Research Article

Forced Oscillations of Half-Linear Elliptic Equations via Picone-Type Inequality

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Picone-type inequality is established for a class of half-linear elliptic equations with forcing term, and oscillation results are derived on the basis of the Picone-type inequality. Our approach is to reduce the multi-dimensional oscillation problems to one-dimensional oscillation problems for ordinary half-linear differential equations.

1. Introduction

The p -Laplacian $\Delta_p v = \nabla \cdot (|\nabla v|^{p-2} \nabla v)$ arises from a variety of physical phenomena such as non-Newtonian fluids, reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaciology, and petroleum extraction (cf. Díaz [1]). It is important to study the qualitative behavior (e.g., oscillatory behavior) of solutions of p -Laplace equations with superlinear terms and forcing terms.

Forced oscillations of superlinear elliptic equations of the form

$$\nabla \cdot (A(x)|\nabla v|^{\alpha-1} \nabla v) + C(x)|v|^{\beta-1} v = f(x) \quad (\beta > \alpha > 0) \quad (1.1)$$

were studied by Jaroš et al. [2], and the more general quasilinear elliptic equation with first-order term

$$\nabla \cdot (A(x)|\nabla v|^{\alpha-1} \nabla v) + (\alpha + 1)B(x) \cdot (|\nabla v|^{\alpha-1} \nabla v) + C(x)|v|^{\beta-1} v = f(x) \quad (1.2)$$

was investigated by Yoshida [3], where the dot $(\cdot)$ denotes the scalar product. There appears to be no known oscillation results for the case where $\alpha = \beta$. The techniques used in [2, 3] are not applicable to the case where $\alpha = \beta$.

The purpose of this paper is to establish a Picone-type inequality for the half-linear elliptic equation with the forcing term:

$$P[v] := \nabla \cdot \left(A(x) |\nabla v|^{\alpha-1} \nabla v \right) + (\alpha + 1) B(x) \cdot \left(|\nabla v|^{\alpha-1} \nabla v \right) + C(x) |v|^{\alpha-1} v = f(x), \quad (1.3)$$

and to derive oscillation results on the basis of the Picone-type inequality. The approach used here is motivated by the paper [4] in which oscillation criteria for second-order nonlinear ordinary differential equations are studied. Our method is an adaptation of that used in [5]. Since the proofs of Theorems 2.2–3.3 are quite similar to those of [5, Theorems 1–4], we will omit them.

2. Picone-Type Inequality

Let G be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G . It is assumed that $\alpha > 0$ is a constant, $A(x) \in C(\overline{G}; (0, \infty))$, $B(x) \in C(\overline{G}; \mathbb{R}^n)$, $C(x) \in C(\overline{G}; \mathbb{R})$, and $f(x) \in C(\overline{G}; \mathbb{R})$.

The domain $\mathfrak{D}_P(G)$ of P is defined to be the set of all functions $v \in C^1(\overline{G}; \mathbb{R})$ with the property that $A(x) |\nabla v|^{\alpha-1} \nabla v \in C^1(G; \mathbb{R}^n) \cap C(\overline{G}; \mathbb{R}^n)$.

Lemma 2.1. *If $v \in \mathfrak{D}_P(G)$ and $|v| \geq k_0$ for some $k_0 > 0$, then the following Picone-type inequality holds for any $u \in C^1(G; \mathbb{R})$:*

$$\begin{aligned} & - \nabla \cdot \left(u \varphi(u) \frac{A(x) |\nabla v|^{\alpha-1} \nabla v}{\varphi(v)} \right) \\ & \geq -A(x) \left| \nabla u - \frac{u}{A(x)} B(x) \right|^{\alpha+1} + (C(x) - k_0^{-\alpha} |f(x)|) |u|^{\alpha+1} \\ & \quad + A(x) \left[\left| \nabla u - \frac{u}{A(x)} B(x) \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha + 1) \left(\nabla u - \frac{u}{A(x)} B(x) \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] \\ & \quad - \frac{u \varphi(u)}{\varphi(v)} (P[v] - f(x)), \end{aligned} \quad (2.1)$$

where $\varphi(s) = |s|^{\alpha-1} s$ ($s \in \mathbb{R}$) and $\Phi(\xi) = |\xi|^{\alpha-1} \xi$ ($\xi \in \mathbb{R}^n$).

