Research Article

On the Existence of Nodal Solutions for a Nonlinear Elliptic Problem on Symmetric Riemannian Manifolds

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Given that $(M, g)$ is a smooth compact and symmetric Riemannian $n$-manifold, $n ≥ 2$, we prove a multiplicity result for antisymmetric sign changing solutions of the problem $-ε^2 Δ_g u + u = |u|^{p-2} u$ in $M$. Here $p > 2$ if $n = 2$ and $2 < p < 2^* = 2n/(n-2)$ if $n ≥ 3$.

1. Introduction

Let $(M, g)$ be a smooth compact connected Riemannian manifold without boundary of dimension $n ≥ 2$. Let us consider the problem

$$-ε^2 Δ_g u + u = |u|^{p-2} u \quad \text{in } M, \quad u ∈ H^1_g(M), \quad \text{(1.1)}$$

where $p > 2$ if $n = 2$, $2 < p < 2n/(n-2)$ if $n ≥ 3$ and $ε$ is a positive parameter. Here $H^1_g(M)$ is the completion of $C^∞(M)$ with respect to

$$\|u\|^2_g := \int_M |∇_g u|^2 dμ_g + \int_M u^2 dμ_g. \quad \text{(1.2)}$$
It is well known that any critical point of the energy functional \( J_\varepsilon : H^1_g(M) \to \mathbb{R} \) constrained to the Nehari manifold \( \mathcal{N}_\varepsilon \) is a solution to (1.1). Here
\[
\begin{align*}
J_\varepsilon(u) := & \frac{1}{\varepsilon^n} \int_M \left( \frac{1}{2} \varepsilon^2 |\nabla_g u|^2 + \frac{1}{2} u^2 - \frac{1}{p} |u|^p \right) d\mu_g, \\
\mathcal{N}_\varepsilon := & \left\{ u \in H^1_g(M) \setminus \{0\} : J'_\varepsilon(u)[u] = 0 \right\}. 
\end{align*}
\] (1.3) (1.4)

In [1] the authors show that the least energy solution of (1.1), that is, the minimum of \( J_\varepsilon \) on \( \mathcal{N}_\varepsilon \) is a positive solution with a spike layer, whose peak converges to the maximum point of the scalar curvature \( S_g \) of \( (M, g) \) as \( \varepsilon \) goes to zero. Successively, in [2] (see also [3, 4]) the authors point out that the topology of the manifold \( M \) influences the multiplicity of positive solutions of (1.1), that is, (1.1) has at least \( \text{cat}(M) \) nontrivial solutions provided that \( \varepsilon \) is small enough. Here \( \text{cat}(M) \) denotes the Lusternik-Schnirelman category of \( M \). Recently, in [5–7] it has been proved that the existence of positive solutions is strongly related to the geometry of \( M \), that is stable critical points of the scalar curvature \( S_g \) generate positive solutions with one or more peaks as \( \varepsilon \) goes to zero.

As far as it concerns the existence of sign changing solutions to (1.1), a few results are known. The first result has been obtained in [7] where it has been constructed solutions with one positive peak and one negative peak, which approach, as \( \varepsilon \) goes to zero, the minimum point and the maximum point of \( S_g \), provided the scalar curvature is not constant. In [8] the authors assume the following:

(S) the manifold \( M \) is a regular submanifold of \( \mathbb{R}^N \) invariant with respect to \( \tau \), where \( \tau : \mathbb{R}^N \to \mathbb{R}^N \) is an orthogonal linear transformation such that \( \tau \neq I \) and \( \tau^2 = I \), \( I \) being the identity of \( \mathbb{R}^N \).

They prove problem (1.1) has at least \( G_\tau - \text{cat}(M - M_\tau) \) pairs of sign changing solutions which change sign exactly once. Here \( G_\tau - \text{cat}(M - M_\tau) \) denotes the \( G_\tau \)-equivariant Lusternik-Schnirelman category for the group \( G_\tau := \{ I, \tau \} \) and \( M_\tau := \{ x \in M : \tau x = x \} \).

In this paper we assume \( M \) satisfies (S) in the particular case \( \tau = -I \). We look for solutions of the problem
\[
\begin{align*}
-\varepsilon^2 \Delta_g u + u = |u|^{p-2} u & \quad \text{in } M, \\
u \in H^1_g(M), \\
u(-x) &= -\nu(x).
\end{align*}
\] (1.5)

We evaluate the number of solutions of problem (1.5) using Morse theory. Our main result reads as following.

