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Research Article

The Second Eigenvalue of the p-Laplacian as p Goes to 1

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The asymptotic behaviour of the second eigenvalue of the p-Laplacian operator as p goes to 1 is investigated. The limit setting depends only on the geometry of the domain. In the particular case of a planar disc, it is possible to show that the second eigenfunctions are nonradial if p is close enough to 1.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with Lipschitz boundary. We consider the following nonlinear eigenvalue problem:

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $p \in (1, +\infty)$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the *p-Laplacian* operator. Notice that for $p = 2$ we recover the usual Laplacian. A real number λ is said to be an *eigenvalue* if there exists a function $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ (called *eigenfunction*) satisfying (1.1) in the weak sense, which means

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \lambda \int_\Omega |u|^{p-2} u v \quad \forall v \in W_0^{1,p}(\Omega). \tag{1.2}$$

A sequence of eigenvalues $\{\lambda_k(p;\Omega)\}_{k=1}^{\infty}$ can be obtained by means of a minimax principle, as shown, for instance, in [1] and explained later in Section 5. The sequence is such that

$$\lambda_1(p;\Omega) < \lambda_2(p;\Omega) \leq \lambda_3(p;\Omega) \leq \cdots \tag{1.3}$$

and $\lambda_k(p;\Omega) \to +\infty$ as $k \to +\infty$.

In this paper we will mainly focus on the asymptotic behaviour of the second eigenvalue of the p-Laplacian as p goes to 1. Our aim is to extend the results found in [2], where it was shown that the first eigenvalue converges to the so-called *Cheeger constant* defined as

$$h_1(\Omega) = \inf_{E\subset\Omega} \frac{\mathrm{Per}(E;\mathbb{R}^n)}{V(E)}. \tag{1.4}$$

Here $\mathrm{Per}(E;\mathbb{R}^n)$ is the perimeter of E measured with respect to $\mathbb{R}^n$ and defined in the distributional sense (see [3]), while $V(E)$ stands for the n-dimensional Lebesgue measure of E. The task of finding a set for which the infimum is attained is called *Cheeger problem*. We will show that a similar result holds also for the second eigenvalue; moreover, we are able to state that the second eigenfunctions of the p-Laplacian in a planar disc cannot be radial, if p is sufficiently close to 1.

The paper is structured as follows. After recalling some known results about the Cheeger problem (Section 2), in Sections 3 and 4 we deal with a geometrical problem, which will turn out to be crucial in order to describe the asymptotic behaviour of $\lambda_2(p;\Omega)$ as $p \to 1$ (Section 5). Finally, in Section 6 we apply the results to the particular case where Ω is a planar disc.

2. Known Facts on the Cheeger Problem

In this section we will recall some known results about the Cheeger problem. We say that a function $u \in L^1(\Omega)$ has *bounded variation* if the quantity

$$\|Du\|(\Omega) = \sup\left\{\int_\Omega u \operatorname{div}\varphi\, dx \mid \varphi \in \mathcal{C}_c^1(\Omega;\mathbb{R}^n),\ |\varphi| \leq 1\right\}, \tag{2.1}$$

called *total variation*, is finite; in this case we will write $u \in BV(\Omega)$. Since we assume Ω to have a Lipschitz boundary, it can be proved that $u \in BV(\Omega)$ implies $u \in BV(\mathbb{R}^n)$ (we consider u equal to zero outside Ω). The space $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$. Moreover, the total variation is a lower semicontinuous functional with respect to the L^1-convergence, that is,

$$u_k \longrightarrow u \quad \text{in } L^1(\Omega) \Longrightarrow \|Du\|(\Omega) \leq \liminf_{k\to\infty} \|Du_k\|(\Omega). \tag{2.2}$$

A set $E \subset \mathbb{R}^n$ has *finite perimeter* (measured with respect to $\mathbb{R}^n$) if

$$\mathrm{Per}(E;\mathbb{R}^n) = \left\|D\chi_E\right\|(\mathbb{R}^n) < \infty, \tag{2.3}$$

where χ_E is the characteristic function of E. For the sake of simplicity, in the following we will set $\mathrm{Per}(E) = \mathrm{Per}(E;\mathbb{R}^n)$.

A set $C \subset \Omega$ such that

$$\frac{\mathrm{Per}(C)}{V(C)} = h_1(\Omega) = \inf_{E\subset\Omega} \frac{\mathrm{Per}(E)}{V(E)} \tag{2.4}$$

is called a *Cheeger set* for Ω. The existence of a Cheeger set for every domain Ω follows easily from the coarea formula and from the fact that actually

$$h_1(\Omega) = \inf_{u\in BV(\Omega)\setminus\{0\}} \frac{\|Du\|(\mathbb{R}^n)}{\|u\|_1}. \tag{2.5}$$

The following proposition is a useful approximation result, whose proof can be found in [4].

Proposition 2.1. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then there exists a sequence of sets of finite perimeter $\{E_k\}_{k=1}^{+\infty}$ such that:*

(1) ∂E_k *is* smooth for every k;

(2) $E_k \subset\subset E$ for every k;

(3) $\chi_{E_k} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to +\infty$;

(4) $\mathrm{Per}(\mathrm{E_k}) \to \mathrm{Per}(\mathrm{E})$ as $k \to +\infty$.