Proof. The following Picone identity holds for any $u \in C^1(G; \mathbb{R})$:

$$\begin{aligned}
 & -\nabla \cdot \left(u\varphi(u) \frac{A(x)|\nabla v|^{\alpha-1}\nabla v}{\varphi(v)} \right) \\
 &= -A(x) \left| \nabla u - \frac{u}{A(x)}B(x) \right|^{\alpha+1} + C(x)|u|^{\alpha+1} \\
 &+ A(x) \left[\left| \nabla u - \frac{u}{A(x)}B(x) \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{u}{A(x)}B(x) \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] \\
 &- \frac{u\varphi(u)}{\varphi(v)} (P[v] - f(x)) - \frac{u\varphi(u)}{\varphi(v)} f(x)
 \end{aligned} \tag{2.2}$$

(see, e.g., Yoshida [6, Theorem 1.1]). Since $|v| \geq k_0$, we obtain

$$|\varphi(v)| = |v|^\alpha \geq k_0^\alpha, \tag{2.3}$$

and therefore

$$\left| \frac{u\varphi(u)}{\varphi(v)} f(x) \right| \leq |u|^{\alpha+1} k_0^{-\alpha} |f(x)|. \tag{2.4}$$

Combining (2.2) with (2.4) yields the desired inequality (2.1). $\square$

Theorem 2.2. Let $k_0 > 0$ be a constant. Assume that there exists a nontrivial function $u \in C^1(\overline{G}; \mathbb{R})$ such that $u = 0$ on ∂G and

$$M_G[u] := \int_G \left[A(x) \left| \nabla u - \frac{u}{A(x)}B(x) \right|^{\alpha+1} - (C(x) - k_0^{-\alpha}|f(x)|)|u|^{\alpha+1} \right] dx \leq 0. \tag{2.5}$$

Then for every solution $v \in \mathfrak{D}_P(G)$ of (1.3), either v has a zero on $\overline{G}$ or

$$|v(x_0)| < k_0 \quad \text{for some } x_0 \in G. \tag{2.6}$$

3. Oscillation Results

In this section we investigate forced oscillations of (1.3) in an exterior domain Ω in $\mathbb{R}^n$, that is, $\Omega \supset E_{r_0}$ for some $r_0 > 0$, where

$$E_r = \{x \in \mathbb{R}^n; |x| \geq r\} \quad (r > 0). \tag{3.1}$$

It is assumed that $\alpha > 0$ is a constant, $A(x) \in C(\Omega; (0, \infty))$, $B(x) \in C(\Omega; \mathbb{R}^n)$, $C(x) \in C(\Omega; \mathbb{R})$, and $f(x) \in C(\Omega; \mathbb{R})$.

The domain $\mathfrak{D}_P(\Omega)$ of P is defined to be the set of all functions $v \in C^1(\Omega; \mathbb{R})$ with the property that $A(x)|\nabla v|^{\alpha-1}\nabla v \in C^1(\Omega; \mathbb{R}^n)$.

A solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3) is said to be *oscillatory* in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap E_r. \quad (3.2)$$

Theorem 3.1. Assume that for any $k_0 > 0$ and any $r > r_0$ there exists a bounded domain $G \subset E_r$ such that (2.5) holds for some nontrivial $u \in C^1(\overline{G}; \mathbb{R})$ satisfying $u = 0$ on ∂G . Then for every solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3), either v is oscillatory in Ω or

$$\liminf_{|x| \rightarrow \infty} |v(x)| = 0. \quad (3.3)$$

Theorem 3.2. Assume that for any $k_0 > 0$ and any $r > r_0$ there exists a bounded domain $G \subset E_r$ such that

$$\widetilde{M}_G[u] := \int_G \left[2^\alpha A(x) |\nabla u|^{\alpha+1} - \left(C(x) - 2^\alpha A(x)^{-\alpha} |B(x)|^{\alpha+1} - k_0^{-\alpha} |f(x)| \right) |u|^{\alpha+1} \right] dx \leq 0 \quad (3.4)$$

holds for some nontrivial $u \in C^1(\overline{G}; \mathbb{R})$ satisfying $u = 0$ on ∂G . Then for every solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3), either v is oscillatory in Ω or satisfies (3.3).