**Theorem 1.1.** Assume that for \( \varepsilon \) small enough all the solutions to problem (1.5) with energy close to \( 2m_\infty \) are nondegenerate. Then there are at least \( P_1(M/G) \) pairs \( (u, -u) \) of nontrivial solutions to (1.5) which change sign exactly once, where
\[
m_\infty := \inf_{J_\varepsilon \neq |\nabla u|^2 + |u|^2} \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} u^2 - \frac{1}{p} |u|^p \right) dx. \] (1.6)

Here \( G = \{ I, -I \} \) and \( P_1(M/G) \) is Poincaré polynomial \( P_t(M/g) \) when \( t = 1 \).
Concerning the assumptions of nondegeneracy of all the critical points with energy close to $2m_{\infty}$, we think that it is true “generically" in some sense with respect to $(\epsilon, g)$ where $\epsilon$ is a positive parameter and $g$ is a Riemannian metric.

We point out that problem (1.1) has been widely studied when the manifold $M$ is replaced by an open bounded and smooth domain in $\mathbb{R}^N$ with Dirichlet or Neumann boundary condition. In particular, it has been studied the effect of the domain topology or the domain geometry on the number of solutions. See, for example, [9–19] for the Dirichlet problem and [20–32] for the Neumann problem.

The paper is organized as follows. In Section 2 we set the problem and we recall some known results; in Section 3 we give the proof of Theorem 1.1; in Section 4 we prove the technical Lemma 4.5, which is crucial for the proof of Theorem 1.1.

2. Setting of the Problem

First of all, we will recall some topological notions which are used in the paper.

**Definition 2.1** (Poincaré polynomial). If $(X,Y)$ is a couple of the topological spaces, the Poincaré polynomial $P_t(X,Y)$ is defined as the following power series in $t$:

$$P_t(X,Y) := \sum_k \dim H_k(X,Y) t^k,$$

where $H_k(X,Y)$ is the $k$th homology group with coefficients in some fields. Moreover, we set

$$P_t(X) := P_t(X,\emptyset) = \sum_k \dim H_k(X) t^k.$$

If $X$ is a compact manifold, we have that $\dim H_k(X) < +\infty$ and in this case $P_t(X)$ is a polynomial and not a formal series.

**Definition 2.2** (Morse index). Let $J$ be a $C^2$-functional on a Banach space $X$ and $u \in X$ an isolated critical point of $J$ with $J(u) = c$. If $J^c := \{v \in X : J(v) \leq c\}$ then the (polynomial) Morse index $i_t(u)$ of $u$ is the following series:

$$i_t(u) := \sum_k \dim H_k(J^c, J^c \setminus \{u\}) t^k,$$

where $H_k(J^c, J^c \setminus \{u\})$ is the $k$th homology group of the couple $(J^c, J^c \setminus \{u\})$. If $u$ is a nondegenerate critical point of $J$ then $i_t(u) = t^{n(u)}$, where $n(u)$ is the (numerical) Morse index of $u$ and it is given by the dimension of the maximal subspace on which the bilinear form $J''(u)[\cdot,\cdot]$ is negatively definite.

It is useful to recall the following result (see [33]).

**Remark 2.3.** Let $X$ and $Y$ be topological spaces. If $f : X \to Y$ and $g : Y \to X$ are continuous maps such that $g \circ f$ is homotopic to the identity map on $X$ then $P_t(Y) = P_t(X) + Z(t)$, where $Z(t)$ is a polynomial with non negative coefficients.
Now, let us point out that the transformation $\tau = -I : M \to M$ induces a transformation on $H^1_g(M)$. We define the linear operator $\tau^*$ as follows:

$$\tau^* : H^1_g(M) \to H^1_g(M), \quad \tau^*(u(x)) := -u(-x). \quad (2.4)$$

The operator $\tau^*$ is selfadjoint with respect to the following scalar product on $H^1_g(M)$, which is equivalent to the usual one:

$$\left\langle u, v \right\rangle_\varepsilon := \frac{1}{\varepsilon^n} \int_M \left( \varepsilon^2 \nabla_g u \nabla_g v + uv \right) d\mu_g, \quad (2.5)$$

which induces the norm

$$\|u\|_\varepsilon^2 := \frac{1}{\varepsilon^n} \int_M \left( \varepsilon^2 |\nabla_g u|^2 + u^2 \right) d\mu_g. \quad (2.6)$$

