Research Article

Multiple Solutions of Quasilinear Elliptic Equations in $\mathbb{R}^N$

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Assume that $Q$ is a positive continuous function in $\mathbb{R}^N$ and satisfies some suitable conditions. We prove that the quasilinear elliptic equation

$$-\Delta_p u + |u|^{p-2}u = Q(z)|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N,$$

admits at least two solutions in $\mathbb{R}^N$ (one is a positive ground-state solution and the other is a sign-changing solution).

1. Introduction

For $N \geq 3$, $2 \leq p < N$, and $p < q < p^* = Np/(N - p)$, we consider the quasilinear elliptic equations

$$-\Delta_p u + |u|^{p-2}u = Q(z)|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N,$$

$$u \in W^{1,p}(\mathbb{R}^N),$$

(1.1)

$$-\Delta_p u + |u|^{p-2}u = Q_{\infty}|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N,$$

$$u \in W^{1,p}(\mathbb{R}^N),$$

(1.2)

where $\Delta_p$ is the $p$-Laplacian operator, that is,

$$\Delta_p u = \sum_{i=1}^N \frac{\partial}{\partial z_i} \left( |\nabla u|^{p-2} \frac{\partial u}{\partial z_i} \right).$$

(1.3)
Let $Q$ be a positive continuous function in $\mathbb{R}^N$ and satisfy

$$Q(z) \geq Q_\infty = \lim_{|z| \to \infty} Q(z) > 0, \quad Q(z) \geq Q_\infty \text{ on a set of positive measure.} \quad (Q1)$$

Associated with (1.1) and (1.2), we define the functionals $a, b, b^\infty, J$, and $J^\infty$, for $u \in W^{1,p}(\mathbb{R}^N)$,

$$a(u) = \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)dz = ||u||^p_{1,p},$$
$$b(u) = \int_{\mathbb{R}^N} Q(z)|u|^qdz,$$
$$b^\infty(u) = \int_{\mathbb{R}^N} Q_\infty|u|^qdz,$$
$$J(u) = \frac{1}{p} a(u) - \frac{1}{q} b(u), \quad J^\infty(u) = \frac{1}{p} a(u) - \frac{1}{q} b^\infty(u). \quad (1.4)$$

It is easy to verify that the functionals $a, b, b^\infty, J$, and $J^\infty$ are $C^1$.

For the case $p = 2$, Lions [1, 2] proved that if $\lim_{|z| \to \infty} Q(z) = Q_\infty$, and $Q(z) \geq Q_\infty > 0$, then (1.1) has a positive ground-state solution in $\mathbb{R}^N$. Benci and Cerami [3] proved that (1.2) does not have any ground-state solution in an exterior domain. Bahri and Li [4] proved that there is at least one positive solution of (1.1) in $\mathbb{R}^N$ (or an exterior domain) when $\lim_{|z| \to \infty} Q(z) = Q_\infty > 0$ and $Q(z) \geq Q_\infty - C \exp(-\delta|z|)$ for $\delta > 2$. Cao [5] has studied the multiplicity of solutions (one is a positive ground-state solution and the other is a nodal solution) of (1.1) with Neumann condition in an exterior domain as follows. Assume that $\lim_{|z| \to \infty} Q(z) = Q_\infty > 0$, and $Q(z) \geq Q_\infty + C|z|^{-m} \exp(-\delta|z|)$ for $C > 0$, $m < (N-1)/2$, $\delta = q/(q+1)$, then (1.1) has at least two nontrivial solutions (one is a positive ground-state solution and the other is a nodal solution) in an exterior domain.

This article is motivated by the above papers. If $Q$ is a positive continuous function in $\mathbb{R}^N$ and satisfies (Q1), then we prove that (1.1) admits a positive ground-state solution in $\mathbb{R}^N$. Combine it with some ideas of Cerami et al. [6] to show that if $Q$ also satisfies $Q(z) \geq Q_\infty + C \exp(-\delta|z|)$ for $0 < \delta < \theta = (p-1)^{-1}/p$, then a nodal solution of (1.1) exists.

2. Preliminaries

We define the Palais-Smale (denoted by (PS)) sequences and (PS)-conditions in $W^{1,p}(\mathbb{R}^N)$ for $J$ as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a (PS)$_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for $J$ if $J(u_n) = \beta + o_n(1)$ and $J'(u_n) = o_n(1)$ strongly in $W^{-1,p'}(\mathbb{R}^N)$ as $n \to \infty$, where $W^{-1,p'}(\mathbb{R}^N)$ is the dual space of $W^{1,p}(\mathbb{R}^N)$ and $1/p + 1/p' = 1$.