Let $\overline{\{Q(x)\}}(r)$ denote the spherical mean of $Q(x)$ over the sphere $S_r = \{x \in \mathbb{R}^n; |x| = r\}$. We define $p(r)$ and $q_{k_0}(r)$ by

$$\begin{aligned} p(r) &= \overline{\{2^\alpha A(x)\}}(r), \\ q_{k_0}(r) &= \overline{\left\{ C(x) - 2^\alpha A(x)^{-\alpha} |B(x)|^{\alpha+1} - k_0^{-\alpha} |f(x)| \right\}}(r). \end{aligned} \quad (3.5)$$

Theorem 3.3. If the half-linear ordinary differential equation

$$\left(r^{n-1} p(r) |y'|^{\alpha-1} y' \right)' + r^{n-1} q_{k_0}(r) |y|^{\alpha-1} y = 0 \quad (3.6)$$

is oscillatory at $r = \infty$ for any $k_0 > 0$, then for every solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3), either v is oscillatory in Ω or satisfies (3.3).

Oscillation criteria for the half-linear differential equation (3.6) were obtained by numerous authors (see, e.g., Došlý and Řehák [7], Kusano and Naito [8], and Kusano et al. [9]).

Now we derive the following Leighton-Wintner-type oscillation result.

Corollary 3.4. *If*

$$\int_{r_0}^{\infty} \left(\frac{1}{r^{n-1}p(r)} \right)^{1/\alpha} dr = \infty, \quad \int_{r_0}^{\infty} r^{n-1} q_{k_0}(r) dr = \infty \quad (3.7)$$

for any $k_0 > 0$, then for every solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3), either v is oscillatory in Ω or satisfies (3.3).

Proof. The conclusion follows from the Leighton-Wintner oscillation criterion (see Došlý and Řehák [7, Theorem 1.2.9]). $\square$

By combining Theorem 3.3 with the results of [8, 9], we obtain Hille-Nehari-type criteria for (1.3) (cf. Došlý and Řehák [7, Section 3.1], Kusano et al. [10], and Yoshida [11, Section 8.1]).

Corollary 3.5. *Assume that $q_{k_0}(r) \geq 0$ eventually and suppose that $p(r)$ satisfies*

$$\int_{r_0}^{\infty} \left(\frac{1}{r^{n-1}p(r)} \right)^{1/\alpha} dr = \infty, \quad (3.8)$$

and $q_{k_0}(r)$ satisfies

$$\liminf_{r \rightarrow \infty} (P(r))^\alpha \int_r^\infty s^{n-1} q_{k_0}(s) ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \quad (3.9)$$

for any $k_0 > 0$, where

$$P(r) = \int_{r_0}^r \left(\frac{1}{s^{n-1}p(s)} \right)^{1/\alpha} ds. \quad (3.10)$$

Then for every solution $v \in \mathfrak{D}_P(\Omega)$ of (1.3), either v is oscillatory in Ω or satisfies (3.3).

Corollary 3.6. *Assume that $q_{k_0}(r) \geq 0$ eventually and suppose that $p(r)$ satisfies*

$$\int_{r_0}^{\infty} \left(\frac{1}{r^{n-1}p(r)} \right)^{1/\alpha} dr < \infty, \quad (3.11)$$

and $q_{k_0}(r)$ satisfies either

$$\int_{r_0}^{\infty} (\pi(r))^{\alpha+1} r^{n-1} q_{k_0}(r) dr = \infty \quad (3.12)$$

or

$$\liminf_{r \rightarrow \infty} \frac{1}{\pi(r)} \int_r^\infty (\pi(s))^{\alpha+1} s^{n-1} q_{k_0}(s) ds > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \quad (3.13)$$

for any $k_0 > 0$, where

$$\pi(r) = \int_r^\infty \left(\frac{1}{s^{n-1} p(s)} \right)^{1/\alpha} ds. \quad (3.14)$$

Then for every solution $v \in \mathfrak{D}_p(\Omega)$ of (1.3), either v is oscillatory in Ω or satisfies (3.3).