In particular, we have

$$\|\tau^* u\|_{\varepsilon,p} = \|u\|_{\varepsilon,p}, \quad \|\tau^* u\|_\varepsilon = \|u\|_\varepsilon, \quad J_\varepsilon(\tau^* u) = J_\varepsilon(u). \quad (2.7)$$

Here

$$\|u\|_{\varepsilon,p} := \frac{1}{\varepsilon^n} \int_M |u|^p d\mu_g \quad (2.8)$$
denotes the norm in $L^p(M)$, which is equivalent to the usual one. Therefore, in virtue of the Palais Principle, the nontrivial solutions of (1.5) are the critical points of the restriction of $J_\varepsilon$ to the $\tau$-invariant Nehari manifold

$$\mathcal{N} \varepsilon := \{ u \in \mathcal{N} : u(-x) = -u(x) \} = \mathcal{N} \cap H^1_g, \quad (2.9)$$

where $H^1_g := \{ u \in H^1_g(M) : u(-x) = -u(x) \}$.

In fact, since $\int -\varepsilon (\tau^* u) = J_\varepsilon(u)$ and $\tau^*$ is a selfadjoint operator, we have

$$\left\langle \nabla J_\varepsilon(\tau^* u), \tau^* \varphi \right\rangle_\varepsilon = \left\langle \nabla J_\varepsilon(u), \varphi \right\rangle_\varepsilon \quad \forall \varphi \in H^1_g(M) \quad (2.10)$$

and so $\nabla J_\varepsilon(u) = \tau^* \nabla J_\varepsilon(\tau^* u) = \tau^* \nabla J_\varepsilon(u)$ if $(\tau^* u)(x) = u(x) = -u(-x)$.

Let us set

$$m_\varepsilon := \inf_{\mathcal{N}} J_\varepsilon, \quad m^\tau_\varepsilon := \inf_{\mathcal{N} \tau} J_\varepsilon \quad (2.11)$$

and let $m_{\infty}$ be as in (1.6).
Let us sketch the proof of our main result.

3. The Main Ingredient of the Proof

By Lemma 4.5 we deduce that

\[ \bar{\mathcal{N}}^\tau_{\sigma}/\mathbb{Z}_2 \to \mathcal{N}^\tau_{\sigma}/\mathbb{Z}_2 \]

where \( \bar{\mathcal{N}}^\tau_{\sigma} \) is homotopic to the identity map and \( M_d/G \) is homotopically equivalent to \( M_{\tilde{g}} \).

Therefore by Remark 2.3 we get

\[ P_1 \left( \bar{\mathcal{N}}^\tau_{\sigma}/\mathbb{Z}_2 \right) = P_1 \left( \frac{M}{G} \right) + Z(t), \]

where \( Z(t) \) is a polynomial with nonnegative integer coefficients.
By our assumption we have that for \( \epsilon \) small enough all the critical points \( u \) such that \( \tilde{f}_\epsilon(u) < 2(m_\infty + \delta) \) are nondegenerate. Moreover the functional \( \tilde{f}_\epsilon \) satisfies the Palais-Smale condition. Then by Morse theory and relations (3.1) and (3.3) we get at least \( P_1(M/G) \) pairs \((u, -u)\) of nontrivial solutions for (1.5). By Remark (4.7) these solutions change sign exactly once. That concludes the proof of Theorem 1.1.

Remark 3.1. By [33, Lemma 5.2] we deduce that

\[
P_1\left( H^*_g \setminus \{0\}, \tilde{f}_\epsilon^{2(m_\infty - \delta)} \right) = tP_1\left( \mathcal{N}_\epsilon^{\tau} / \mathbb{Z}_2 \right),
\]

(3.4)

Since \( P^\infty \) is homeomorphic to \( \mathcal{N}_\epsilon^{\tau} / \mathbb{Z}_2 \) we get \( P_1(\mathcal{N}_\epsilon^{\tau} / \mathbb{Z}_2) = P_1(P^\infty) \). Provided the homology is evaluated with \( \mathbb{Z}_2 \)-coefficients (see, e.g., [35, Theorem 7.4]), we have \( P_1(P^\infty) = +\infty \). Then, if all the critical points are nondegenerate, we get infinitely many pairs \((u, -u)\) of nontrivial solutions for (1.5).