Proposition 2.2. *The following equalities hold:*

$$\inf_{E\subset\Omega} \frac{\mathrm{Per(E)}}{V(E)} = \inf_{E\subset\subset\Omega} \frac{\mathrm{Per(E)}}{V(E)} = \inf_{\substack{E\subset\subset\Omega \\ \partial E \text{ smooth}}} \frac{\mathrm{Per(E)}}{V(E)}. \tag{2.6}$$

Proof. It is clear that

$$\inf_{E\subset\Omega} \frac{\mathrm{Per}(E)}{V(E)} \le \inf_{E\subset\subset\Omega} \frac{\mathrm{Per}(E)}{V(E)} \le \inf_{\substack{E\subset\subset\Omega \\ \partial E \text{ smooth}}} \frac{\mathrm{Per}(E)}{V(E)}. \tag{2.7}$$

Let F be a Cheeger set for Ω; by Proposition 2.1 we can approximate F with a sequence of smooth sets $F_k \subset\subset F$ such that $\mathrm{Per}(F_k) \to \mathrm{Per}(F)$ and $V(F_k) \to V(F)$. This yields

$$\inf_{\substack{E\subset\subset\Omega \\ \partial E \text{ smooth}}} \frac{\mathrm{Per}(E)}{V(E)} \le \frac{\mathrm{Per}(F)}{V(F)} = \inf_{E\subset\Omega} \frac{\mathrm{Per}(E)}{V(E)}, \tag{2.8}$$

so that the claim is proved. □

In the following we will mention some geometric properties of Cheeger sets.

Proposition 2.3. *Let E be a Cheeger set for Ω; then $\partial E \cap \partial\Omega \neq \emptyset$.*

Proof. Let us suppose that this is not the case. Then E is compactly contained in Ω, which means that there exists a number $\lambda > 1$ such that the set $\lambda E = \{\lambda x \mid x \in E\}$ is contained in Ω. But then

$$\frac{\mathrm{Per}(\lambda E)}{V(\lambda E)} = \frac{1}{\lambda}\frac{\mathrm{Per}(E)}{V(E)} < \frac{\mathrm{Per}(E)}{V(E)} \tag{2.9}$$

which contradicts the fact that E is a Cheeger set. □

Proposition 2.4. *Let E be a Cheeger set for Ω; then*

(1) *$\partial E \cap \Omega$ is analytical, up to a singular set of Hausdorff dimension $n - 8$;*

(2) *the mean curvature in every regular point of $\partial E \cap \Omega$ is equal to $h_1(\Omega)$;*

(3) *let $x \in \partial E \cap \partial\Omega$ be a regular point for $\partial\Omega$; then x is a regular point for ∂E.*

Proof. The proof can be found in [5]. As a consequence, the boundary of E must touch the boundary of Ω tangentially. □

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Then there exists an unique Cheeger set E for Ω. Moreover, E is convex.*

Proof. A proof of the existence of a convex Cheeger set can be found in [2, Remark 10]. Uniqueness has been proved in [6] for the case $n = 2$, and in [7] for general n. □

Remark 2.6. If $n = 2$ and Ω is convex, then according to [6] the Cheeger set is the union of balls of suitable radius contained in Ω (where "suitable" means in this case equal to $h_1(\Omega)^{-1}$). It seems that the hypothesis of convexity cannot be dropped; there are examples of star-shaped domains which admit infinitely many Cheeger sets (see [8]). However, it was proved that "almost all" bounded domains admit a unique Cheeger set (see [9]).

We will often make use of the following property.

Proposition 2.7. *Let $\Omega \subset \mathbb{R}^n$, and let $B \subset \mathbb{R}^n$ be a ball such that $|B| = |\Omega|$. Then*

$$h_1(B) \leq h_1(\Omega). \tag{2.10}$$

Proof. The proof is a consequence of the well-known isoperimetric property of the ball (see e.g., [10]). □

Remark 2.8. There are some domains whose Cheeger set coincides with the whole Ω; this is of course the case of balls, but also of annuli and other domains satisfying a condition on the curvature of the boundary (see [6]).

3. Higher Cheeger Constants

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary. We define, for $k \in \mathbb{N}$,

$$h_k(\Omega) = \inf\left\{ \lambda \in \mathbb{R}^+ \mid \exists E_1, \dots, E_k \subset \Omega,\ E_i \cap E_j = \emptyset \text{ for } i \neq j,\ \max_{i=1,\dots,k} \frac{\mathrm{Per}(E_i)}{V(E_i)} \leq \lambda \right\}, \tag{3.1}$$

with the convention that

$$\frac{\mathrm{Per}(E)}{V(E)} = +\infty \tag{3.2}$$

whenever $V(E) = 0$. We will call $h_k(\Omega)$ the kth *Cheeger constant* for Ω. Notice that, for $k = 1$, we recover the definition of the Cheeger constant $h_1(\Omega)$. By Proposition 2.1 it is possible to take the infimum on sets compactly contained in Ω, or even on sets compactly contained in Ω with smooth boundary.

Theorem 3.1. *For every k, there exist k pairwise disjoint subsets $E_1, \dots, E_k$ contained in Ω such that*

$$\max_{i=1,\dots,k} \frac{Per(E_i)}{V(E_i)} \leq h_k(\Omega). \tag{3.3}$$

Proof. Let us consider minimizing sequences of pairwise disjoint sets $E_{1,n}, \dots, E_{k,n}$ for $n = 1, 2, \dots$, corresponding to the value μ_n, where

$$\mu_n = \max_{i=1,\dots,k} \frac{\mathrm{Per}(E_{i,n})}{V(E_{i,n})}. \tag{3.4}$$

Set $\chi_{i,n} = \chi_{E_{i,n}}$ for $i = 1, \dots, k$. Fix R as the radius of k equal disjoint balls of fixed arbitrary volume $V_0 > 0$ contained in Ω. We are going to show that we can consider $V(E_{i,n}) \geq V_0$ for every i, n. Indeed, if we had $V(E_{\hat{i},\hat{n}}) < V_0$ for some values of $\hat{i}$ and $\hat{n}$, then by Proposition 2.7 we would surely have

$$\frac{\mathrm{Per}\left(E_{\hat{i},\hat{n}}\right)}{V\left(E_{\hat{i},\hat{n}}\right)} \geq h_1(B_r), \tag{3.5}$$

where B_r is a ball with the same volume as $V(E_{\hat{i},\hat{n}})$ and so of radius $r < R$. As a consequence, $\mu_{\hat{n}} > h_1(B_R)$, which means that we can actually discard the k-tuple of sets $E_{1,\hat{n}}, \dots, E_{k,\hat{n}}$. Because of the compact embedding of $BV(\Omega)$ in $L^1(\Omega)$, there exist $E_1, \dots, E_k$ such that, up to a subsequence, $\chi_{i,n} \to \chi_{E_i}$ almost everywhere on Ω. Moreover, $V(E_i) \geq V_0 > 0$. Denote with N the negligible set of nonconvergence. From the lower semicontinuity of the total variation, it follows that