Remark 3.7. If the following hypotheses are satisfied:

$$\begin{aligned} C(x) - 2^\alpha A(x)^{-\alpha} |B(x)|^{\alpha+1} &> 0 \text{ (eventually),} \\ \lim_{|x| \rightarrow \infty} \frac{|f(x)|}{C(x) - 2^\alpha A(x)^{-\alpha} |B(x)|^{\alpha+1}} &= 0, \end{aligned} \quad (3.15)$$

then we observe that $q_{k_0}(r) > 0$ eventually.

Example 3.8. We consider the half-linear elliptic equation

$$\nabla \cdot (A(x) |\nabla v|^{\alpha-1} \nabla v) + (\alpha+1) B(x) \cdot (|\nabla v|^{\alpha-1} \nabla v) + C(x) |v|^{\alpha-1} v = f(x), \quad x \in \Omega, \quad (3.16)$$

where $n = 2$, $\Omega = E_1$, $A(x) = 2|x|^{-1}$, $B(x) = 2|x|^{-1-\alpha/(\alpha+1)} (\cos |x|, \sin |x|)$, $C(x) = |x|^{-1} (5/2 + \sin |x|)$, and $f(x) = |x|^{-1} e^{-|x|}$. It is easy to verify that

$$\begin{aligned} \int_1^\infty \left(\frac{1}{r p(r)} \right)^{1/\alpha} dr &= \infty, \\ q_{k_0}(r) &= \frac{1}{r} \left(\frac{1}{2} + \sin r - k_0^{-\alpha} e^{-r} \right), \end{aligned} \quad (3.17)$$

and therefore

$$\int_1^\infty r q_{k_0}(r) dr = \int_1^\infty \left(\frac{1}{2} + \sin r - k_0^{-\alpha} e^{-r} \right) dr = \infty \quad (3.18)$$

for any $k_0 > 0$. Hence, from Corollary 3.4, we see that for every solution v of (3.16), either v is oscillatory in Ω or satisfies (3.3).

Example 3.9. We consider the half-linear elliptic equation

$$\nabla \cdot (A(x) |\nabla v|^{\alpha-1} \nabla v) + (\alpha+1) B(x) \cdot (|\nabla v|^{\alpha-1} \nabla v) + C(x) |v|^{\alpha-1} v = f(x), \quad x \in \Omega, \quad (3.19)$$

where $n = 2$, $\Omega = E_1$, $A(x) = 2|x|^{-1}$, $B(x) = |x|^{-\alpha/(\alpha+1)}(\sin|x|, \cos|x|)$, $C(x) = 3 + \cos|x|$, and $f(x) = e^{-|x|} \sin|x|$. It is easily checked that

$$\lim_{|x| \rightarrow \infty} \frac{|f(x)|}{C(x) - 2^\alpha A(x)^{-\alpha} |B(x)|^{\alpha+1}} = \lim_{|x| \rightarrow \infty} \frac{e^{-|x|} |\sin|x||}{2 + \cos|x|} = 0, \quad (3.20)$$

and therefore $q_{k_0}(r) > 0$ eventually by Remark 3.7. Furthermore, we observe that

$$\begin{aligned} \int_1^\infty \left(\frac{1}{rp(r)} \right)^{1/\alpha} dr &= \infty, \\ q_{k_0}(r) &= 2 + \cos r - k_0^{-\alpha} e^{-r} |\sin r|, \\ (P(r))^\alpha &= 2^{-(\alpha+1)} (r-1)^\alpha, \\ \int_r^\infty s q_{k_0}(s) ds &= \int_r^\infty s (2 + \cos s - k_0^{-\alpha} e^{-s} |\sin s|) ds \\ &\geq \int_r^\infty s (1 - k_0^{-\alpha} e^{-s}) ds = \infty \end{aligned} \quad (3.21)$$

for any $k_0 > 0$. Hence we obtain

$$\liminf_{r \rightarrow \infty} (P(r))^\alpha \int_r^\infty s^{n-1} q_{k_0}(s) ds = \infty. \quad (3.22)$$

It follows from Corollary 3.5 that for every solution v of (3.19), either v is oscillatory in Ω or satisfies (3.3).