4. Technical Results

Let \( \chi_r \) be a smooth cut-off function such that

\[
\chi_r(z) = 1 \quad \text{if} \quad z \in B\left(0, \frac{r}{2}\right), \quad \chi_r(z) = 0 \quad \text{if} \quad z \in \mathbb{R}^N \setminus B(0, r), \quad |\nabla \chi_r(z)| \leq 2 \quad \forall z \in \mathbb{R}^N.
\]

(4.1)

Fixing a point \( q \in M \) and \( \epsilon > 0 \), let us define the function \( w_{\epsilon,q} \) on \( M \) as

\[
w_{\epsilon,q}(x) := U_q\left(x \chi_{\epsilon}^{-1}(x)\right) \chi_{\epsilon}(x) \quad \text{if} \quad x \in B_{g}(q, r) \quad w_{\epsilon,q}(x) := 0 \quad \text{otherwise}.
\]

(4.2)

We choose \( r \) smaller than the injectivity radius of \( M \) and such that \( B_{g}(q, r) \cap B_{g}(-q, r) = \emptyset \) for any \( q \in M \). For any \( \epsilon > 0 \) we can define a positive number \( t(w_{\epsilon,q}) \) such that

\[
\Phi_{\epsilon}(q) := t(w_{\epsilon,q})w_{\epsilon,q} \in H^1_{g}(M) \cap \mathcal{N}_\epsilon \quad \text{for any} \quad q \in M.
\]

(4.3)

Namely, \( t(w_{\epsilon,q}) \) verifies

\[
t(w_{\epsilon,q}) = \left[ \frac{\int_{M} \left( \epsilon^2 |\nabla_g w_{\epsilon,q}|^2 + w_{\epsilon,q}^2 \right) d\mu_g}{\int_{M} w_{\epsilon,q}^2 d\mu_g} \right]^{1/p-2}.
\]

(4.4)

In [2, Proposition 4.2] the following lemma has been proved.

**Lemma 4.1.** Given \( \epsilon > 0 \) the map \( \Phi_{\epsilon} : M \to H^1_{g}(M) \cap \mathcal{N}_\epsilon \) is continuous. Moreover, given \( \delta > 0 \) there exists \( \epsilon_0(\delta) \) such that if \( \epsilon \in (0, \epsilon_0(\delta)) \) then \( \Phi_{\epsilon}(q) \in \mathcal{N}_\epsilon \cap J_{r}^{m_{\infty} + \delta} \).
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Now, fixing a point \( q \in M \) let us define the function

\[
\Phi^*_\varepsilon(q) := t(w_{\varepsilon,q})w_{\varepsilon,q} - t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q}.
\]  

(4.5)

It holds that

\[
\int_M |w_{\varepsilon,q}|^2 p = \int_M |w_{\varepsilon,\tau q}|^2 p, \quad \int_M |\nabla_g w_{\varepsilon,q}|^2 d\mu_g = \int_M |\nabla_g w_{\varepsilon,\tau q}|^2 d\mu_g.
\]

(4.6)

By (4.4) and (4.6), we deduce

\[
t(w_{\varepsilon,q}) = t(w_{\varepsilon,\tau q}).
\]

(4.7)

The proof of the next results follows the same arguments as in [8].

**Lemma 4.2.** Given \( \varepsilon > 0 \) the map \( \Phi^*_\varepsilon : M \to H^1_g(M) \cap \mathcal{M}^\tau_\varepsilon \) is continuous. Moreover, given \( \delta > 0 \) there exists \( \varepsilon_0(\delta) \) such that if \( \varepsilon \in (0, \varepsilon_0(\delta)) \) then \( \Phi^*_\varepsilon(q) \in \mathcal{M}^\tau_\varepsilon \cap \mathcal{I}^{(m+\delta)}_\varepsilon \).

