar{\varphi}_m(t)\bar{\varphi}_m(s), & 0 \leq t \leq s \leq \beta_m, \\ \bar{\varphi}_m(t)\bar{\varphi}_m(s), & 0 \leq s \leq t \leq \beta_m. \end{cases} \quad (2.26)$$

By the similar method to prove Case 1, we may get the desired results.

Case 3. Let $(\alpha_m, 1)$ be a subinterval of $[0, 1]$ satisfying

- (i) $I \subset (\alpha_m, 1)$;
- (ii) $y_m(\alpha_m) = 0$, $y'_m(1) = 0$;
- (iii) $y_m(t) > 0$ for all $t \in (\alpha_m, 1)$.

Using the same method to prove Case 2, with obvious changes, we may show that (2.8) is true.

Case 4. Let $(\alpha_m, \beta_m) = (0, 1)$. We may assume that $y_m(t) > 0$ for all $(0, 1)$.

Let $\psi(t)$ and $\varphi(t)$ be the unique solution of the problems

$$\begin{aligned} -(ky')' + q(t)y &= 0, & t \in (0, 1), \\ y(0) &= 0, & y'(0) = 1, \\ -(ky')' + q(t)y &= 0, & t \in (0, 1), \\ y(1) &= 0, & y'(1) = -1, \end{aligned} \quad (2.27)$$

respectively. Then, it is easy to verify that ψ is strictly increasing on $[0, 1]$ and φ is strictly decreasing on $[0, 1]$. Using the same method to deal with Case 1, we may get the desired results. $\square$

3. Proof of the Results

Recall the proof of Theorem A.

By [6, Remark 1], the hypotheses (H1)–(H3) imply that

$$\mathcal{S} \cap \left((-\infty, 0] \times C^1([0, 1]) \right) = (-\infty, 0] \times \{0\}. \quad (3.1)$$

Actually, such continua can be obtained as upper limits in the sense of Kuratowski of sequences of solution continua from associated continuous problems. To this end one sets

$$d_f := \min\{\inf f_+, \inf |f_-|\} \quad (3.2)$$

and selects an approximation sequence $\{f_l\} \in C([0, 1] \times \mathbb{R}, \mathbb{R})^{\mathbb{N}}$ of F satisfying

- (A1) $f_l(x, y) = ly$ for $x \in [0, 1]$ and $y \in [-d_f/2l, d_f/2l]$;
- (A2) $f_l(x, y) \times \text{sgn}(y) \geq d_f/2$ for $x \in [0, 1]$ and $|y| \geq d_f/2l$; $f_l \leq f_+$ on $[0, 1] \times [d_f/2l, d_f/l]$; $f_l \geq f_-$ on $[0, 1] \times [-d_f/l, -d_f/2l]$;
- (A3) $f_l(x, y) = f_+(x, y)$ for $x \in [0, 1]$ and $y \geq d_f/l$; $f_l(x, y) = f_-(x, y)$ for $x \in [0, 1]$ and $y \leq -d_f/l$;
- (A4) $\{f_l(x, y)\}_{l \in \mathbb{N}}$ is nondecreasing in l for $(x, y) \in [0, 1] \times (0, \infty)$; $\{f_l(x, y)\}_{l \in \mathbb{N}}$ is nonincreasing in l for $(x, y) \in [0, 1] \times (-\infty, 0)$.

Remark 3.1. Let

$$\xi(x, u) := g(x, u) - g_1(x)u. \quad (3.3)$$

We may show that there exists a positive constant $\bar{\gamma}$, independent of l , such that for each $l \in \mathbb{N}$,

$$\frac{f_l(x, u)}{u} - \frac{\xi(x, u)}{\gamma u} \geq \rho_0, \quad \forall \gamma \geq \bar{\gamma} \quad (3.4)$$

for some constant $\rho_0 > 0$.

In fact, it is easy to see from the definition of f_l that

$$\frac{f_l(x, u)}{u} \geq \rho_1, \quad u \neq 0 \quad (3.5)$$

for some positive constant ρ_1 , independent of l .

Applying (H2) and (H2'), it concludes that

$$0 \leq \left| \frac{\xi(x, u)}{u} \right| \leq \rho_2 \quad (3.6)$$

for some positive constant ρ_2 . Therefore, if we take

$$\bar{\gamma} := \frac{2\rho_2}{\rho_1}, \quad \rho_0 = \frac{\rho_1}{2}, \quad (3.7)$$

then (3.4) holds.