Example 3.10. We consider the half-linear elliptic equation

$$\nabla \cdot \left(A(x) |\nabla v|^{\alpha-1} \nabla v \right) + (\alpha+1) B(x) \cdot \left(|\nabla v|^{\alpha-1} \nabla v \right) + C(x) |v|^{\alpha-1} v = f(x), \quad x \in \Omega, \quad (3.23)$$

where $n = 2$, $\Omega = E_1$, $A(x) = |x|^{-1} e^{|x|}$, $B(x) = |x|^{-\alpha/(\alpha+1)} e^{|x|} (\cos|x|, \sin|x|)$, $C(x) = e^{2|x|}$, and $f(x)$ is a bounded function. It is easy to see that

$$\begin{aligned} C(x) - 2^\alpha A(x)^{-\alpha} |B(x)|^{\alpha+1} &= e^{2|x|} - 2^\alpha e^{|x|}, \\ \lim_{|x| \rightarrow \infty} \frac{|f(x)|}{C(x) - 2^\alpha A(x)^{-\alpha} |B(x)|^{\alpha+1}} &= \lim_{|x| \rightarrow \infty} \frac{|f(x)|}{e^{2|x|} - 2^\alpha e^{|x|}} = 0, \end{aligned} \quad (3.24)$$

and hence $q_{k_0}(r) > 0$ eventually by Remark 3.7. Since $f(x)$ is bounded, there exists a constant $M > 0$ such that $|f(x)| \leq M$. Moreover, we see that

$$\begin{aligned} \int_1^\infty \left(\frac{1}{rp(r)} \right)^{1/\alpha} dr &= \int_1^\infty \left(\frac{1}{4e^r} \right)^{1/2} dr < \infty, \\ \pi(r) &= \alpha 2^{-2/\alpha} e^{-r/\alpha}, \\ q_{k_0}(r) &= e^{2r} - 2^\alpha e^r - k_0^{-\alpha} \{ |f(x)| \}(r) \geq e^{2r} - 2^\alpha e^r - k_0^{-\alpha} M. \end{aligned} \quad (3.25)$$

If $\alpha > 1$, then

$$\int_1^\infty (\pi(r))^{\alpha+1} r q_{k_0}(r) dr \geq \frac{\alpha^{\alpha+1}}{2^{2(\alpha+1)/\alpha}} \int_1^\infty r \left(e^{((\alpha-1)/\alpha)r} - 2^\alpha e^{-r/\alpha} - k_0^{-\alpha} M e^{-((\alpha+1)/\alpha)r} \right) dr = \infty, \quad (3.26)$$

and if $0 < \alpha < 1$, then

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{1}{\pi(r)} \int_r^\infty (\pi(s))^{\alpha+1} s q_{k_0}(s) ds \\ \geq \liminf_{r \rightarrow \infty} \frac{\alpha^\alpha}{4} e^{r/\alpha} \int_r^\infty s \left(e^{((\alpha-1)/\alpha)s} - 2^\alpha e^{-s/\alpha} - k_0^{-\alpha} M e^{-((\alpha+1)/\alpha)s} \right) ds \\ \geq \liminf_{r \rightarrow \infty} \frac{\alpha^\alpha}{4} e^{r/\alpha} \int_r^\infty \left(e^{((\alpha-1)/\alpha)s} - 2^\alpha e^{-s/\alpha} - k_0^{-\alpha} M e^{-((\alpha+1)/\alpha)s} \right) ds \\ = \liminf_{r \rightarrow \infty} \frac{\alpha^\alpha}{4} \left(\frac{\alpha}{1-\alpha} e^r - 2^\alpha \alpha - k_0^{-\alpha} M \frac{\alpha}{\alpha+1} e^{-r} \right) = \infty \end{aligned} \quad (3.27)$$

for any $k_0 > 0$. Corollary 3.6 implies that for every solution v of (3.23), either v is oscillatory in Ω or satisfies (3.3).

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