**Proof.** Since \( U(x) \chi_r \) is a radially symmetric function, we set \( \tilde{U}_\varepsilon(|z|) := U_\varepsilon(z) \chi_r(z) \). Moreover, since we have

\[
\begin{align*}
|\exp^{-1}_q(\tau x)| &= d^*_g(-x,-q) = d^*_g(x,q) = |\exp^{-1}_q(x)|, \\
|\exp^{-1}_q(\tau x)| &= d^*_g(-x,q) = d^*_g(x,-q) = |\exp^{-1}_q(x)|,
\end{align*}
\]

(4.8)

we get

\[
\tau^*\Phi^*_\varepsilon(q)(x)
\]

\[
= -t(w_{\varepsilon,q})w_{\varepsilon,q}(-x) + t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q}(-x)
\]

(4.10)

\[
= -t(w_{\varepsilon,q})\tilde{U}_\varepsilon \left( |\exp^{-1}_q(-x)| \right) + t(w_{\varepsilon,\tau q})\tilde{U}_\varepsilon \left( |\exp^{-1}_q(-x)| \right)
\]

\[
= t(w_{\varepsilon,\tau q})\tilde{U}_\varepsilon \left( |\exp^{-1}_q(x)| \right) - t(w_{\varepsilon,q})\tilde{U}_\varepsilon \left( |\exp^{-1}_q(x)| \right)
\]

(4.11)

\[
= t(w_{\varepsilon,\tau q})\tilde{U}_\varepsilon \left( |\exp^{-1}_q(x)| \right) - t(w_{\varepsilon,q})\tilde{U}_\varepsilon \left( |\exp^{-1}_q(x)| \right)
\]

\[
= \Phi^*_\varepsilon(q)(x),
\]

(4.12)

because by (4.7) we have \( t(w_{\varepsilon,q}) = t(w_{\varepsilon,\tau q}) \). Hence \( \Phi^*_\varepsilon(q) \in \mathcal{M}^\tau_\varepsilon \).
To get that $\Phi^*_\varepsilon(q) \in J^2_{\varepsilon}([m_\infty + \delta])$, it is enough to prove that $J_\varepsilon(\Phi^*_\varepsilon(q)) = 2J_\varepsilon(\Phi_\varepsilon(q))$, by Lemma 4.1 the statement will follow. Since the support of the function $\Phi^*_\varepsilon(q)$ is $B_q(q, r) \cup B_q(-q, r)$ and $B_q(q, r) \cap B_q(-q, r) = \emptyset$, by (4.6) and the definition of the function $\Phi^*_\varepsilon$, we get

$$J_\varepsilon(\Phi^*_\varepsilon(q)) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^n} \int_M |\Phi^*_\varepsilon(q)|^p d\mu_g$$

$$= \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^n} \left(\int_{B_q(q, r)} |\Phi_\varepsilon(q)|^p d\mu_g + \int_{B_q(-q, r)} |\Phi_\varepsilon(\tau q)|^p d\mu_g\right)$$

(4.13)

$$= 2J_\varepsilon(\Phi_\varepsilon(q)).$$

That concludes the proof.

\[\square\]

Lemma 4.3. One has that $\lim_{\varepsilon \to 0} m^*_\varepsilon = 2m_\infty$.

Proof. By Lemma 4.2 and (4.12) we have that for any $\delta > 0$ there exists $\varepsilon_0(\delta)$ such that for any $\varepsilon \in (0, \varepsilon_0(\delta))$ it holds that

$$2m_\varepsilon \leq m^*_\varepsilon \leq J_\varepsilon(\Phi^*_\varepsilon(q)) = 2J_\varepsilon(\Phi_\varepsilon(q)) \leq 2(m_\infty + \delta).$$

(4.14)

Since $\lim_{\varepsilon \to 0} m_\varepsilon = 2m_\infty$ (see [2, Remark 5.9]) we get the claim.

\[\square\]

For any function $u \in \mathcal{N}^*_\varepsilon$ we can define a point $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) := \frac{\int_M x|u^*(x)|^p d\mu_g}{\int_M |u^*(x)|^p d\mu_g}.\quad (4.15)$$

Lemma 4.4. There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, for any $\varepsilon \in (0, \varepsilon_0(\delta))$ (as in Lemma 4.2), and for any function $u \in \mathcal{N}^*_\varepsilon \cap J^2_{\varepsilon}([m_\infty + \delta])$, it holds that $\beta(u) \in M_d$, where $M_d := \{x \in \mathbb{R}^N : d(x, M) < d\}$.