(ii) $J$ satisfies the (PS)$_\beta$-condition in $W^{1,p}(\mathbb{R}^N)$ if every (PS)$_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for $J$ contains a convergent subsequence.

Lemma 2.2. Let $\beta \in \mathbb{R}$ and let $\{u_n\}$ be a (PS)$_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for $J$, then $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Moreover, $a(u_n) = b(u_n) + o_n(1) = (qp/(q-p))\beta + o_n(1)$ as $n \to \infty$ and $\beta \geq 0$. 

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Proof. Since \( p \geq 2 \), we have that \( \sqrt[p]{a(u_n)} \leq 1 \) if \( a(u_n) \leq 1 \) and \( \sqrt[p]{a(u_n)} \leq \sqrt{a(u_n)} \) if \( a(u_n) > 1 \). For sufficiently large \( n \), we get

\[
|\beta| + 2 + \sqrt{a(u_n)} \geq |\beta| + 1 + \sqrt{a(u_n)}
\]

It follows that \( \{u_n\} \) is bounded in \( W^{1,p}(\mathbb{R}^N) \). Then \( \langle J'(u_n), u_n \rangle = o_n(1) \) as \( n \to \infty \). Thus,

\[
\beta + o_n(1) = J(u_n) = \left( \frac{1}{p} - \frac{1}{q} \right) a(u_n) + o_n(1) = \left( \frac{1}{p} - \frac{1}{q} \right) b(u_n) + o_n(1),
\]

that is, \( a(u_n) = b(u_n) + o_n(1) = (qp/(q-p))\beta + o_n(1) \) as \( n \to \infty \) and \( \beta \geq 0 \).

Define

\[
a\left( \mathbb{R}^N \right) = \inf_{u \in M(\mathbb{R}^N)} J(u),
\]

where \( M(\mathbb{R}^N) = \{ u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid a(u) = b(u) \} \), and

\[
a^\infty \left( \mathbb{R}^N \right) = \inf_{u \in M^\infty(\mathbb{R}^N)} J^\infty(u),
\]

where \( M^\infty(\mathbb{R}^N) = \{ u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid a(u) = b^\infty(u) \} \).

Lemma 2.3. Let \( u \) be a sign-changing solution of (1.1). Then \( J(u) \geq 2a(\mathbb{R}^N) \).

Proof. Define \( u^+ = \max\{u,0\} \) and \( u^- = \max\{-u,0\} \). Since \( u \) is a sign-changing solution of (1.1), then \( u^- \) is nonnegative and nonzero. Multiply (1.1) by \( u^- \) and integrate it to obtain

\[
\int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla u^- + |u|^{p-2} u u^-) dz = \int_{\mathbb{R}^N} Q(z)|u|^{p-2} u u^- dz,
\]

that is, \( u^- \in M(\mathbb{R}^N) \) and \( J(u^-) \geq a(\Omega) \). Similarly, \( J(u^+) \geq a(\mathbb{R}^N) \). Hence,

\[
J(u) = J(u^+) + J(u^-) \geq 2a(\mathbb{R}^N).
\]

Lemma 2.4. (i) For each \( u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \), there exists a positive number \( s_u \) such that \( s_u u \in M(\mathbb{R}^N) \) and \( \sup_{s \geq 0} J(su) = J(s_u u) \).

(ii) Let \( \beta > 0 \) and let \( \{u_n\} \) be a sequence in \( W^{1,p}(\mathbb{R}^N) \setminus \{0\} \) for \( J \) such that \( a(u_n) = b(u_n) + o(1) \) and \( J(u_n) = \beta + o(1) \). Then there is a sequence \( \{s_n\} \) in \( \mathbb{R}^+ \) such that \( s_n = 1 + o(1) \), \( \{s_n u_n\} \subset M(\mathbb{R}^N) \), and \( J(s_n u_n) = \beta + o(1) \) as \( n \to \infty \).
Proof. (i) For each $u \in W_0^{1,p}(\mathbb{R}^N) \setminus \{0\}$ and $s \geq 0$, let

$$h_u(s) = J(su) = \frac{s^p}{p} a(u) - \frac{s^q}{q} b(u). \quad (2.7)$$

Thus, $h_u'(s) = s^{p-1}a(u) - s^{q-1}b(u)$. Define $s_u = (a(u)/b(u))^{1/(q-p)} > 0$, then $h_u'(s_u) = 0$, that is, $s_u u \in M(\mathbb{R}^N)$.

(ii) By (i), there exists a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\} \subset M(\mathbb{R}^N)$, that is, $s_n^p a(u_n) = s_n^q b(u_n)$ for each $n$. Since $a(u_n) = b(u_n) + o(1)$ and $J(u_n) = \beta + o(1)$, we have that $s_n = 1 + o(1)$. Hence, $J(s_n u_n) = \beta + o(1)$ as $n \to \infty$. $\square$

**Lemma 2.5.** There exists $c > 0$ such that $\|u\|_{1,p} \geq c > 0$ for each $u \in M(\mathbb{R}^N)$, where $c$ is independent of $u$.

