Research Article

Double Integral Operators Concerning Starlike of Order β

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Double integral operators which were considered by S. S. Miller and P. T. Mocanu (Integral Transform. Spec. Funct. **19**(2008), 591–597) are discussed. In order to show the analytic function $f(z)$ is starlike of order β in the open unit disk $\mathbb{U}$, the theory of differential subordinations for analytic functions is applied. The object of the present paper is to discuss some interesting conditions for $f(z)$ to be starlike of order β in $\mathbb{U}$ concerned with second-order differential inequalities and double integral operators.

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1. Introduction, Definition and Preliminaries

Let $\mathcal{A} = \mathcal{A}(\mathbb{U})$ denote the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}. \quad (1.1)$$

For a positive integer n and $a \in \mathbb{C}$, we define the following classes of analytic functions:

$$\mathcal{A}[a, n] = \left\{ f \in \mathcal{A} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in \mathbb{U} \right\} \quad (1.2)$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{A} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots, z \in \mathbb{U} \right\} \quad (1.3)$$

with $\mathcal{A}_1 = \mathcal{A}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order β in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \mathbb{U}), \quad (1.4)$$

for some $\beta(0 \leq \beta < 1)$. We denote by $\mathcal{S}^*(\beta)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are starlike of order β in $\mathbb{U}$.

By the familiar principle of differential subordinations between analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ ($z \in \mathbb{U}$).

We denote this subordination by

$$f(z) < g(z) \quad (z \in \mathbb{U}). \quad (1.5)$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then it is known that

$$f(z) < g(z), \quad (z \in \mathbb{U}) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \quad (1.6)$$

To obtain some results of this paper, we need the following two lemmas concerning the differential subordinations.

Lemma 1.1 (see [1, Hallenbeck and Ruscheweyh]). *Let $h(z)$ be a convex function with $h(0) = a$ and let $\operatorname{Re} \gamma > 0$. If $p(z) \in \mathcal{A}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} < h(z), \quad (1.7)$$

then

$$p(z) < q(z) < h(z), \quad (1.8)$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt. \quad (1.9)$$

This result is sharp.

Lemma 1.2 (see [2, Al-Amiri and Mocanu]). *Let n be a positive integer, and let α be real with $0 \leq \alpha < n$. Let $q(z) \in \mathcal{A}$ with $q(0) = 0, q'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}. \quad (1.10)$$

If $p \in \mathcal{H}[0, n]$ satisfies

$$zp'(z) - \alpha p(z) < nzq'(z) - \alpha q(z), \quad (1.11)$$

then $p(z) < q(z)$ and this result is sharp.

By making use of these lemmas, Miller and Mocanu [3] have investigated some second-order differential inequality that implies starlikeness and deduced the following lemma.

Lemma 1.3. Let $f(z) \in \mathcal{A}_n$ and let $0 \leq \alpha < n$. If $f(z)$ satisfies

$$|zf''(z) - \alpha(f'(z) - 1)| < n - \alpha, \quad (1.12)$$

then $f(z)$ is starlike.

Furthermore, by using Lemma 1.3, Miller and Mocanu [3] obtained some result concerning the double integral starlike operator as follows.

Lemma 1.4. Let $0 \leq \alpha < n$ and let $g(z) \in \mathcal{H}$. If $g(z)$ satisfies

$$|g(z)| \leq n - \alpha, \quad (1.13)$$

then the function $f(z)$ given by

$$f(z) = z + z^{n+1} \iint_0^1 g(rs) r^{n-\alpha-1} s^n dr ds \quad (1.14)$$

is starlike.

2. Some Second-Order Differential Inequalities for Starlike of Order β

In this section, we deduced some conditions concerning the second-order differential inequality to show that $f(z)$ is starlike of order β in $\mathbb{U}$.

Theorem 2.1. Let $f(z) \in \mathcal{A}_n$ and let $0 \leq \alpha < n$ and $0 \leq \beta < 1$. If $f(z)$ satisfies

$$|zf''(z) - \alpha(f'(z) - 1)| < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}, \quad (2.1)$$

then $f(z)$ is starlike of order β in $\mathbb{U}$.