$$\frac{\mathrm{Per}(E_i)}{V(E_i)} \leq h_k(\Omega) \tag{3.6}$$

for every $i = 1, \ldots, k$. We are going to show that the E_i are pairwise disjoint: suppose $i \neq j$, then $x \in E_i \setminus N \Rightarrow \chi_{E_i}(x) = 1$, which implies $\chi_{i,n}(x) = 1$ definitely. This means $\chi_{j,n}(x) = 0$ definitely, hence $\chi_{E_j}(x) = 0$, that is, $x \notin E_j \setminus N$. If $x \in N$, we can assign arbitrary values to the characteristic functions (this does not affect the total variation). Hence we obtain the claim. □

Definition 3.2. Any k-tuple of sets $E_1, \ldots, E_k$ as in Theorem 3.1 will be called a k-tuple of *multiple Cheeger sets*. If $k = 2$, we will also speak of *coupled Cheeger sets*.

Remark 3.3. The proof of the theorem shows that we can always consider a minimizing sequence of k-tuples of sets for $h_k(\Omega)$, where the volumes of the sets are uniformly bounded from below.

Remark 3.4. Proceeding as in Proposition 2.3, one can show that at least one of the minimizing sets must touch the boundary.

In the following we will give a different characterization of the higher Cheeger constants.

Proposition 3.5. *Let $\mathcal{P}_k$ be the set of all partitions of Ω with k subsets $E_1, \ldots, E_k$. Then*

$$h_k(\Omega) = \inf_{\mathcal{P}_k} \max_{i=1,\ldots,k} h_1(E_i). \tag{3.7}$$

Proof. Set $\widehat{h}_k(\Omega) = \inf_{\mathcal{P}_k} \max_{i=1,\ldots,k} h_1(E_i)$. Let us suppose $\widehat{h}_k(\Omega) < h_k(\Omega)$; then there exists a partition $E_1, \ldots, E_k$ of Ω such that

$$\max_{i=1,\ldots,k} h_1(E_i) < h_k(\Omega) \tag{3.8}$$

which is a contradiction. Thus $\widehat{h}_k(\Omega) \geq h_k(\Omega)$. On the other hand, if $C_1, \ldots, C_k$ are a k-tuple of multiple Cheeger sets (which exist by Theorem 3.1), we can find a partition $E_1, \ldots, E_k$ of Ω with the property that $C_i \subset E_i$ for every $i = 1, \ldots, k$. Hence, for every i,

$$h_1(E_i) \leq \frac{\mathrm{Per}(C_i)}{V(C_i)} \leq h_k(\Omega), \tag{3.9}$$

and consequently

$$\max_{i=1,\ldots,k} h_1(E_i) \leq h_k(\Omega), \tag{3.10}$$

that is,

$$\widehat{h}_k(\Omega) \leq h_k(\Omega) \tag{3.11}$$

which finally yields

$$\widehat{h}_k(\Omega) = h_k(\Omega). \tag{3.12}$$

□

Remark 3.6. The sets realizing $h_k(\Omega)$ can be supposed to be connected. Indeed, if $E = E_1 \cup E_2$, with $\overline{E}_1 \cap \overline{E}_2 = \emptyset$, we have

$$\frac{\mathrm{Per}(E)}{V(E)} = \frac{\mathrm{Per}(E_1) + \mathrm{Per}(E_2)}{V(E_1) + V(E_2)} \geq \min\left\{\frac{\mathrm{Per}(E_1)}{V(E_1)}, \frac{\mathrm{Per}(E_2)}{V(E_2)}\right\}. \tag{3.13}$$

This follows from the fact that, for any $a, b, c, d > 0$,

$$\frac{a}{c} \leq \frac{a+b}{c+d} \leq \frac{b}{d} \Longleftrightarrow \frac{a}{c} \leq \frac{b}{d}. \tag{3.14}$$

So one connected component of E has a lower or equal ratio perimeter/area. If $E_1 \cap E_2 = \emptyset$, but $\overline{E}_1 \cap \overline{E}_2 \neq \emptyset$, we modify E on a set of measure zero (this does not affect the total variation) to get a connected set E' defined as

$$E' = E_1 \cup E_2 \cup (\partial E_1 \cap \partial E_2). \tag{3.15}$$

Theorem 3.7. *It is possible to find multiple Cheeger sets such that the part of their boundary contained in Ω is a piecewise smooth hypersurface of piecewise constant mean curvature.*

Proof. We will give the proof for the case $k = 2$. Let E_1 and E_2 be two coupled Cheeger sets, which exist according to Theorem 3.1. Since E_1 minimizes perimeter (measured in $\mathbb{R}^n$) in $\Omega \setminus \overline{E_2}$ with a volume constraint, it will have interior regularity according to [5]. More precisely, $\partial E_1 \cap (\Omega \setminus \overline{E_2})$ is an analytic hypersurface up to a singular set with Hausdorff dimension $n-8$, whose regular points have constant mean curvature. The same can be stated for E_2. Then we have to consider the possibly nonempty contact surface; also in this case, [5, Theorem 2] can be applied to state that the contact surface (if it exists) enjoys the same regularity as the interior boundary of the two sets and has constant mean curvature. □

Definition 3.8. Let E_1 and E_2 be a pair of coupled Cheeger sets. The *free boundary* of E_1 is defined as $\partial E_1 \cap (\Omega \setminus \overline{E_2})$ (analogously for E_2). The *contact surface* between E_1 and E_2 is $\partial E_1 \cap \partial E_2 \cap \Omega$.