It is easy to see thanks to (H2) and (A1) that

$$\begin{aligned} -(kv')'(x) + g(x, v(x)) &= \mu f_l(x, v(x)), \quad x \in [0, 1], \\ v'(0) &= 0, \quad v'(1) = 0 \end{aligned} \quad (3.8_l)$$

falls into the scope of the Sturm-Liouville version of the celebrated Rabinowitz bifurcation theorem (cf. [10] for a more general, but somewhat different setting).

Indeed, denote the strictly increasing sequence of simple eigenvalues of

$$\begin{aligned} -(k\psi')'(x) + g_1(x)\psi(x) &= \lambda\psi(x), \quad x \in [0, 1], \\ \psi'(0) &= 0, \quad \psi'(1) = 0, \end{aligned} \quad (3.9)$$

by $\{\lambda_n\}_{n \in \mathbb{Z}_+}$ and set

$$\mu_{n,l} := \frac{\lambda_n}{l}. \quad (3.10)$$

Then $(\mu_{n,l}, \mathbf{0})$ is a bifurcation point of the solution set of (3.8_l) for every $n \in \mathbb{Z}_+$, and for each $(n, l) \in \mathbb{Z}_+ \times \mathbb{N}$, there exist two unbounded closed connected subsets $C_{n,l}^\pm$ of the solution set of (3.8_l) with the following.

- (a) $C_{n,l}^+ \cap C_{n,l}^- = \{(\mu_{n,l}, \mathbf{0})\}$. Moreover, $(\mu_{n,l}, \mathbf{0})$ is the only bifurcation point contained in $C_{n,l}^\pm$.
- (b) If $(\mu, \vartheta) \in C_{n,l}^+$ and $\vartheta \neq 0$, then ϑ possesses exactly n simple zeroes (and no multiple zeroes) in $(0, 1)$ and is positive on $(0, \delta)$ for some $\delta > 0$.
- (c) If $(\mu, \vartheta) \in C_{n,l}^-$ and $\vartheta \neq 0$, then ϑ possesses exactly n simple zeroes (and no multiple zeroes) in $(0, 1)$ and is negative on $(0, \delta)$ for some $\delta > 0$.

Combining the above with the fact

$$\lim_{l \rightarrow \infty} (\mu_{n,l}, \mathbf{0}) = (0, \mathbf{0}) \quad (3.11)$$

and utilizing Lemma 2.4, it concludes that there exists an unbounded component C_n^v with

$$\begin{aligned} (0, \mathbf{0}) &\in C_n^v, \\ C_n^v &\subseteq \limsup_{l \rightarrow \infty} C_{n,l}^v, \quad v \in \{+, -\}. \end{aligned} \quad (3.12)$$

As an immediate consequence of [6, Lemma 4-6], we have the following

Lemma 3.2. *If $(\mu, u) \in C_n^\pm$, then (μ, u) is a solution of (1.1) and $u \in W^{2,\infty}([0, 1])$. Moreover, if $(\mu, u) \in C_n^+$ with $u \neq 0$, u has exactly n simple zeroes in $[0, 1]$, and u is positive on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0, 1]$; if $(\mu, u) \in C_n^-$ with $u \neq 0$, u has exactly n simple zeroes in $[0, 1]$, and u is negative on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0, 1]$.*

Lemma 3.3. *Let (H1)–(H4), (H2') and (A1)–(A4) be fulfilled. Then for each $(n, l) \in \mathbb{Z}_+ \times \mathbb{N}$, the connected component $C_{n,l}^\pm$ joins $(\mu_{n,l}, \mathbf{0})$ with (η_n, ∞) .*

Proof. Assume that $\{(r_k, y_k)\} \subset C_{n,l}^+$ for some fixed $(n, l) \in \mathbb{Z}_+ \times \mathbb{N}$ with

$$|r_k| + \|y_k\|_{C^1} \rightarrow \infty. \quad (3.13)$$

The case $\{(r_k, y_k)\} \subset C_{n,l}^-$ can be treated by the same way.

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$r_k \in (0, M], \quad (3.14)$$

then $C_{n,l}^+$ joins $(\mu_{n,l}, 0)$ with (η_n, ∞) . In this case it follows that

$$\|y_k\|_{C^1} \rightarrow \infty. \quad (3.15)$$

Define

$$\zeta_l(r, x, u) := r[f_l(x, u) - b(x)u] - [g(x, u) - g_2(x)u]. \quad (3.16)$$

Then $\{(r_k, y_k)\}$ satisfies the problem:

$$\begin{aligned} -(ky'_k)'(x) + g_2(x)y_k(x) &= r_k b(x)y_k(x) + \zeta_l(r_k, x, y_k(x)), \quad x \in [0, 1], \\ y'_k(0) &= 0, \quad y'_k(1) = 0. \end{aligned} \quad (3.17_l)$$

