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Research Article

Existence of Solutions for Integrodifferential Equations of Fractional Order with Antiperiodic Boundary Conditions

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We discuss the existence of solutions for a nonlinear antiperiodic boundary value problem of integrodifferential equations of fractional order $q \in (1,2]$. The contraction mapping principle and Krasnoselskii's fixed point theorem are applied to establish the results.

1. Introduction

Recently, the subject of fractional differential equations has emerged as an important area of investigation. Fractional differential equations arise in many engineering and scientific disciplines as the fractional derivatives describe numerous events and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth. For some recent development on the subject, see [1–15] and the references therein.

Integrodifferential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible. For details, see [16–20] and the references therein.

Antiperiodic boundary value problems have recently received considerable attention as antiperiodic boundary conditions appear in numerous situations, for instance, see [21–25].

In this paper, we prove some existence and uniqueness results for the following antiperiodic fractional boundary value problem:

$$\begin{gathered} {}^{c}D^{q}x(t) = f\big(t, x(t), (\chi x)(t)\big), \quad t \in [0,T],\ T > 0,\ 1 < q \le 2, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \end{gathered} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,T] \times X \times X \to X$, and for $\gamma : [0,T] \times [0,T] \to [0,\infty)$,

$$(\chi x)(t) = \int_0^t \gamma(t,s)x(s)ds, \tag{1.2}$$

with $\gamma_0 = \max\{\int_0^t \gamma(t,s)ds : (t,s) \in [0,T] \times [0,T]\}$. Here, $(X, \|\cdot\|)$ is a Banach space and $C = C([0,T],X)$ denotes the Banach space of all continuous functions from $[0,T] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

2. Preliminaries

First of all, we recall some basic definitions [26–28].

Definition 2.1. For a function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \quad n-1 < q < n,\ n = [q]+1, \tag{2.1}$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \tag{2.2}$$

provided that the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q]+1, \tag{2.3}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.4 (see [8]). *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [q]+1$).

In view of Lemma 2.4, it follows that

$$I^q\, {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

Now, we state a known result due to Krasnoselskii [29] which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.5. *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (i)$Ax + By \in M$ *whenever* $x, y \in M$; (ii)A *is compact and continuous;* (iii)B *is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Lemma 2.6. *For any $\omega \in C[0,T]$, the unique solution of the boundary value problem*

$$\begin{gathered} {}^cD^q x(t) = \omega(t), \quad 0 < t < T,\ 1 < q \le 2, \\ x(0) = -x(T), \qquad x'(0) = -x'(T) \end{gathered} \tag{2.6}$$

is given by

$$x(t) = \int_0^T G(t,s)\omega(s)ds, \tag{2.7}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{q-1} - (1/2)(T-s)^{q-1}}{\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)}, & s \le t, \\ -\dfrac{(T-s)^{q-1}}{2\Gamma(q)} + \dfrac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)}, & t \le s. \end{cases} \tag{2.8}$$

Proof. Using (2.5), for some constants $b_1, b_2 \in \mathbb{R}$, we have

$$x(t) = I^q\omega(t) - b_1 - b_2 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\omega(s)ds - b_1 - b_2 t. \tag{2.9}$$

In view of the relations ${}^cD^q I^q x(t) = x(t)$ and $I^q\ I^p x(t) = I^{q+p}x(t)$ for $q, p > 0$, $x \in L(0,1)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\omega(s)ds - b_2. \tag{2.10}$$

Applying the boundary conditions $x(0) = -x(T)$, $x'(0) = -x'(T)$, we find that

$$b_1 = \frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}\omega(s)ds - \frac{T}{4\Gamma(q-1)}\int_0^T (T-s)^{q-2}\omega(s)ds,$$
$$b_2 = \frac{1}{2\Gamma(q-1)}\int_0^T (T-s)^{q-2}\omega(s)ds. \tag{2.11}$$

Thus, the unique solution of (2.6) is

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\omega(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\omega(s)ds \\ &\quad + \frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\omega(s)ds \\ &= \int_0^T G(t,s)\omega(s)ds, \end{aligned} \tag{2.12}$$

where $G(t,s)$ is given by (2.8). This completes the proof. □

3. Main Results

To prove the main results, we need the following assumptions:

(A_1) $\|f(t,x(t),(\chi x)(t)) - f(t,y(t),(\chi y)(t))\| \le L_1\|x-y\| + \overline{L}_1\|\chi x - \chi y\|$, for all $t \in [0,T]$, $x, y \in X$;

(A_2) $\|f(t,x(t),(\chi x)(t))\| \le \mu(t)$, for all $(t,x,\chi x) \in [0,T]\times X\times X$ and $\mu \in L^1([0,T],R^+)$.