Theorem 3.9. *It is possible to find two coupled Cheeger sets E_1 and E_2 such that the following holds. Suppose that $\partial E_1 \cap \partial E_2 \neq \emptyset$. Let us denote by c_1 the mean curvature of the free boundary of E_1, by c_2 the mean curvature of the free boundary of E_2, and by c_3 the mean curvature of the contact surface, measured from E_1. Then the relation*

$$c_1 - c_2 - 2c_3 = 0 \tag{3.16}$$

holds.

Proof. We follow [11, pages 10-11]. Take $x_1 \in (\partial E_1 \setminus \partial E_2) \cap \Omega$, $x_2 \in (\partial E_2 \setminus \partial E_1) \cap \Omega$, and $x_3 \in \partial E_1 \cap \partial E_2 \cap \Omega$. Suppose that the boundaries of E_1 and E_2 can be locally described by the graph of a function u defined in an open subset $\omega = \omega_1 \cup \omega_2 \cup \omega_3$ of $\mathbb{R}^{n-1}$, where ω_1, ω_2, and ω_3 are disjoint open neighborhoods of x_1, x_2, and x_3, respectively. For $i = 1, 2, 3$, let v_i be a function defined in ω_i with compact support and such that the following conditions are satisfied:

$$\begin{aligned} \int_{\omega_1} v_1 + \int_{\omega_3} v_3 &= 0, \\ \int_{\omega_2} v_2 - \int_{\omega_3} v_3 &= 0. \end{aligned} \tag{3.17}$$

Since E_1 and E_2 are coupled Cheeger sets, we can suppose that u is such that

$$\begin{aligned} \int_{\omega_1 \cup \omega_3} \sqrt{1 + |\nabla u|^2} &\leq \int_{\omega_1 \cup \omega_3} \sqrt{1 + |\nabla u + \varepsilon \nabla (v_1 + v_3)|^2}, \\ \int_{\omega_2 \cup \omega_3} \sqrt{1 + |\nabla u|^2} &\leq \int_{\omega_2 \cup \omega_3} \sqrt{1 + |\nabla u + \varepsilon \nabla (v_2 + v_3)|^2} \end{aligned} \tag{3.18}$$

for small $\varepsilon > 0$. It follows that

$$\begin{aligned} 0 &\leq \int_{\omega_1} \frac{\nabla u \nabla v_1}{\sqrt{1 + |\nabla u|^2}} + \int_{\omega_2} \frac{\nabla u \nabla v_2}{\sqrt{1 + |\nabla u|^2}} + 2 \int_{\omega_3} \frac{\nabla u \nabla v_3}{\sqrt{1 + |\nabla u|^2}} \\ &= -\int_{\omega_1} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v_1 - \int_{\omega_2} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v_2 - 2 \int_{\omega_3} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v_3 \\ &= -c_1 \int_{\omega_1} v_1 - c_2 \int_{\omega_2} v_2 - 2c_3 \int_{\omega_3} v_3. \end{aligned} \tag{3.19}$$

Since also the functions $-v_1$, $-v_2$, and $-v_3$ are admissible, it follows that

$$c_1 \int_{\omega_1} v_1 + c_2 \int_{\omega_2} v_2 + 2c_3 \int_{\omega_3} v_3 = 0 \tag{3.20}$$

for arbitrary v_1, v_2, v_3 satisfying conditions (3.17); hence we obtain

$$c_1 - c_2 - 2c_3 = 0. \tag{3.21}$$

□

Remark 3.10. The condition on the mean curvat

ures is similar to the one given in [12] for the double bubble problem: find two regions in $\mathbb{R}^n$ which enclose two given amounts of volume, such that they minimize the sum of the

surface measures. However, in that problem the quantity to minimize is slightly different, so also the condition on the mean curvatures differs and reads $c_1 - c_2 - c_3 = 0$.

Proposition 3.11. *Let $\Omega \subset \mathbb{R}^2$ be a convex planar domain; then it is possible to find two coupled Cheeger sets E_1, E_2 such that they satisfy condition* (3.16) *in Theorem 3.9 and such that, if $\partial E_1 \cap \partial E_2 \neq \emptyset$, then their boundaries meet tangentially.*

Proof. We can suppose that $c_1, c_2 \geq 0$; otherwise, since Ω is convex, it would be possible to modify the sets suitably in order to decrease their perimeter and increase their volume. As a consequence, at least one of the two sets (say E_1) is convex. Let us suppose that ∂E_1 and ∂E_2 meet each other in a nonsmooth way. Then one could consider the Cheeger set C_1 of E_1, which is convex and has a $\mathcal{C}^1$ boundary, and then find a perimeter-minimizing set C_2 in $\Omega \setminus \overline{C}_1$ under the volume constraint $|C_2| = |E_2|$. The boundaries ∂C_1 and ∂C_2 will meet tangentially as proved in [5]. Then one can apply again Theorem 3.9 to get the condition on the curvatures. □

Proposition 3.12. *Let $\Omega \subset \mathbb{R}^n$ admit a unique Cheeger set. Then*

$$h_1(\Omega) < h_2(\Omega). \tag{3.22}$$

Proof. Let us suppose that $h_1(\Omega) = h_2(\Omega)$; then there exist two disjoint subsets $C_1, C_2 \subset \Omega$ such that

$$\max\left\{\frac{\mathrm{Per}(C_1)}{V(C_1)}, \frac{\mathrm{Per}(C_2)}{V(C_2)}\right\} = h_1(\Omega) \tag{3.23}$$

which means, by definition of $h_1(\Omega)$,

$$\frac{\mathrm{Per}(C_1)}{V(C_1)} = \frac{\mathrm{Per}(C_2)}{V(C_2)} = h_1(\Omega). \tag{3.24}$$

This is a contradiction to the uniqueness of the Cheeger set for Ω. □

Remark 3.13. It is worth noting that there exist nonconvex domains for which $h_1(\Omega) = h_2(\Omega)$; think, for example, of a "barbell domain" made of two identical rectangles connected by a thin rope. For instance, we could consider the planar set

$$\Omega = \{(0,1) \times (0,2)\} \cup \{[1,2] \times (0,\varepsilon)\} \cup \{(2,3) \times (0,2)\}, \tag{3.25}$$

where $\varepsilon > 0$ is small enough.