Proof. We can rewrite the inequality (2.1) in terms of subordination as

$$zf''(z) - \alpha(f'(z) - 1) < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta} z. \quad (2.2)$$

If we set

$$\begin{aligned} P(z) &= f'(z) - (1 + \alpha) \frac{f(z)}{z} \\ &= -\alpha + (n - \alpha)a_{n+1}z^n + (n + 1 - \alpha)a_{n+2}z^{n+1} + \cdots \in \mathcal{H}[-\alpha, n], \end{aligned} \quad (2.3)$$

then the subordination (2.2) becomes

$$P(z) + zP'(z) < -\alpha + \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}z. \quad (2.4)$$

Applying Lemma 1.1 to this first-order differential subordination, we obtain

$$P(z) < \frac{1}{nz^{1/n}} \int_0^z \left\{ -\alpha + \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}t \right\} t^{1/n-1} dt = -\alpha + \frac{(1-\beta)(n-\alpha)}{n+1-\beta}z, \quad (2.5)$$

or equivalently

$$f'(z) - (1 + \alpha) \frac{f(z)}{z} < -\alpha + \frac{(1-\beta)(n-\alpha)}{n+1-\beta}z. \quad (2.6)$$

If we consider

$$q(z) = \frac{1-\beta}{n+1-\beta}z \quad \left(q(0) = 0, \quad q'(0) = \frac{1-\beta}{n+1-\beta} \neq 0 \right) \quad (2.7)$$

and

$$p(z) = \frac{f(z)}{z} - 1 = a_{n+1}z^n + a_{n+2}z^{n+1} + \cdots \in \mathcal{H}[0, n], \quad (2.8)$$

then the subordination (2.6) can be written as

$$zp'(z) - \alpha p(z) < \frac{(1-\beta)(n-\alpha)}{n+1-\beta}z = nzq'(z) - \alpha q(z). \quad (2.9)$$

Since $0 \leq \alpha < n$ and the function $q(z)$ satisfies

$$\operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 = 1 > \frac{\alpha}{n}, \quad (2.10)$$

using Lemma 1.2, we obtain the subordination $p(z) < q(z)$, or

$$\frac{f(z)}{z} - 1 < \frac{1-\beta}{n+1-\beta}z. \quad (2.11)$$

It follows from the subordination (2.6) that

$$\left| f'(z) - (1 + \alpha) \frac{f(z)}{z} \right| < \alpha + \frac{(1 - \beta)(n - \alpha)}{n + 1 - \beta} = \frac{n(1 + \alpha - \beta)}{n + 1 - \beta}, \quad (2.12)$$

while from the subordination (2.11) that

$$\left| \frac{f(z)}{z} \right| > 1 - \frac{1 - \beta}{n + 1 - \beta} = \frac{n}{n + 1 - \beta}. \quad (2.13)$$

Combining these last two inequalities, we see that

$$\begin{aligned} \frac{n}{n + 1 - \beta} \left| \frac{zf'(z)}{f(z)} - (1 + \alpha) \right| &< \left| \frac{f(z)}{z} \right| \left| \frac{zf'(z)}{f(z)} - (1 + \alpha) \right| \\ &= \left| f'(z) - (1 + \alpha) \frac{f(z)}{z} \right| < \frac{n(1 + \alpha - \beta)}{n + 1 - \beta}, \end{aligned} \quad (2.14)$$

which simplifies to

$$\left| \frac{zf'(z)}{f(z)} - (1 + \alpha) \right| < \frac{n(1 + \alpha - \beta)}{n + 1 - \beta} \cdot \frac{n + 1 - \beta}{n} = 1 + \alpha - \beta. \quad (2.15)$$

This gives us that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > (1 + \alpha) - (1 + \alpha - \beta) = \beta, \quad (2.16)$$

which proves that $f(z)$ is starlike of order β in $\mathbb{U}$. □

We introduce the following example for Theorem 2.1.