Theorem 3.1. *Let $f : [0,T]\times X\times X \to X$ be a jointly continuous function satisfying the assumption (A_1) with $(L_1 + \gamma_0\overline{L}_1) \le \Gamma(q+1)/T^q(3+q/2)$. Then the antiperiodic boundary value problem* (1.1) *has a unique solution.*

Proof. Define $\Theta : C \to C$ by

$$\begin{aligned} (\Theta x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s),(\chi x)(s))ds \\ &\quad - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s),(\chi x)(s))ds \\ &\quad + \frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s),(\chi x)(s))ds, \quad t \in [0,T]. \end{aligned} \tag{3.1}$$

Setting $\sup_{t\in[0,T]}\|f(t,0,0)\| = M$ and choosing $r \geq (MT^q/\Gamma(q+1))(3+q/2)$, we show that $\Theta B_r \subset B_r$, where $B_r = \{x \in C : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}
\|(\Theta x)(t)\| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),(\chi x)(s))\|ds \\
&\quad + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),(\chi x)(s))\|ds \\
&\quad + \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s),(\chi x)(s))\|ds \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\|f(s,x(s),(\chi x)(s)) - f(s,0,0)\| + \|f(s,0,0)\|]ds \\
&\quad + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}[\|f(s,x(s),(\chi x)(s)) - f(s,0,0)\| + \|f(s,0,0)\|]ds \\
&\quad + \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}[\|f(s,x(s),(\chi x)(s)) - f(s,0,0)\| + \|f(s,0,0)\|]ds \\
&\leq \left(\left(L_1+\gamma_0\overline{L}_1\right)r + M\right)\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}ds\right. \\
&\qquad \left. + \frac{1}{4\Gamma(q-1)}|T-2t|\int_0^T (T-s)^{q-2}ds\right] \\
&\leq \left(\left(L_1+\gamma_0\overline{L}_1\right)r + M\right)\left[\frac{T^q}{2\Gamma(q+1)}\left(3+\frac{q}{2}\right)\right] \leq r.
\end{aligned} \tag{3.2}$$

Now, for $x, y \in C$ and for each $t \in [0,T]$, we obtain

$$\begin{aligned}
\|(\Theta x)(t) - (\Theta y)(t)\| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),(\chi x)(s)) - f(s,y(s),(\chi y)(s))\|ds \\
&\quad + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\|f(s,x(s),(\chi x)(s)) - f(s,y(s),(\chi y)(s))\|ds \\
&\quad + \frac{1}{4}|T-2t|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s),(\chi x)(s)) - f(s,y(s),(\chi y)(s))\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(L_1+\gamma_0\overline{L}_1\right)\|x-y\|\left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds+\frac{1}{2\Gamma(q)}\int_0^T (T-s)^{q-1}ds\right.\\
&\qquad\left.+\frac{1}{4\Gamma(q-1)}|T-2t|\int_0^T (T-s)^{q-2}ds\right]\\
&\leq \frac{\left(L_1+\gamma_0\overline{L}_1\right)T^q}{2\Gamma(q+1)}\left(3+\frac{q}{2}\right)\|x-y\|\\
&\leq \Lambda_{L_1,\overline{L}_1,\gamma_0,T,q}\|x-y\|,
\end{aligned}
\tag{3.3}
$$

where $\Lambda_{L_1,\overline{L}_1,\gamma_0,T,q}=((L_1+\gamma_0\overline{L}_1)T^q/2\Gamma(q+1))(3+q/2)$, which depends only on the parameters involved in the problem. As $\Lambda_{L_1,\overline{L}_1,\gamma_0,T,q}<1$, therefore Θ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Theorem 3.2. *Let $f:[0,T]\times X\times X\to X$ be a jointly continuous function mapping bounded subsets of $[0,1]\times X\times X$ into relatively compact subsets of X, and the assumptions (A_1)-(A_2) hold with $((L_1+\gamma_0\overline{L}_1)T^q/2\Gamma(q+1))(1+q/2)<1$. Then the antiperiodic boundary value problem* (1.1) *has at least one solution on $[0,T]$.*