Proposition 3.14. *Let us denote by ω_n the volume of the unit ball in $\mathbb{R}^n$. Then*

$$h_k(\Omega) \geq n\left(\frac{k\omega_n}{|\Omega|}\right)^{1/n}. \tag{3.26}$$

Proof. Let $E_1, \ldots, E_k$ be a family of multiple Cheeger sets, so that

$$\max_{i=1,\ldots,k} h_1(E_i) \leq h_k(\Omega). \tag{3.27}$$

The volume of each E_i cannot be smaller than the volume of a ball with Cheeger constant $h_k(\Omega)$, which is exactly $\omega_n(n/h_k(\Omega))^n$. In fact, let $\widetilde{B}$ be a ball such that $|E_i| = |\widetilde{B}|$, and B a ball such that $h_1(B) = h_k(\Omega)$; if $|\widetilde{B}| < |B|$ we would have, applying Proposition 2.7,

$$h_k(\Omega) = h_1(B) < h_1\left(\widetilde{B}\right) \leq h_1(E_i) \leq h_k(\Omega) \tag{3.28}$$

which is a contradiction. So we obtain

$$k\omega_n\left(\frac{n}{h_k(\Omega)}\right)^n \leq |\Omega| \Longrightarrow h_k(\Omega) \geq n\left(\frac{k\omega_n}{|\Omega|}\right)^{1/n}. \tag{3.29}$$

□

Corollary 3.15. *It holds*

$$h_k(\Omega) \longrightarrow +\infty \tag{3.30}$$

as $k \to +\infty$.

Remark 3.16. The lower bound in Proposition 3.14 for $k = 1$ follows directly from Proposition 2.7, and is obviously optimal if Ω is a ball. For the higher Cheeger constants, it can be easily seen that the estimate is optimal for the union of k balls with equal radii. If we try to minimize $h_k(\Omega)$ among connected sets, it turns out that the infimum is the same (consider a family of k balls of equal radii connected by thin strips whose width goes to 0). An interesting question would be to minimize $h_k(\Omega)$ among convex sets. If we focus on $h_2(\Omega)$, it seems that a stadium (the convex hull of two tangent balls with both radii equal to R) is very near to be a minimizer; namely, it is possible to show that

$$\frac{1.874}{R} \leq h_2(\Omega) \leq \frac{1.912}{R}. \tag{3.31}$$

The lower bound follows directly from Proposition 3.14. To obtain the upper bound, one can note that the common tangent divides Ω into two equal convex halves, whose Cheeger set E is given by the union of balls of constant radius $x \leq R$. E satisfies then the conditions

$$\begin{aligned} \operatorname{Per}(E) &= 4R + \pi R - 4x + \pi x, \\ V(E) &= \frac{1}{2}\pi R^2 + 2R^2 - 2x^2 + \frac{1}{2}\pi x^2, \end{aligned} \tag{3.32}$$

and since it must be

$$\frac{\operatorname{Per}(E)}{V(E)} = \frac{1}{x}, \tag{3.33}$$

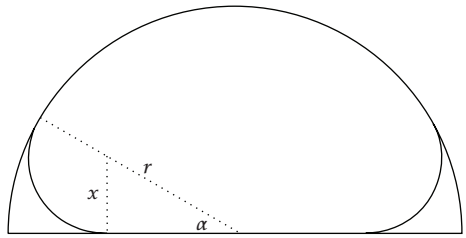

Figure 1: The candidate Cheeger set for a half-disc.

we get $x = 0.523R$. This yields the estimate from above. However, it should be mentioned that the stadium does not minimize the second eigenvalue of the Laplacian among convex planar domains, as proved in [13].

4. Coupled Cheeger Sets for a Disc

In this section we will determine the coupled Cheeger sets of a disc $\Omega \subset \mathbb{R}^2$ with radius r. As a first step we will compute the Cheeger set E for a half-disc Ω' of same radius. According to the results in Section 2, the Cheeger set must have the geometry shown in Figure 1.

We will denote by α the inner angle and by x the radius of the inner arc. Thus we have the relation

$$(r - x)\sin\alpha = x \tag{4.1}$$

which gives the existence condition $0 \le x \le r/2$. Then

$$\begin{aligned} \operatorname{Per}(E) &= 2(r - x)\cos\alpha + 2x\left(\frac{\pi}{2} + \alpha\right) + r(\pi - 2\alpha), \\ V(E) &= x(r - x)\cos\alpha + x^2\left(\frac{\pi}{2} + \alpha\right) + \frac{r^2}{2}(\pi - 2\alpha). \end{aligned} \tag{4.2}$$

Remember that $\alpha = \arcsin(x/(r - x))$ and $\cos\alpha = \sqrt{1 - (x/(r - x))^2}$, since we consider $0 \le \alpha \le \pi/2$. Numerical resolution of the equation

$$\frac{\operatorname{Per}(E)}{V(E)} = \frac{1}{x}\left(= \text{possible } h_1(\Omega')\right) \tag{4.3}$$

gives, for $r = 1$,

$$x = 0.317028\ldots \tag{4.4}$$

which means

$$h_1(\Omega') = 3.15429\ldots \tag{4.5}$$

This is the best configuration with convex subsets to compute $h_2(\Omega)$; indeed, a convex partition of a convex set can be obtained only cutting the set with hyperplanes (otherwise we would have a point of nonzero curvature which gives convexity from one side but concavity from the other one). The Cheeger sets of each of the two partitioning subsets are then convex. Conversely, two convex subsets can be separated by a hyperplane thanks to the Hahn-Banach theorem. The Cheeger constant of a circular segment strictly contained in a half-disc is then strictly higher, due to uniqueness reasons. So the above configuration is the best among convex subsets of the disc.