Proof. For each $u \in M(\mathbb{R}^N)$, by the Sobolev inequality, we obtain that

$$\|u\|_{1,p}^p = \int_{\mathbb{R}^N} Q(z)|u|^p dz \leq c_1 \|u\|_{1,p}^q. \quad (2.8)$$

This implies that $\|u\|_{1,p} \geq c_1^{-1/(q-p)} = c > 0$ for each $u \in M(\mathbb{R}^N)$. $\square$

By Lemma 2.5, $\alpha(\mathbb{R}^N) > 0$.

**Lemma 2.6.** Let $u \in M(\mathbb{R}^N)$ such that

$$J(u) = \min_{v \in M(\mathbb{R}^N)} J(v) = \alpha(\mathbb{R}^N), \quad (2.9)$$

then $u$ is a nonzero solution of (1.1) in $\mathbb{R}^N$.

Proof. Suppose that $\varphi(v) = \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) dz - \int_{\mathbb{R}^N} Q(z)|v|^q dz$, then

$$\langle \varphi'(v), v \rangle = (p-q) \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) dz < 0 \quad \text{for each } v \in M(\mathbb{R}^N). \quad (2.10)$$

Since $J(u) = \min_{v \in M(\mathbb{R}^N)} J(v)$, by the Lagrange multiplier theorem, there is a $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda \varphi'(u)$ in $W^{-1,q'}(\mathbb{R}^N)$. Then we have

$$0 = \langle J'(u), u \rangle = \lambda \langle \varphi'(u), u \rangle. \quad (2.11)$$

Thus, $\lambda = 0$ and $J'(u) = 0$ in $W^{-1,q'}(\mathbb{R}^N)$. Therefore, $u$ is a nonzero solution of (1.1) in $\mathbb{R}^N$ with $J(u) = \alpha(\mathbb{R}^N)$. $\square$
Lemma 2.7. There is a \((PS)_{a(\mathbb{R}^N)}\)-sequence in \(W^{1,p}(\mathbb{R}^N)\) for \(J\).

Proof. Let \(\{u_n\} \subset M(\mathbb{R}^N)\) be a minimizing sequence of \(a(\mathbb{R}^N)\). Applying the Ekeland principle, there exists a sequence \(\{v_n\} \subset M(\mathbb{R}^N)\) such that \(\|v_n - u_n\|_{1,p} < 1/n\), \(J(v_n) = a(\mathbb{R}^N) + o(1)\), and \(J_{\mid M(\mathbb{R}^N)}(v_n) = o(1)\) strongly in \(W^{-1,p}(\mathbb{R}^N)\) as \(n \to \infty\). Let \(\varphi(u) = a(u) - b(u)\) for each \(u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}\), then

\[
M(\mathbb{R}^N) = \left\{ u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid \varphi(u) = 0 \right\}.
\]

(2.12)

Thus, there exists a sequence \(\{\theta_n\} \subset \mathbb{R}\) such that \(J'(v_n) = \theta_n \varphi'(v_n) + o_n(1)\), where \(o_n(1) \to 0\) as \(n \to \infty\). Since \(v_n \in M(\mathbb{R}^N)\), we have that

\[
0 = \langle J'(v_n), v_n \rangle = \theta_n \langle \varphi'(v_n), v_n \rangle + (o_n(1), v_n),
\]

\[
\langle \varphi'(v_n), v_n \rangle = (p - q)a(v_n) \not\equiv 0 \ \forall n.
\]

(2.13)

Hence, \(\theta_n \to 0\) as \(n \to \infty\). This implies that \(J'(v_n) = o(1)\) strongly in \(W^{-1,p}(\mathbb{R}^N)\) as \(n \to \infty\), that is, \(\{v_n\} \subset M(\mathbb{R}^N)\) is a \((PS)_{a(\Omega)}\)-sequence in \(W^{1,p}(\mathbb{R}^N)\) for \(J\).

\[ Q.E.D. \]

Remark 2.8. The above definitions and lemmas also hold for \(J^\infty, M^\infty(\mathbb{R}^N), \) and \(a^\infty(\mathbb{R}^N)\).

3. Existence of a Ground-State Solution

Using the arguments by Lions [1, 2], Benci and Cerami [3], Struwe [7], and Alves [8], we have the following decomposition lemma.