Example 2.2. For the function $f(z) = z + ((1 - \beta)/(n + 1 - \beta))z^{n+1}$ ($0 \leq \beta < 1$), we have

$$\begin{aligned} |zf''(z) - \alpha(f'(z) - 1)| &= \left| \frac{n(n+1)(1-\beta)}{n+1-\beta} z^n - \alpha \frac{(n+1)(1-\beta)}{n+1-\beta} z^n \right| \\ &= \left| \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta} \right| |z|^n < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}. \end{aligned} \quad (2.17)$$

Furthermore, we see that

$$\begin{aligned} \operatorname{Re} \frac{zf'(z)}{f(z)} &= \operatorname{Re} \left(\frac{1 + ((n+1)(1-\beta)/(n+1-\beta))z^n}{1 + ((1-\beta)/(n+1-\beta))z^n} \right) \\ &> \frac{1 - (n+1)(1-\beta)/(n+1-\beta)}{1 - ((1-\beta)/(n+1-\beta))} = \beta \quad (z \in \mathbb{U}). \end{aligned} \quad (2.18)$$

Remark 2.3. Letting $\beta = 0$ in Theorem 2.1, we obtain Lemma 1.3 given by Miller and Mocanu [3].

Also, setting $\alpha = 0$ in Theorem 2.1, we have the following.

Corollary 2.4. Let $f(z) \in \mathcal{A}_n$ and let $0 \leq \beta < 1$. If $f(z)$ satisfies

$$|zf''(z)| < \frac{n(n+1)(1-\beta)}{n+1-\beta}, \quad (2.19)$$

then $f(z)$ is starlike of order β in $\mathbb{U}$.

For the case $n = 1$ in the above corollary, we find the following.

Remark 2.5. For $f(z) \in \mathcal{A}$, we have that

$$|zf''(z)| < \frac{2(1-\beta)}{2-\beta} \implies f(z) \in \mathcal{S}^*(\beta). \quad (2.20)$$

The case $\beta = 0$ was first discussed by Obradović [4].

Next, by making use of Theorem 2.1, we obtain the following result concerning the double, integral operator for starlike of order β .

Theorem 2.6. Let a function $g(z) \in \mathcal{H}$ satisfy

$$|g(z)| \leq \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta}, \quad (2.21)$$

for some $0 \leq \alpha < n$ and $0 \leq \beta < 1$. Then the function $f(z)$ given by

$$f(z) = z + z^{n+1} \iint_0^1 g(rs) r^{n-\alpha-1} s^n dr ds \quad (2.22)$$

is starlike of order β in $\mathbb{U}$.

Proof. We first consider the function $f(z) \in \mathcal{A}_n$ satisfying the differential equation

$$zf''(z) - \alpha(f'(z) - 1) = z^n g(z). \quad (2.23)$$

Then, it is clear that

$$|zf''(z) - \alpha(f'(z) - 1)| = |z|^n |g(z)| < \frac{(n+1)(1-\beta)(n-\alpha)}{n+1-\beta} \quad (z \in \mathbb{U}). \quad (2.24)$$

Thus, from Theorem 2.1, we see that the solution of the differential equation (2.23) must be starlike of order β . The solution of (2.23) can be obtained in two integrations. If we set $\varphi(z) = f'(z) - 1$, then the equation (2.23) can be simplified to

$$z\varphi'(z) - \alpha\varphi(z) = z^n g(z), \quad (2.25)$$

which has the solution $\varphi(z)$ given by

$$\varphi(z) = z^\alpha \int_0^z g(\zeta) \zeta^{n-\alpha-1} d\zeta = z^n \int_0^1 g(rz) r^{n-\alpha-1} dr. \quad (2.26)$$

Since $\varphi(z) = f'(z) - 1$, we have

$$f'(z) - 1 = z^n \int_0^1 g(rz) r^{n-\alpha-1} dr, \quad (2.27)$$

that is,

$$f(z) = z + \int_0^z \zeta^n \left(\int_0^1 g(r\zeta) r^{n-\alpha-1} dr \right) d\zeta = z + z^{n+1} \iint_0^1 g(rs\zeta) r^{n-\alpha-1} s^n dr ds. \quad (2.28)$$

□

Remark 2.7. Taking $\beta = 0$ in Theorem 2.6, we find Lemma 1.4 given by Miller and Mocanu [3]. However, (1.14) and (2.22) are double integral operators of the same form.

Moreover, making $n = 1$ and $\alpha = 0$ in Theorem 2.6, we have the following.