Observe that the two coupled Cheeger sets E_1 and E_2 must have a contact surface. If it was not the case, we can suppose without loss of generality that

$$\frac{\mathrm{Per}(E_1)}{V(E_1)} \leq \frac{\mathrm{Per}(E_2)}{V(E_2)}, \tag{4.6}$$

and that E_1 is a Cheeger set for $\Omega \setminus E_2$. Notice that E_2 is automatically a Cheeger set for $\Omega \setminus E_1$. Due to the properties of Cheeger sets, the free boundaries of E_1 and E_2 must be circular arcs which meet $\partial\Omega$ tangentially. The only possibility is that E_1 and E_2 are discs, and the best configuration is given by to equal discs with radius $r/2$, which is clearly not optimal for Ω.

We are now going to prove that the contact surface cannot be closed; if it was the case, then one of the two coupled Cheeger sets, say E_1, would be a disc of radius $r' < r$, as in Figure 2. The other set E_2 will be then contained in $\Omega \setminus E_1$. Suppose that E_2 has a free boundary consisting of arcs with constant curvature $c_2 \geq 0$. An easy computation shows that the case $c_2 = 0$ is never optimal; so we can suppose that the arcs have constant curvature $c_2 > 0$. Due to the fact that ∂E_1 is the contact surface, these arcs cannot start on $\partial\Omega$ and end on ∂E_1; the only possibility is that the free boundary "encloses" E_1 as the dashed line in Figure 2. But in this case, the choice $E_2 = \Omega \setminus E_1$ would give a lower ratio perimeter/area. So the optimal choice is the pair consisting of E_1 and its complement. By modifying r' suitably, one can easily convince himself that the optimal configuration is achieved when the ratios perimeter/area of E_1 and E_2 are equal. This implies

$$\frac{\mathrm{Per}(E_1)}{V(E_1)} = \frac{\mathrm{Per}(E_2)}{V(E_2)} \Longrightarrow \frac{2}{r'} = \frac{2}{r - r'} \Longrightarrow r' = \frac{r}{2} \tag{4.7}$$

which yields, for $r = 1$,

$$h_1(E_1) = h_1(E_2) = 4. \tag{4.8}$$

This gives a worse configuration than the one found before. As a consequence, the contact surface cannot be a closed line.

We will now use the regularity results about the coupled Cheeger sets; in particular, by Proposition 3.11 we can suppose that the boundary of each of the two sets meets the boundary of the other set tangentially. Suppose that the separating surface is an arc PQ with constant curvature c_3. From the point P two arcs of curvature c_1 and c_2, respectively, will depart, in such a way that the centres of curvature lie on the chord AB orthogonal to PQ and such that $P \in AB$. Notice that we can suppose, without loss of generality, that $c_1, c_2 \geq 0$.

Let E_1 be the "candidate" Cheeger set containing the segment AP and such that the curvature of its free boundary is c_1; let E_2 be the set containing the segment PB and with

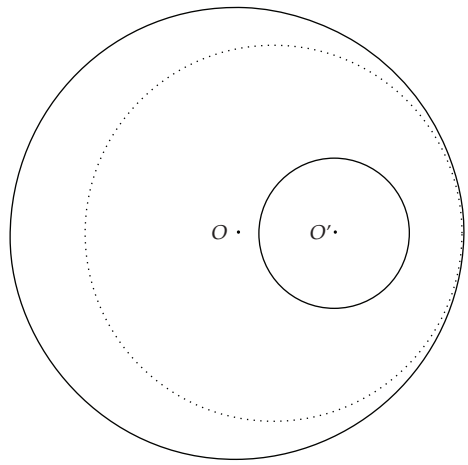

Figure 2: The contact surface cannot be closed.

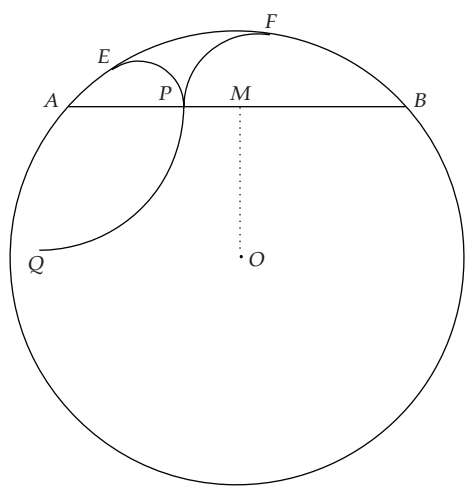

Figure 3: The case $c_3 > 0$.

curvature of the free boundary equal to c_2. Without loss of generality, we can suppose that $\overline{AP} \leq \overline{PB}$. Let M be the middle point of the segment AB. If $P \neq M$, it is impossible that $c_3 \geq 0$ (as in Figure 3); indeed, since E_1 would be a subset of a circular segment strictly contained in a half-disc, this would contradict the fact that the configuration of the Cheeger sets of the two half-discs is better. So it must be $c_3 < 0$.

Let C and D the centers of curvature of the free boundaries of E_1 and E_2, respectively, and E, F as in Figure 4 such that $\overline{CP} = \overline{EC}$ and $\overline{PD} = \overline{DF}$. Since $c_3 < 0$, from Theorem 3.9 it must be $c_1 < c_2$, that is $\overline{PC} > \overline{PD}$. This is impossible for geometrical reasons; indeed, take a point C' on AB such that $\overline{AC} = \overline{C'B}$; it follows $\overline{PC'} > \overline{PC} > \overline{PD}$, which means that the point D must lie between P and C'. If E' is the intersection of the circle with the line OC', it is clear that $\overline{DF} > \overline{C'E'}$. This is a contradiction because we would have $\overline{C'E'} = \overline{CE} > \overline{DF} > \overline{C'E'}$.

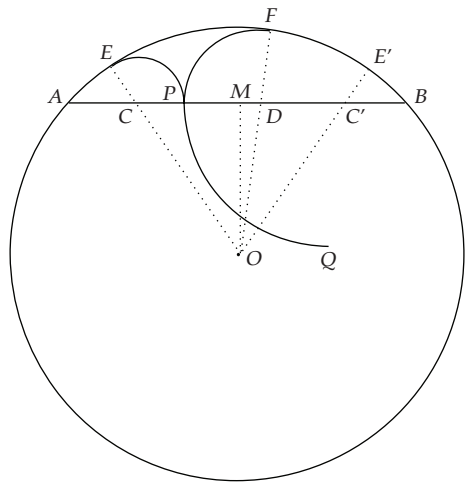

Figure 4: The case $c_3 < 0$.