Lemma 3.1 (Palais-Smale Decomposition Lemma for \(J\)). Assume that \(Q\) is a positive continuous function in \(\mathbb{R}^N\) and \(\lim_{|z| \to \infty} Q(z) = Q_\infty > 0\). Let \(\{u_n\}\) be a \((PS)_p\)-sequence in \(W^{1,p}(\mathbb{R}^N)\) for \(J\). Then there are a subsequence \(\{u_n\}_1\), a positive integer \(l\), sequences \(\{\gamma^i_n\}_{n=1}^\infty\) in \(\mathbb{R}^N\), functions \(u\) in \(W^{1,p}(\mathbb{R}^N)\), and \(\omega^i \not\equiv 0\) in \(W^{1,p}(\mathbb{R}^N)\) for \(1 \leq i \leq l\) such that

\[
\left| \gamma^i_n \right| \to \infty \ \text{for} \ 1 \leq i \leq l,
\]

\[
-\Delta_p u + |u|^{p-2} u = Q(z)|u|^{q-2} u \ \text{in} \ \mathbb{R}^N,
\]

\[
-\Delta_p \omega^i + |\omega^i|^{p-2} \omega^i = Q_\infty |\omega^i|^{q-2} \omega^i \ \text{in} \ \mathbb{R}^N,
\]

\[
u_n = u + \sum_{i=1}^l \omega^i \left( - \gamma^i_n \right) + o_n(1) \ \text{strongly in} \ W^{1,p}(\mathbb{R}^N),
\]

\[
J(u_n) = J(u) + \sum_{i=1}^l J(\omega^i) + o_n(1).
\]

(3.1)

In addition, if \(u_n \geq 0\), then \(u \geq 0\) and \(\omega^i \geq 0\) for \(1 \leq i \leq l\).
**Lemma 3.2.** Let \( \{u_n\} \subset M(\mathbb{R}^N) \) be a \((PS)_\beta\)-sequence in \( W^{1,p}(\mathbb{R}^N) \) for \( 0 < \beta < \alpha^\infty(\mathbb{R}^N) \). Then there exist a subsequence \( \{u_n\} \) and a non-zero \( u \in W^{1,p}(\mathbb{R}^N) \) such that \( u_n \to u \) strongly in \( W^{1,p}(\mathbb{R}^N) \) and \( J(u) = \beta \), that is, \( J \) satisfies the \((PS)_\beta\)-condition in \( W^{1,p}(\mathbb{R}^N) \).

**Proof.** Since \( \{u_n\} \subset M(\mathbb{R}^N) \) is a \((PS)_\beta\)-sequence in \( W^{1,p}(\mathbb{R}^N) \) for \( 0 < \beta < \alpha^\infty(\mathbb{R}^N) \), by Lemma 2.2, \( \{u_n\} \) is bounded in \( W^{1,p}(\mathbb{R}^N) \). Thus, there exist a subsequence \( \{u_n\} \) and \( u \in W^{1,p}(\mathbb{R}^N) \) such that \( u_n \to u \) weakly in \( W^{1,p}(\mathbb{R}^N) \). It is easy to check that \( u \) is a solution of (1.1) in \( \mathbb{R}^N \). Applying Palais-Smale Decomposition Lemma 3.1, we get

\[
\alpha^\infty > \beta = J(u_n) \geq l\alpha^\infty. \tag{3.2}
\]

Then \( l = 0 \) and \( u \neq 0 \). Hence, \( u_n \to u \) strongly in \( W^{1,p}(\mathbb{R}^N) \) and \( J(u) = \beta \). \( \Box \)

Let \( w \in W^{1,p}(\mathbb{R}^N) \) be the positive ground-state solution of (1.2) in \( \mathbb{R}^N \). Using the same arguments by Li and Yan [9] and Marcos do Ó [10, Lemma 3.8], or see Serrin and Tang [11, page 899] and Li and Zhao [12, Theorem 1.1], we obtain the following results:

(i) \( w \in L^\infty(\mathbb{R}^N) \cap C^{1,p}_{\text{loc}}(\mathbb{R}^N) \) for some \( 0 < \gamma_0 < 1 \) and \( \lim_{|z| \to \infty} w(z) = 0 \);

(ii) for any \( \epsilon > 0 \), there exist positive numbers \( C_1 \) and \( C_2 \) such that

\[
C_2 \exp(-(\theta + \epsilon)|z|) \leq w(z) \leq C_1 \exp(-(\theta - \epsilon)|z|) \quad \forall z \in \mathbb{R}^N, \tag{3.3}
\]

where \( \theta = (p - 1)^{-1/p} \).

**Remark 3.3.** Similarly, we also show that all positive solutions of (1.1) in \( \mathbb{R}^N \) have exponential decay.

By Lemma 3.2, we can prove the following theorem.