Proof. Let us fix

$$
r\geq \frac{\|\mu\|_{L^1}T^q}{2\Gamma(q+1)}\left(3+\frac{q}{2}\right),
\tag{3.4}
$$

and consider $B_r=\{x\in C:\|x\|\leq r\}$. We define the operators Φ and Ψ on B_r as

$$
\begin{aligned}
(\Phi x)(t)&=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f\left(s,x(s),(\chi x)(s)\right)ds,\\
(\Psi x)(t)&=-\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}f\left(s,x(s),(\chi x)(s)\right)ds\\
&\quad+\frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f\left(s,x(s),(\chi x)(s)\right)ds.
\end{aligned}
\tag{3.5}
$$

For $x,y\in B_r$, we find that

$$
\|\Phi x+\Psi y\|\leq \frac{\|\mu\|_{L_1}T^q}{2\Gamma(q+1)}\left(3+\frac{q}{2}\right)\leq r.
\tag{3.6}
$$

Thus, $\Phi x + \Psi y \in B_r$. It follows from the assumption (A_1) that Ψ is a contraction mapping for $((L_1 + \gamma_0 \overline{L}_1)T^q/2\Gamma(q+1))(1+q/2) < 1$. Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\| \leq \frac{\|\mu\|_{L^1} T^q}{\Gamma(q+1)}. \tag{3.7}$$

Now we prove the compactness of the operator Φ. In view of (A_1), we define $\sup_{(t,x,\chi x)\in\Omega}\|f(t,x,(\chi x))\| = f_{\max}$, $\Omega = [0,T] \times B_r \times B_r$, and consequently we have

$$\begin{aligned}
\|(\Phi x)(t_1) - (\Phi x)(t_2)\| &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} \left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right] f(s, x(s), (\chi x)(s)) ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, x(s), (\chi x)(s)) ds \right\| \\
&\leq \frac{f_{\max}}{\Gamma(q+1)} \left| 2(t_2 - t_1)^q + t_1^q - t_2^q \right|,
\end{aligned} \tag{3.8}$$

which is independent of x. So Φ is relatively compact on B_r. Hence, by Arzela Ascoli theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 2.5 are satisfied and the conclusion of Theorem 2.5 implies that the antiperiodic boundary value problem (1.1) has at least one solution on $[0,T]$. □

Example 3.3. Consider the following antiperiodic boundary value problem:

$$\begin{gathered}
{}^cD^q x(t) = \frac{1}{(t+4)^2} \frac{\|x\|}{1+\|x\|} + \int_0^t \frac{e^{-(s-t)}}{16} x(s) ds, \quad t \in [0,1],\ 1 < q \leq 2, \\
x(0) = -x(1), \qquad x'(0) = -x'(1).
\end{gathered} \tag{3.9}$$

Here, $f(t,x) = (1/(t+4)^2)(\|x\|/(1+\|x\|))$, $\gamma(t,s) = e^{-(s-t)}/16$, $T = 1$. Clearly,

$$\|f(t,x,\chi x) - f(t,y,\chi y)\| \leq \frac{1}{16}(\|x-y\| + \|\chi x - \chi y\|). \tag{3.10}$$

So (A_1) is satisfied with $L_1 = \overline{L}_1 = 1/16$. Further

$$\frac{\left(L_1 + \gamma_0 \overline{L}_1\right) T^q}{\Gamma(q+1)} \left(3 + \frac{q}{2}\right) < 1 \Longleftrightarrow \frac{(3+q/2)}{\Gamma(q+1)} < \frac{16}{e}. \tag{3.11}$$

Thus, by Theorem 3.1, the boundary value problem (3.9) has a unique solution on $[0,1]$.

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