Set

$$\tilde{\zeta}_l(u) = \max_{0 \leq |s| \leq u, x \in [0, 1], r \in [0, M]} |\zeta_l(r, x, s)| \quad (3.18)$$

then $\tilde{\zeta}$ is nondecreasing, and (H4) and (H2') yields

$$\lim_{u \rightarrow \infty} \frac{\tilde{\zeta}(u)}{u} = 0. \quad (3.19)$$

Now, we divide (3.17_l) by $\|y_k\|_{C^1}$ and set $\bar{y}_k = (y_k / \|y_k\|_{C^1})$. Since $\bar{y}_k$ is bounded in $C^2[0, 1]$, after taking a subsequence if necessary, we have that $\bar{y}_k \rightarrow \bar{y}$ for some $\bar{y} \in C^1[0, 1]$ with $\|\bar{y}\|_{C^1} = 1$. Moreover, from the definition of f_l and (3.19) and the fact that $\tilde{\zeta}$ is nondecreasing, we have that

$$\lim_{k \rightarrow \infty} \frac{|\zeta_l(r_k, x, y_k(x))|}{\|y_k\|_{C^1}} = 0 \quad (3.20)$$

since

$$\frac{|\zeta_l(r_k, x, y_k(x))|}{\|y_k\|_{C_1}} \leq \frac{\tilde{\zeta}(\|y_k(x)\|)}{\|y_k\|_{C_1}} \leq \frac{\tilde{\zeta}(\|y_k\|_{\infty})}{\|y_k\|_{C_1}} \leq \frac{\tilde{\zeta}(\|y_k\|_{C_1})}{\|y_k\|_{C_1}}. \quad (3.21)$$

By standard limit procedure, we get

$$\begin{aligned} -(k\bar{y}')'(x) + g_2(x)\bar{y}(x) &= \bar{r}b(x)\bar{y}(x), \quad x \in [0, 1], \\ \bar{y}'(0) &= 0, \quad \bar{y}'(1) = 0, \end{aligned} \quad (3.22)$$

where $\bar{r} := \lim_{k \rightarrow \infty} r_k$, again choosing a subsequence and relabeling if necessary. Moreover, the fact that $y_k, k \in \mathbb{Z}_+$, has exactly n simple zeroes in $[0, 1]$ implies that $\bar{y}$ has exactly n simple zeroes in $[0, 1]$, too. Therefore $\bar{r} = \eta_n$.

Step 2. We show that there exists a constant M such that $r_k \in (0, M]$, for all n . Suppose there is no such M , choosing a subsequence and relabeling if necessary, it follows that

$$\lim_{k \rightarrow \infty} r_k = \infty. \quad (3.23)$$

Let

$$\tau(1, k) < \dots < \tau(n, k) \quad (3.24)$$

denote the zeroes of y_k , and set

$$0 = \tau(0, k), \quad \tau(n+1, k) = 1. \quad (3.25)$$

Then, after taking a subsequence if necessary,

$$\lim_{k \rightarrow \infty} \tau(l, k) := \tau(l, \infty), \quad l \in \{0, 1, \dots, n+1\}. \quad (3.26)$$

We claim that for all $l \in \{0, 1, \dots, n\}$

$$\tau(l+1, \infty) - \tau(l, \infty) = 0. \quad (3.27)$$

Suppose on the contrary that there exists $l_0 \in \{0, 1, \dots, n\}$ such that

$$\tau(l_0, \infty) < \tau(l_0 + 1, \infty). \quad (3.28)$$

Define a function $p : [0, 1] \rightarrow \mathbb{R}$ by

$$p_l(x) := \begin{cases} \frac{f_l(x, y_k(x))}{y_k(x)} - \frac{\xi(x, y_k(x))}{r_k y_k(x)}, & x \in [0, 1], y_k(x) \neq 0, \\ l, & y_k(x) = 0. \end{cases} \quad (3.29)$$

Then by Remark 3.1, there exists ρ_0 , such that

$$p_l(x) \geq \rho_0, \quad x \in [0, 1]. \quad (3.30)$$

Now we choose a closed interval $I \subset (\tau(l_0, \infty), \tau(l_0 + 1, \infty))$ with positive length, then we know from Lemma 2.7 that y_k (after taking a subsequence if necessary) must change sign on I . However, this contradicts the fact that for all k sufficiently large, we have $I \subset (\tau(l_0, k), \tau(l_0 + 1, k))$ and

$$(-1)^{l_0} \nu y_k(x) > 0, \quad x \in (\tau(l_0, k), \tau(l_0 + 1, k)). \quad (3.31)$$

Therefore, (3.27) holds.