**Theorem 3.4.** Assume that \( Q \) is a positive continuous function in \( \mathbb{R}^N \) and satisfies (Q1). Then there exists a positive ground-state solution \( w_0 \) of (1.1) in \( \mathbb{R}^N \).

**Proof.** Let \( w \in W^{1,p}(\mathbb{R}^N) \) be the positive ground-state solution of (1.2) in \( \mathbb{R}^N \), then \( w \) is a minimizer of \( \alpha^\infty(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} |(\nabla w|^p + w^p)dz = \int_{\mathbb{R}^N} Q_{\infty}w^p dz. \tag{3.4}
\]
By Lemma 2.4(i), there exists a positive number $s_w$ such that $s_w w \in M(\mathbb{R}^N)$, that is, 
\[ \int_{\mathbb{R}^N} ((\nabla (s_w w))^p + (s_w w)^p)dz = \int_{\mathbb{R}^N} Q(z)(s_w w)^q dz. \] Since $Q(z) > Q_\infty$ on a set of positive measure, we can deduce that $s_w < 1$. Therefore,
\[
a(\mathbb{R}^N) \leq J(s_w w) = \left( \frac{1}{p} - \frac{1}{q} \right) (s_w)^p \int_{\mathbb{R}^N} (|\nabla w|^p + w^p)dz
\]
\[
< \left( \frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^N} (|\nabla w|^p + w^p)dz
\]
\[
= \left( \frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^N} Q_\infty w^q dz = a_\infty (\mathbb{R}^N).
\] (3.5)

Applying Lemma 3.2, there exists $u_0 \in W^{1,p}(\mathbb{R}^N)$ such that $J(u_0) = a(\mathbb{R}^N)$. From the results of Lemmas 2.6 and 2.3, by Maximum Principle, $u_0$ is a positive ground-state solution of (1.1) in $\mathbb{R}^N$.

\[ \square \]

4. Existence of a Nodal Solution

In this section, assume that $Q$ is a positive continuous function in $\mathbb{R}^N$ and satisfies (Q1). In order to prove Lemma 4.8, $Q$ also satisfies the following condition (Q2): there exist some constants $C > 0$ and $0 < \delta < \theta = (p - 1)^{-1/p}$ such that
\[
Q(z) \geq Q_\infty + C \exp(-\delta |z|) \forall z \in \mathbb{R}^N.
\] (Q2)

Let $h$ be a functional in $W^{1,p}(\mathbb{R}^N)$ defined by
\[
h(u) = \begin{cases} \frac{b(u)}{a(u)} & \text{for } u \neq 0, \\ 0 & \text{for } u = 0. \end{cases}
\] (4.1)

We define
\[
M_0 = \{ u \in W^{1,p}(\mathbb{R}^N) \mid h(u^+) = 1, \ h(u^-) = 1 \} \subset M(\mathbb{R}^N),
\]
\[
\mathcal{M} = \{ u \in W^{1,p}(\mathbb{R}^N) \mid |h(u^+) - 1| < \frac{1}{2} \} \subset M_0,
\] (4.2)

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.  

Lemma 4.1. (i) If $u \in W^{1,p}(\mathbb{R}^N)$ changes sign, then there exist positive numbers $s^+(u) = s^+$ such that $s^+ u^+ \in M(\mathbb{R}^N)$ and $s^- u^- \in M(\mathbb{R}^N)$.

(ii) There exists $c' > 0$ such that $\|u^+\|_{1,p} \geq c' > 0$ for each $u \in \mathcal{M}$. 
Proof. (i) Since $u^+$ and $u^-$ are nonzero and nonnegative, by Lemma 2.4(i), it is easy to obtain the result.

(ii) For each $u \in \mathcal{A}$, by Lemma 2.4(i), there exists $s^+(u) = s^+ > 0$ such that $s^+ u^+ \in \mathbf{M}(\mathbb{R}^N)$. Then

$$\frac{1}{2} < (s^+)^{p-q} = \frac{b(u^+)}{a(u^+)} < \frac{3}{2}$$ for each $u \in \mathcal{A}$. \hfill (4.3)

By Lemma 2.5, we have

$$\|s^+ u^+\|_{1,p} \geq c$$ for some $c > 0$ and each $u \in \mathcal{A}$. \hfill (4.4)

Hence, $\|u^+\|_{1,p} \geq c/s^+ \geq c' > 0$ for each $u \in \mathcal{A}$.

Consider these minimization problem

$$\gamma(\mathbb{R}^N) = \inf_{u \in \mathbf{M}_0} J(u).$$ \hfill (4.5)

By Lemma 4.1, $\gamma(\mathbb{R}^N) > 0$.