It follows that necessarily $P = M$. For symmetry reasons, this implies $c_1 = c_2$ and hence, again from Theorem 3.9, $c_3 = 0$. So we recover the optimal configuration consisting of the Cheeger sets of the two half-discs.

5. The Second Eigenvalue as $p \to 1$

Let us consider now the eigenvalue problem (1.1). The natural question which arises is how one can find eigenvalues of the p-Laplacian. A possibility is to use the direct method of Calculus of Variations by minimizing the so-called *Rayleigh quotient*; working this way we obtain the first eigenvalue

$$\lambda_1(p;\Omega) = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{5.1}$$

One can prove (see e.g., [14]) that there exists, up to a nonzero multiplicative constant, one and only one eigenfunction $e_{1,p}$ associated to $\lambda_1(p;\Omega)$. Moreover, $e_{1,p}$ is of only one sign.

We will now describe how higher eigenvalues of the p-Laplacian can be obtained. Let $A \subset W_0^{1,p}(\Omega)$ be a closed, symmetric subset. The *Krasnoselskii genus* $\gamma(A)$ of A is defined as

$$\gamma(A) = \min\left\{m \in \mathbb{N} \mid \exists \varphi : A \longrightarrow \mathbb{R}^m \setminus \{0\},\ \varphi \text{ is continuous and odd}\right\}. \tag{5.2}$$

Let us denote by Γ_k the set

$$\Gamma_k = \left\{A \subset W_0^{1,p}(\Omega) \setminus \{0\} \mid A \cap \left\{\|u\|_p = 1\right\} \text{ is compact, } A \text{ symmetric, } \gamma(A) \geq k\right\}. \tag{5.3}$$

Then, for every $k \in \mathbb{N}$, the following numbers are eigenvalues:

$$\lambda_k(p;\Omega) = \inf_{A\in\Gamma_k} \max_{u\in A} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{5.4}$$

In the literature they are sometimes called *variational eigenvalues*. It can be easily seen that the two definitions of $\lambda_1(p;\Omega)$ given so far coincide. It is still an open question whether other eigenvalues can exist. As shown, for instance, in [15, Lemma 3.1], eigenfunctions associated to higher eigenvalues of the p-Laplacian must be sign-changing. Moreover, there does not exist any eigenvalue between $\lambda_1(p;\Omega)$ and $\lambda_2(p;\Omega)$, which means that $\lambda_1(p;\Omega)$ is isolated (see [16]).

A *nodal domain* of a function $u : \Omega \to \mathbb{R}$ is a connected component of the set $\{x \in \Omega \mid u(x) \neq 0\}$. It is not known whether the zero set of an eigenfunction of the p-Laplacian has Hausdorff dimension $n-1$, or if it can be even an open subset. A generalization of Courant's nodal domain theorem states that every eigenfunction associated to $\lambda_k(p;\Omega)$ has at most $2k-2$ nodal domains ([17, Theorem 3.3]). As an easy consequence it follows that any second eigenfunction has exactly two nodal domains.

We are now ready to prove the main result of this paper.

Lemma 5.1. *Let $E \subset \mathbb{R}^n$ be a set with Lipschitz boundary, and let E^ε be the ε-strip around E defined as*

$$E^\varepsilon = \{x \in \mathbb{R}^n \setminus E \mid \operatorname{dist}(x,E) \le \varepsilon\}. \tag{5.5}$$

Then

$$V(E^\varepsilon) = \varepsilon Per(E) + o(\varepsilon), \tag{5.6}$$

where $(o(\varepsilon))/\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. The proof can be found in [18]. □

Theorem 5.2. *It holds*

$$\limsup_{p\to 1} \lambda_2(p;\Omega) \le h_2(\Omega). \tag{5.7}$$

Proof. Let $C_1, C_2 \subset\subset \Omega$ be two subsets such that $C_1 \cap C_2 = \emptyset$, and

$$\max\left\{\frac{\operatorname{Per}(C_1)}{V(C_1)}, \frac{\operatorname{Per}(C_2)}{V(C_2)}\right\} \le h_2(\Omega) + \frac{1}{2k}. \tag{5.8}$$

By Proposition 2.1 it is possible to find E_1, E_2 with the property that, for $i = 1,2$, $E_i \subset\subset C_i$, ∂E_i is smooth, and

$$\frac{\operatorname{Per}(E_i)}{V(E_i)} \le h_2(\Omega) + \frac{1}{k}. \tag{5.9}$$

Let $\varepsilon > 0$, and let v_i $(i = 1,2)$ be two functions such that: $v_i = 1$ on E_i, $v_i = 0$ outside a ε-neighbourhood of E_i, and $|\nabla v_i| = \varepsilon^{-1}$ on the ε-layer E_i^ε outside E_i. ε should be chosen in a way that $E_1^\varepsilon \cap E_2^\varepsilon = \emptyset$. Set

$$A_0 = \{\alpha v_1 + \beta v_2 \mid |\alpha|^p + |\beta|^p = 1\}. \tag{5.10}$$

Then $A_0 \in \Gamma_2$ (see also [19, Lemma 2.1]). Thus we have

$$\begin{aligned}
\lambda_2(p;\Omega) &\le \sup_{u\in A_0} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \le \sup_{|\alpha|^p+|\beta|^p=1} \frac{\varepsilon^{-p}|\alpha|^p V(E_1^\varepsilon) + \varepsilon^{-p}|\beta|^p V(E_2^\varepsilon)}{|\alpha|^p V(E_1) + |\beta|^p V(E_2)} \\
&= \sup_{|\alpha|^p+|\beta|^p=1} \frac{\varepsilon^{1-p}|\alpha|^p \mathrm{Per}(E_1) + \varepsilon^{1-p}|\beta|^p \mathrm{Per}(E_2) + \varepsilon^{-p} o(\varepsilon)}{|\alpha|^p V(E_1) + |\beta|^p V(E_2)} \\
&\le \varepsilon^{1-p}\left(h_2(\Omega) + \frac{1}{k}\right) + \frac{\varepsilon^{-p} o(\varepsilon)}{\min\{V(E_1), V(E_2)\}},
\end{aligned} \tag{5.11}$$