Proof. Let $u \in \mathcal{N}^*_\varepsilon \cap J^2_{\varepsilon}([m_\infty + \delta])$. Since $u(x) = -u(-x)$ we set $M^+ := \{x \in M : u(x) > 0\}$ and $M^- := \{x \in M : u(x) < 0\}$. It is easy to see that $M^+ = \{-x : x \in M^\cdot\}$. Then we have

$$J_\varepsilon(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^n} \int_M |u|^p d\mu_g$$

$$= \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^n} \left(\int_{M^+} |u|^p d\mu_g + \int_{M^-} |u^-|^p d\mu_g\right) = 2J_\varepsilon(u^*).$$

(4.16)

Since $J_\varepsilon(u) \leq 2(m_\infty + \delta)$, we have $J_\varepsilon(u^*) \leq m_\infty + \delta$ and by [2, Proposition 5.10] we get the claim.

\[\square\]
It is easy to check that $\Phi_\varepsilon^r(q) = -\Phi_\varepsilon^r(q)$ and $\beta(-u) = -\beta(u)$. Moreover, by Lemmas 4.1 and 4.2, we can define a map $\tilde{\Phi}_\varepsilon : M/G \to \tilde{J}_\varepsilon^{2(m_\infty + \delta)} \cap \mathcal{M}_\varepsilon / \mathbb{Z}_2$ by

$$\tilde{\Phi}_\varepsilon([q]) := \{\Phi_\varepsilon^r(q), \Phi_\varepsilon^r(-q)\}. \quad (4.17)$$

By Lemma 4.4 we can define a map $\tilde{\beta} : \tilde{J}_\varepsilon^{2(m_\infty + \delta)} \cap \mathcal{M}_\varepsilon / \mathbb{Z}_2 \to M_d/G$ by

$$\tilde{\beta}([u]) := \{\beta(u), \beta(-u)\}. \quad (4.18)$$

**Lemma 4.5.** There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the map

$$I_\varepsilon := \tilde{\beta} \circ \tilde{\Phi}_\varepsilon : \frac{M}{G} \to \frac{M_d}{G} \quad (4.19)$$

is well defined, continuous, and homotopic to the identity map.

**Proof.** By Lemmas 4.2 and 4.4, $I_\varepsilon$ is well defined. In order to show that $I_\varepsilon$ is homotopic to the identity, we estimate the following difference:

$$|\beta\Phi_\varepsilon^r(q) - q| = \frac{\int_M (x - q) \left| (\Phi_\varepsilon^r(q))^{\gamma} \right|^p \, d\mu_g}{\int_M \left| (\Phi_\varepsilon^r(q))^{\gamma} \right|^p \, d\mu_g}$$

$$= \frac{\int_{B(0,r)} |U(y/y^\varepsilon)\chi_r(|y|)|^p |g_q(y)|^{1/2} \, dy}{\int_{B(0,r)} |U(y/y^\varepsilon)\chi_r(|y|)|^p |g_q(y)|^{1/2} \, dy} \quad (4.20)$$

$$= \frac{\varepsilon \int_{B(0,r/\varepsilon)} |U(z)\chi_r(|ez|)|^p |g_q(ez)|^{1/2} \, d\mu_g}{\int_{B(0,r/\varepsilon)} |U(z)\chi_r(|ez|)|^p |g_q(ez)|^{1/2} \, d\mu_g}.$$

Hence $|\beta\Phi_\varepsilon^r(q) - q|, |\beta\Phi_\varepsilon^r(-q) + q| \leq ce$, because $\beta\Phi_\varepsilon^r(-q) = -\beta\Phi_\varepsilon^r(q)$, for a constant $c$ which does not depend on the point $q$. Therefore $|I_\varepsilon(q) - q| < ce$; that concludes the proof.

**Remark 4.6.** We have only to prove that any solution $u$ of (1.5) such that $J_\varepsilon(u) < 2(m_\infty + \delta)$ changes sign exactly once. In fact, assume that the set $\{u \in M : u(x) > 0\}$ has $h$ connected components $M_1, \ldots, M_h$. Set $u_i(x) := u(x)$ if $x \in M_i \cup (-M_i)$ and $u_i(x) := 0$ otherwise. We have $u_i \in \mathcal{M}_\varepsilon$ and

$$\frac{3}{2} hm_\infty \leq m_\varepsilon \leq J_\varepsilon(u) = \sum_{i=1}^h J_\varepsilon(u_i) \leq 2(m_\infty + \delta) < 3m_\infty. \quad (4.21)$$

Then $h = 1$. This concludes the proof.
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