**Lemma 4.2.** There exists a sequence $\{u_n\} \subset \mathcal{A}$ such that $J(u_n) = \gamma(\mathbb{R}^N) + o_n(1)$ and $J'(u_n) = o_n(1)$ strongly in $W^{1,p}(\mathbb{R}^N)$ as $n \to \infty$.

*Proof.* It is similar to the proof of Zhu [13]. \hfill \Box

**Lemma 4.3.** Let $f$ and $g$ be real-valued functions in $\mathbb{R}^N$. If $g(z) > 0$ in $\mathbb{R}^N$, then one has the following inequalities:

(i) $(f + g)^+ \geq f^+$,
(ii) $(f + g)^- \leq f^-$,
(iii) $(f - g)^+ \leq f^+$,
(iv) $(f - g)^- \geq f^-$.

**Lemma 4.4.** Let $\{u_n\} \subset \mathcal{A}$ be a $(PS)_{\gamma(\mathbb{R}^N)}$-sequence in $W^{1,p}(\mathbb{R}^N)$ for $J$ satisfying

$$a(\mathbb{R}^N) < \gamma(\mathbb{R}^N) < a(\mathbb{R}^N) + a^{\infty}(\mathbb{R}^N) \left( < 2a^{\infty}(\mathbb{R}^N) \right).$$ \hfill (4.6)

Then there exists $u^* \in \mathbf{M}_0$ such that $u_n$ converges to $u^*$ strongly in $W^{1,p}(\mathbb{R}^N)$ and $u^*$ is a higher-energy solution of (1.1) such that $J(u^*) = \gamma(\mathbb{R}^N)$.

*Proof.* By the definition of the $(PS)_{\gamma(\mathbb{R}^N)}$-sequence in $W^{1,p}(\mathbb{R}^N)$ for $J$, it is easy to see that $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and satisfies

$$\int_{\mathbb{R}^N} (|\nabla u_n^+|^p + |u_n^+|^p) \, dz = \int_{\mathbb{R}^N} Q(z)|u_n^+|^q \, dz + o_n(1).$$ \hfill (4.7)
By Lemma 4.1(ii), there exists \( c' > 0 \) such that

\[
c' \leq \int_{\mathbb{R}^N} (|\nabla u_n|^p + |u_n|^p)\,dz = \int_{\mathbb{R}^N} Q(z)|u_n|^q\,dz + o_n(1). \tag{4.8}
\]

Using the Palais-Smale Decomposition Lemma 3.1, then we have \( \gamma(\mathbb{R}^N) = J(u^*) + \sum_{i=1}^l J(\omega_i) \), where \( u^* \) is a solution of (1.1) in \( \mathbb{R}^N \) and \( \omega_i \) is a solution of (1.2) in \( \mathbb{R}^N \). Since \( J(\omega_i) \geq \alpha^\infty(\mathbb{R}^N) \) for each \( i \in \mathbb{N} \) and \( \alpha(\mathbb{R}^N) < \alpha^\infty(\mathbb{R}^N) \), we have \( l \leq 1 \). Now we want to show that \( l = 0 \). On the contrary, suppose that \( l = 1 \).

(i) \( \omega_1 \) is a sign-changing solution of (1.2): by Lemma 2.3 and Remark 2.8, we have \( \gamma(\mathbb{R}^N) \geq 2\alpha^\infty(\mathbb{R}^N) \), which is a contradiction.

(ii) \( \omega_1 \) is a constant-sign solution of (1.2): we may assume that \( \omega_1 > 0 \). Applying the Decomposition Lemma 3.1, there exists a sequence \( \{z_n^1\} \) in \( \mathbb{R}^N \) such that \( |z_n^1| \to \infty \), and

\[
\left\| u_n - u^* - \omega_1 \left( \cdot - z_n^1 \right) \right\|_{L^1_p} = o_n(1). \tag{4.9}
\]

By the Sobolev continuous embedding inequality, we obtain

\[
\left\| u_n - u^* - \omega_1 \left( \cdot - z_n^1 \right) \right\|_{L^q} = o_n(1). \tag{4.10}
\]

Since \( \omega_1 > 0 \), by Lemma 4.3, then

\[
\left\| (u_n - u^*)^+ \right\|_{L^q} = o_n(1) \quad \text{as} \quad n \to \infty. \tag{4.11}
\]

(a) Suppose that \( u^* \equiv 0 \); we obtain \( \|u_n^+\|_{L^q} = o_n(1) \) as \( n \to \infty \). Then

\[
0 < c' \leq \int_{\mathbb{R}^N} Q(z)|u_n^-|^q\,dz = o_n(1), \tag{4.12}
\]

which is a contradiction.