On the other hand, it follows

$$1 = \tau(n + 1, k) - \tau(0, k) = \sum_{l=0}^n (\tau(l + 1, k) - \tau(l, k)) \quad (3.32)$$

that

$$1 = \sum_{l=0}^n (\tau(l + 1, \infty) - \tau(l, \infty)) \quad (3.33)$$

which contradicts (3.27).

Therefore

$$|r_k| \leq M \quad (3.34)$$

for some constant number $M > 0$, independent of $k \in \mathbb{N}$. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We only prove that C_n^+ has the desired property, the case of C_n^- can be treated by the same way.

Assume that $\{(\mu_k, z_k)\} \subset C_n^+$ is a sequence with

$$|\mu_k| + \|z_k\|_{C^1} \rightarrow \infty. \quad (3.35)$$

We claim that

$$\lim_{k \rightarrow \infty} (\mu_k, z_k) = (\eta_n, \infty). \quad (3.36)$$

Assume on the contrary that (3.36) is not true. We divide the proof into two cases.

Case 1. $\lim_{k \rightarrow \infty} \mu_k \neq \eta_n$. In this case, we may take a subsequence of $\{\mu_k\}$, denote it by $\{\mu_k\}$ again, with the property that there exists $\varepsilon_0 > 0$, such that for each $k \in \mathbb{N}$,

$$|\mu_k - \eta_n| \geq \varepsilon_0. \quad (3.37)$$

Since $\{(\mu_k, z_k)\} \subset C_n^+$, it follows that for each $k \in \mathbb{Z}_+$, there exists a sequence $\{(\gamma_{k_j}, z_{k_j})\} \subset C_{n, k_j}^+$ such that

$$\lim_{j \rightarrow \infty} \gamma_{k_j} = \mu_k, \quad \lim_{j \rightarrow \infty} z_{k_j} = z_k. \quad (3.38)$$

Now let us consider the sequence $\{(\gamma_{k_k}, z_{k_k})\}$. Obviously, we have that

$$\begin{aligned} (\gamma_{k_k}, z_{k_k}) &\in C_{n, k_k}^+, \\ |\gamma_{k_k}| + \|z_{k_k}\|_{C^1} &\longrightarrow \infty. \end{aligned} \quad (3.39)$$

Equation (3.39) implies that

$$\begin{aligned} -\left(kz'_{k_k}\right)'(x) + g_2(x)z_{k_k}(x) &= \gamma_{k_k}b(x)z_{k_k}(x) + \zeta_{k_k}(\gamma_{k_k}, x, z_{k_k}(x)), \quad x \in [0, 1], \\ z'_{k_k}(0) &= 0, \quad z'_{k_k}(1) = 0, \end{aligned} \quad (3.40)$$

Noticing that ρ_0 in (3.30) is independent of l and using Remark 3.1 and the method to prove Lemma 3.3 and with obvious changes, we may show that $\{\gamma_{k_k}\}$ is bounded, and subsequently

$$\lim_{k \rightarrow \infty} \gamma_{k_k} = \eta_n. \quad (3.41)$$

However, this contradicts (3.37).

Case 2. $\lim_{k \rightarrow \infty} \|z_k\|_{C^1} \neq \infty$. In this case, after taking a subsequence of $\{z_k\}$ and relabeling if necessary, we may assume that

$$\|z_k\|_{C^1} \leq M_0 \quad (3.42)$$

for some constant $M_0 > 0$. Equation (3.35) together with (3.42) implies

$$\lim_{k \rightarrow \infty} \mu_k = +\infty. \quad (3.43)$$

Using the same notations as those in Case 1, we have from (3.43) that

$$\lim_{k \rightarrow \infty} \gamma_{k_k} = +\infty. \quad (3.44)$$

Combining this with (3.40) and using Remark 3.1 and the similar method to prove Step 2 of Lemma 3.3 and noticing that ρ_0 in (3.30) is independent of l , it concludes that $\{\gamma_{k_k}\}$ is bounded. This is a contradiction. $\square$

Remark 3.4. It is easy to see from Theorem 1.1 and its proof that the “jumping” of F at $u = 0$: $f_+(x, 0) - f_-(x, 0) (=:\Delta(x))$ does not affect the asymptotic behavior of $C_n^\pm$ at infinity. In other words, for any nonnegative function $\Delta(x)$, the asymptotic behavior of $C_n^\pm$ at infinity is the same.

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