Corollary 2.8. *If $g(z) \in \mathcal{H}$ and*

$$|g(z)| \leq \frac{2(1-\beta)}{2-\beta} \quad (z \in \mathbb{U}) \quad (2.29)$$

for some $0 \leq \beta < 1$, then

$$f(z) = z + z^2 \iint_0^1 g(rs\zeta) s dr ds \in \mathcal{S}^*(\beta). \quad (2.30)$$

As examples of Corollary 2.8, we get the following.

Example 2.9. For the function $g(z) = 2(1 - \beta)/(2 - \beta)$ ($0 \leq \beta < 1$), we find

$$f(z) = z + \frac{1 - \beta}{2 - \beta} z^2 \in \mathcal{S}^*(\beta), \quad (2.31)$$

because

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \operatorname{Re} \left(\frac{1 + (2(1 - \beta)/(2 - \beta))z}{1 + ((1 - \beta)/(2 - \beta))z} \right) > \frac{1 - (2(1 - \beta)/(2 - \beta))}{1 - ((1 - \beta)/(2 - \beta))} = \beta \quad (z \in \mathbb{U}). \quad (2.32)$$

Example 2.10. For the function $g(z) = (2(1 - \beta)/(2 - \beta))z$ ($0 \leq \beta < 1$), we have

$$f(z) = z + \frac{1 - \beta}{3(2 - \beta)} z^3. \quad (2.33)$$

Then, we see that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \operatorname{Re} \left(\frac{1 + ((1 - \beta)/(2 - \beta))z^2}{1 + ((1 - \beta)/3(2 - \beta))z^2} \right) > \frac{3}{5 - 2\beta} > \beta \quad (z \in \mathbb{U}), \quad (2.34)$$

that is,

$$f(z) = z + \frac{1 - \beta}{3(2 - \beta)} z^3 \in \mathcal{S}^* \left(\frac{3}{5 - 2\beta} \right) \subset \mathcal{S}^*(\beta). \quad (2.35)$$

3. Other Result for Starlikeness of Order β

To obtain that $f(z)$ is starlike of order β in Theorem 2.1, we showed that

$$\left| \frac{zf'(z)}{f(z)} - (1 + \alpha) \right| < 1 + \alpha - \beta \quad (z \in \mathbb{U}). \quad (3.1)$$

In this section, to obtain that $f(z)$ is starlike of order β , we consider some second-order differential inequality concerning the order β and show the following inequality:

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \quad (z \in \mathbb{U}) \quad (3.2)$$

for some $0 < \beta < 1$.

Remark 3.1. For some $\beta (0 < \beta < 1)$, we see that

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \iff \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta. \quad (3.3)$$

Now, we consider the following theorem.

Theorem 3.2. Let $f(z) \in \mathcal{A}_n$ and let $0 < \beta < 1$. If $f(z)$ satisfies

$$|zf''(z) + (1 - 2\beta)(f'(z) - 1)| < \begin{cases} \frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} & \left(0 < \beta < \frac{1}{2}\right) \\ \frac{(1-\beta)(n+1)(n+1-2\beta)}{n+1-\beta}, & \left(\frac{1}{2} \leq \beta < 1\right), \end{cases} \quad (3.4)$$

then

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \quad (z \in \mathbb{U}), \quad (3.5)$$

or $f(z)$ is starlike of order β in $\mathbb{U}$.

Proof. (i) For the case $0 < \beta < 1/2$, the inequality (3.4) can be written as follows:

$$zf''(z) + (1 - 2\beta)(f'(z) - 1) < \frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} z. \quad (3.6)$$

If we set

$$\begin{aligned} P(z) &= f'(z) - 2\beta \frac{f(z)}{z} \\ &= (1 - 2\beta) + (n+1-2\beta)a_{n+1}z^n + (n+2-2\beta)a_{n+2}z^{n+1} + \cdots \in \mathcal{H}[1 - 2\beta, n], \end{aligned} \quad (3.7)$$

then the subordination (3.6) becomes

$$P(z) + zP'(z) < (1 - 2\beta) + \frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} z. \quad (3.8)$$