(b) Suppose that \( u^* \neq 0 \). We have \( \gamma(\mathbb{R}^N) = J(u^*) + J^\infty(\omega_1) \geq \alpha(\mathbb{R}^N) + \alpha^\infty(\mathbb{R}^N) \), which is a contradiction.

By (i) and (ii), then \( l = 0 \). Thus, \( \|u_n - u^*\|_{L^1_p} = o_n(1) \) as \( n \to \infty \) and \( J(u^*) = \gamma(\mathbb{R}^N) \). Finally, we claim that \( u^* \) is a sign-changing solution of (1.1) in \( \mathbb{R}^N \). If \( u^* > 0 \) (or \( < 0 \)), by Lemma 4.3, then \( \|u_n^+\|_{L^q} = o_n(1) \) (or \( \|u_n^-\|_{L^q} = o_n(1) \)). Similarly, we have the inequality (4.12), which is a contradiction. Moreover, by Lemma 2.3, \( 2\alpha(\mathbb{R}^N) \leq \gamma(\mathbb{R}^N) \).

Recall that \( \omega \) is the positive ground-state solution of (1.2) in \( \mathbb{R}^N \). For any \( \varepsilon > 0 \), there exist positive numbers \( C_1 \) and \( C_2 \) such that

\[
C_2 \exp\left( -(|\theta + \varepsilon)|z| \right) \leq \omega(z) \leq C_1 \exp\left( -(|\theta - \varepsilon)|z| \right) \quad \forall z \in \mathbb{R}^N, \tag{4.13}
\]
where \( \theta = (p - 1)^{-1/p}. \) Define
\[
\omega_n(z) = \omega(z - z_n) \quad \text{where } z_n = (0, \ldots, 0, n) \in \mathbb{R}^N.
\] (4.14)

Clearly, \( \omega_n(z) \in W^{1,p}(\mathbb{R}^N). \)

**Lemma 4.5.** There are \( n_0 \in \mathbb{N} \) and real numbers \( t_1^* \) and \( t_2^* \) such that for \( n \geq n_0 \)
\[
t_1^* u_0 - t_2^* w_n \in M_0, \quad \gamma\left(\mathbb{R}^N\right) \leq J(t_1^* u_0 - t_2^* w_n),
\] (4.15)

where \( 1/p \leq t_1^*, t_2^* \leq p, \) and \( u_0 \) is the positive ground-state solution of (1.1) in \( \mathbb{R}^N. \)

**Proof.** Applying the mean value theorem by Miranda [14], the proof is similar to that of Zhu [13] (or see Hsu [15, page 728]). \( \Box \)

We need the following lemmas to prove that \( \sup_{1/p < t_1^*, t_2^* < 1/p} J(t_1^* u_0 - t_2^* w_n) < \alpha(\mathbb{R}^N) + \alpha^\infty(\mathbb{R}^N) \) for sufficiently large \( n. \)

**Lemma 4.6.** Let \( E \) be a domain in \( \mathbb{R}^N. \) If \( f : E \to \mathbb{R} \) satisfies
\[
\int_E |f(z) e^{\sigma |z|}| dz < \infty \quad \text{for some } \sigma > 0,
\] (4.16)

then
\[
\left(\int_E f(z) e^{-\sigma |z|} e^{|z||z|} dz\right) e^{\sigma |z|} = \int_E f(z) e^{\sigma (|z| + |z|)} dz + O(1) \quad \text{as } |z| \to \infty.
\] (4.17)

**Proof.** Since \( \sigma |z| \leq \sigma |z| + \sigma |z - z|, \) we have
\[
\left| f(z) e^{-\sigma |z|} e^{\sigma |z|}\right| \leq \left| f(z) e^{\sigma |z|}\right|.
\] (4.18)

Since \( -\sigma |z - z| + \sigma |z| = \sigma(|z| + |z|) + O(1) \) as \( |z| \to \infty, \) then the lemma follows from the Lebesgue-dominated convergence theorem. \( \Box \)

**Lemma 4.7.** For all \( x, y \in \mathbb{R}^N, \) one has the following inequality:
\[
|x - y|^{\rho} \leq \left(|x|^{\rho - 2} x - |y|^{\rho - 2} y\right)(x - y), \quad \text{where } \rho \geq 2.
\] (4.19)

**Proof.** See Yang [16, Lemma 4.2.]. \( \Box \)

**Lemma 4.8.** There exists an \( n_0^* \in \mathbb{N} \) such that for \( n \geq n_0^* \geq n_0 \)
\[
\gamma\left(\mathbb{R}^N\right) \leq \sup_{1/p < t_1^*, t_2^* < 1/p} J(t_1^* u_0 - t_2^* w_n) < \alpha(\mathbb{R}^N) + \alpha^\infty(\mathbb{R}^N),
\] (4.20)

where \( u_0 \) is a positive ground-state solution of (1.1) in \( \mathbb{R}^N. \)
Proof. By Lemma 4.7, then

\[
J(t_1^* u_0 - t_2^* w_n)
\]

\[
= \frac{1}{p} \left\| t_1^* u_0 - t_2^* w_n \right\|_{1,p}^p - \frac{1}{q} b(t_1^* u_0 - t_2^* w_n)
\]

\[
\leq \frac{1}{p} \left\{ \int_{\mathbb{R}^N} \left( |\nabla (t_1^* u_0)|^{p-2} \nabla (t_1^* u_0) - |\nabla (t_2^* w_n)|^{p-2} \nabla (t_2^* w_n) \right) \right\}
\]

\[
+ \frac{1}{p} \left\{ \int_{\mathbb{R}^N} \left( |t_1^* u_0|^{p-2} (t_1^* u_0) - |t_2^* w_n|^{p-2} (t_2^* w_n) \right) \right\} - \frac{1}{q} b(t_1^* u_0 - t_2^* w_n)
\]

\[
\leq J(t_1^* u_0) + J^\infty(t_2^* w) - \frac{(t_2^*)^q}{q} \int_{\mathbb{R}^N} (Q(z) - Q_\infty) w(z-z_n)^q dz
\]

\[
- \frac{1}{q} b(t_1^* u_0 - t_2^* w_n) + \frac{1}{q} b(t_1^* u_0) + \frac{1}{q} b(t_2^* w_n).
\]

(4.21)

Since \( \sup_{t \geq 0} J(t u_0) = \alpha(\mathbb{R}^N) \) and \( \sup_{t \geq 0} J^\infty(t w) = \alpha^\infty(\mathbb{R}^N) \), using the inequality

\[
|c_1 - c_2|^q > c_1^q + c_2^q - K \left( c_1^{q-1} c_2 + c_1 c_2^{q-1} \right),
\]

(4.22)

for any \( c_1, c_2 > 0 \), and some positive constant \( K \), then

\[
\sup_{1/p \leq 1, t_2^* \leq \infty} J(t_1^* u_0 - t_2^* w_n) \leq \alpha(\mathbb{R}^N) + \alpha^\infty(\mathbb{R}^N) - \frac{1}{pq} \int_{\mathbb{R}^N} (Q(z) - Q_\infty) w(z-z_n)^q dz
\]

\[
+ K' \left[ \int_{\mathbb{R}^N} \left( u_0^{q-1} w_n + w_0^{q-1} u_0 \right) dz \right].
\]

(4.23)

(i) Since \( Q(z) \geq Q_\infty + C \exp(-\delta |z|) \) for some constants \( C > 0 \) and \( 0 < \delta < \theta \), by Lemma 4.6, we have that there exists an \( n_1 \geq n_0 \) such that for \( n \geq n_1 \)

\[
\int_{\mathbb{R}^N} (Q(z) - Q_\infty) w(z-z_n)^q dz \geq C' \exp\left( -\min\{\delta, q(\theta + \epsilon)\} |\Xi| \right) \geq C' \exp(-\delta n).
\]

(4.24)

(ii) Applying Lemma 4.6, there exists an \( n_2 \geq n_1 \) such that for \( n \geq n_2 \)

\[
\int_{\mathbb{R}^N} u_0^{q-1} w_n dz \leq C_1 \int_{\mathbb{R}^N} \exp\left( -(q-1)(\theta - \epsilon)|z|\right) \exp\left( -(\theta - \epsilon)|z-z_n|\right) dz \leq C_1' \exp\left( -(\theta - \epsilon)n\right).
\]

(4.25)
Similarly, we also obtain that there exists an \( n_3 \geq n_2 \) such that for \( n \geq n_3 \)

\[
\int_{\mathbb{R}^N} u_{n}^{q-1} u_0 \, dz \leq C_1^n \exp(-\varepsilon n) \tag{4.26}
\]

By (i) and (ii), choosing \( 0 < \varepsilon < \theta - \delta \), we can find an \( n_0^* \geq n_3 \geq n_0 \) such that for \( n \geq n_0^* \)

\[
\sup_{1/p \leq t \leq p} J(t^1 u_0 - t^2 u_n) < a\big(\mathbb{R}^N\big) + a^\infty\big(\mathbb{R}^N\big). \tag{4.27}
\]

**Theorem 4.9.** Assume that \( Q \) is a positive continuous function in \( \mathbb{R}^N \) and satisfies (Q1) and (Q2), then (1.1) has a positive solution and a nodal solution in \( \mathbb{R}^N \).

**Proof.** By Lemmas 4.2, 4.4, 4.5, and 4.8 and Theorem 3.4, we obtain the proof.

**References**