Applying Lemma 1.1 as well as the proof of Theorem 2.1, we obtain that

$$P(z) < \frac{1}{nz^{1/n}} \int_0^z \left\{ (1 - 2\beta) + \frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} t \right\} t^{1/n-1} dt = (1 - 2\beta) + \frac{\beta(n+1-2\beta)}{n+1-\beta} z, \quad (3.9)$$

or equivalently, that

$$f'(z) - 2\beta \frac{f(z)}{z} < (1 - 2\beta) + \frac{\beta(n+1-2\beta)}{n+1-\beta} z. \quad (3.10)$$

Also, if the function $p(z)$ is defined by

$$p(z) = f'(z) - 1 = (n+1)a_{n+1}z^n + (n+2)a_{n+2}z^{n+1} + \cdots \in \mathcal{H}[0, n], \quad (3.11)$$

then the subordination (3.6) becomes

$$zp'(z) + (1 - 2\beta)p(z) < \frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} z, \quad (3.12)$$

namely,

$$p(z) + \frac{zp'(z)}{1-2\beta} < \frac{\beta(n+1)(n+1-2\beta)}{(1-2\beta)(n+1-\beta)} z. \quad (3.13)$$

Since $0 < 1 - 2\beta < 1$, we can use Lemma 1.1 as $\gamma = 1 - 2\beta$ and obtain

$$p(z) < \frac{1-2\beta}{nz^{(1-2\beta)/n}} \int_0^z \frac{\beta(n+1)(n+1-2\beta)}{(1-2\beta)(n+1-\beta)} t^{(1-2\beta)/n} dt = \frac{\beta(n+1)}{n+1-\beta} z, \quad (3.14)$$

that is,

$$f'(z) - 1 < \frac{\beta(n+1)}{n+1-\beta} z. \quad (3.15)$$

From the subordination (3.10), we find

$$\left| f'(z) - 2\beta \frac{f(z)}{z} \right| < (1 - 2\beta) + \frac{\beta(n+1-2\beta)}{n+1-\beta} = \frac{n(1-\beta) + (1-2\beta)}{n+1-\beta}, \quad (3.16)$$

while, from the subordination (3.15), we get

$$|f'(z)| > 1 - \frac{\beta(n+1)}{n+1-\beta} = \frac{n(1-\beta) + (1-2\beta)}{n+1-\beta}. \quad (3.17)$$

By combining these last two inequalities, we obtain that

$$\begin{aligned} \frac{n(1-\beta) + (1-2\beta)}{n+1-\beta} \left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| &< |f'(z)| \left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| \\ &= \frac{1}{2\beta} \left| f'(z) - 2\beta \frac{f(z)}{z} \right| < \frac{n(1-\beta) + (1-2\beta)}{2\beta(n+1-\beta)} \quad (z \in \mathbb{U}), \end{aligned} \quad (3.18)$$

which simplifies to

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \quad (z \in \mathbb{U}). \quad (3.19)$$

(ii) For the case $1/2 \leq \beta < 1$, we can rewrite (3.4) in terms of the subordination as

$$zf''(z) + (1-2\beta)(f'(z) - 1) < \frac{(1-\beta)(n+1)(n+1-2\beta)}{n+1-\beta} z. \quad (3.20)$$

Applying Lemma 1.1 in similar to case (i), we can obtain that

$$f(z) - 2\beta \frac{f(z)}{z} < (1-2\beta) + \frac{(1-\beta)(n+1-2\beta)}{n+1-\beta} z. \quad (3.21)$$

Furthermore, if we set

$$q(z) = \frac{(1-\beta)(n+1)}{n+1-\beta} z \quad \text{and} \quad p(z) = f'(z) - 1 \in \mathcal{H}[0, n], \quad (3.22)$$

then the subordination (3.20) can be written as

$$zp'(z) - (2\beta - 1)p(z) < n z q'(z) - (2\beta - 1)q(z). \quad (3.23)$$

Since $0 \leq 2\beta - 1 < 1 \leq n$ and the function $q(z)$ satisfies

$$q(0) = 0, \quad q'(0) = \frac{(1-\beta)(n+1)}{n+1-\beta} \neq 0 \quad \text{and} \quad \operatorname{Re} \frac{z f''(z)}{f'(z)} + 1 = 1 > \frac{2\beta - 1}{n}, \quad (3.24)$$

we can use Lemma 1.2 and obtain the following fact:

$$zp'(z) - (2\beta - 1)p(z) < n z q'(z) - (2\beta - 1)q(z) \implies p(z) < q(z). \quad (3.25)$$

This implies that

$$f'(z) - 1 < \frac{(1-\beta)(n+1)}{n+1-\beta} z. \quad (3.26)$$

From subordinations (3.21) and (3.26), we find

$$\left| f'(z) - 2\beta \frac{f(z)}{z} \right| < \left| (1-2\beta) - \frac{(1-\beta)(n+1-2\beta)}{n+1-\beta} \right| = \frac{\beta n}{n+1-\beta} \quad (z \in \mathbb{U}) \quad (3.27)$$

and

$$|f'(z)| > 1 - \frac{(1-\beta)(n+1)}{n+1-\beta} = \frac{\beta n}{n+1-\beta} \quad (z \in \mathbb{U}). \quad (3.28)$$

Therefore, we obtain

$$\begin{aligned} \frac{\beta n}{n+1-\beta} \left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| &< |f'(z)| \left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| \\ &= \frac{1}{2\beta} \left| f'(z) - 2\beta \frac{f(z)}{z} \right| < \frac{\beta n}{2\beta(n+1-\beta)} \quad (z \in \mathbb{U}), \end{aligned} \quad (3.29)$$

which simplifies to

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \quad (z \in \mathbb{U}). \quad (3.30)$$

This completes the proof of this theorem. $\square$

Making use of Theorem 3.2, we obtain the following result concerning the double integral operator for starlike of order β .

Theorem 3.3. *If $g(z) \in \mathcal{H}$ satisfies*

$$|g(z)| \leq \begin{cases} \frac{\beta(n+1)(n+1-2\beta)}{n+1-\beta} & \left(0 < \beta < \frac{1}{2}\right) \\ \frac{(1-\beta)(n+1)(n+1-2\beta)}{n+1-\beta} & \left(\frac{1}{2} \leq \beta < 1\right), \end{cases} \quad (3.31)$$

for some $0 < \beta < 1$, then

$$f(z) = z + z^{n+1} \iint_0^1 g(rs) r^{n-2\beta} s^n \, dr \, ds, \quad (3.32)$$

is a starlike function of order β .

This theorem can be shown as well as the proof of Theorem 2.6.

Making $n = 1$ in Theorem 3.3, we have

Corollary 3.4. *If $g(z) \in \mathcal{H}$ satisfies*

$$|g(z)| \leq \begin{cases} \frac{4\beta(1-\beta)}{2-\beta} & \left(0 < \beta < \frac{1}{2}\right) \\ \frac{4(1-\beta)^2}{2-\beta} & \left(\frac{1}{2} \leq \beta < 1\right), \end{cases} \quad (3.33)$$

for some $0 < \beta < 1$, then

$$f(z) = z + z^2 \iint_0^1 g(rs) r^{1-2\beta} s \, dr \, ds \in \mathcal{S}^*(\beta). \quad (3.34)$$

Let us consider an example for Corollary 3.4.

Example 3.5. If we consider $g(z) = 4\beta(1-\beta)/(2-\beta)$ ($0 < \beta < 1/2$), then we find

$$f(z) = z + \frac{\beta}{2-\beta} z^2 \in \mathcal{S}^*\left(\frac{2-3\beta}{2(1-\beta)}\right) \subset \mathcal{S}^*(\beta), \quad (3.35)$$

because

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \operatorname{Re} \left(\frac{1 + (2\beta/(2-\beta))z}{1 + (\beta/(2-\beta))z} \right) > (2-3\beta)/2(1-\beta) > \beta \quad (z \in \mathbb{U}). \quad (3.36)$$

Further, if we take $g(z) = 4(1-\beta)^2/(2-\beta)$ ($1/2 \leq \beta < 1$), then we see that

$$f(z) = z + \frac{1-\beta}{2-\beta} z^2 \in \mathcal{S}^*(\beta). \quad (3.37)$$

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