as we have from Lemma 5.1

$$V(E_i^\varepsilon) = \varepsilon \mathrm{Per}(E_i) + o(\varepsilon), \tag{5.12}$$

where $(o(\varepsilon))/\varepsilon \to 0$ as $\varepsilon \to 0$. The last inequality follows from (3.14). If we send $p \to 1$, we obtain

$$\limsup_{p\to 1} \lambda_2(p;\Omega) \le h_2(\Omega) + \frac{1}{k} + \frac{\varepsilon^{-1} o(\varepsilon)}{\min\{V(E_1), V(E_2)\}}, \tag{5.13}$$

and if $\varepsilon \to 0$

$$\limsup_{p\to 1} \lambda_2(p;\Omega) \le h_2(\Omega) + \frac{1}{k}. \tag{5.14}$$

The claim follows if we send $k \to \infty$. The fact that E_1 and E_2 depend on k does not constitute a problem, since we can estimate $V(E_i)$ uniformly from below (see Remark 3.3). □

Remark 5.3. The theorem can be easily generalized to the kth variational eigenvalue obtaining

$$\limsup_{p\to 1} \lambda_k(p;\Omega) \le h_k(\Omega). \tag{5.15}$$

The so-called *Cheeger's inequality*, whose proof can be found in [20], yields the following lower bound for the first eigenvalue:

$$\lambda_1(p;\Omega) \ge \left(\frac{h_1(\Omega)}{p}\right)^p. \tag{5.16}$$

In the following theorem we show that a similar estimate holds also for the second eigenvalue.

Theorem 5.4. *The following Cheeger-type inequality holds:*

$$\lambda_2(p;\Omega) \geq \left(\frac{h_2(\Omega)}{p}\right)^p. \tag{5.17}$$

Proof. Let $e_{2,p}$ be a second eigenfunction of the p-Laplacian. From [17, Theorem 3.3], we know that $e_{2,p}$ has exactly two nodal domains $N_{1,p}$, $N_{2,p}$. $e_{2,p}$ is also a first eigenfunction on each of the two nodal domains; from Cheeger's inequality it follows, for $i = 1, 2$,

$$\lambda_2(p;\Omega) = \lambda_1(p;N_{i,p}) \geq \left(\frac{h_1(N_{i,p})}{p}\right)^p. \tag{5.18}$$

But as $N_{1,p} \cap N_{2,p} = \emptyset$, we have

$$\max\{h_1(N_{1,p}), h_1(N_{2,p})\} \geq h_2(\Omega) \tag{5.19}$$

due to the definition of $h_2(\Omega)$. So we obtain the claim. □

Theorem 5.5. *It holds*

$$\lim_{p\to 1}\lambda_2(p;\Omega) = h_2(\Omega). \tag{5.20}$$

Proof. The claim follows easily from Theorems 5.2 and 5.4. □

The Second Eigenfunction as $p \to 1$

In the following we will focus on the asymptotic behaviour of the second eigenfunctions as $p \to 1$.

Theorem 5.6. *Let $N_{1,p}$, $N_{2,p}$ be the nodal domains of a second eigenfunction of the p-Laplacian. Then*

$$\lim_{p\to 1}\max\{h_1(N_{1,p}), h_1(N_{2,p})\} = h_2(\Omega). \tag{5.21}$$

Proof. By definition of $h_2(\Omega)$ we have

$$h_2(\Omega) \leq \max\{h_1(N_{1,p}), h_1(N_{2,p})\}. \tag{5.22}$$

It remains to prove that for every $\varepsilon > 0$, there exists $p_0 > 1$ such that for every $1 < p < p_0$,

$$\max\{h_1(N_{1,p}), h_1(N_{2,p})\} \leq h_2(\Omega) + \varepsilon. \tag{5.23}$$

Suppose that this is not the case; then there exists $\varepsilon > 0$ such that, without loss of generality, $h_1(N_{1,p_k}) > h_2(\Omega) + \varepsilon$ for a subsequence $p_k \to 1$. From Cheeger's inequality

$$\lambda_2(p_k) \geq \left(\frac{h_1(N_{1,p_k})}{p_k}\right)^{p_k} > \left(\frac{h_2(\Omega) + \varepsilon}{p_k}\right)^{p_k} > h_2(\Omega) + \frac{\varepsilon}{2} \tag{5.24}$$

for k large enough. But this contradicts the fact that $\lim_{p\to 1}\lambda_2(p;\Omega) = h_2(\Omega)$. Hence the claim follows. □

Corollary 5.7. *For $p \to 1$, the volume of each of the nodal sets is uniformly bounded from below by $\omega_n(n/2h_2(\Omega))^n$.*

Proof. From the preceding theorem there exists $p_0 > 1$ such that, for every $1 < p < p_0$,

$$\max\{h_1(N_{1,p}), h_1(N_{2,p})\} \leq 2h_2(\Omega). \tag{5.25}$$

Arguing as in Proposition 3.14, the volume of the nodal sets cannot be smaller than the volume of a ball with Cheeger constant $2h_2(\Omega)$, which is exactly $\omega_n(n/2h_2(\Omega))^n$. Thus, for $i = 1, 2$,

$$|N_{i,p}| \geq |B| = \omega_n\left(\frac{n}{2h_2(\Omega)}\right)^n \tag{5.26}$$

as claimed. □

We are now going to investigate the asymptotic behaviour of the second eigenfunctions as $p \to 1$. First, we state some technical lemmas.

Lemma 5.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz boundary, $p_j \to 1$ as $j \to \infty$ ($p_j \geq 1$), $u_j \in W_0^{1,p_j}(\Omega)$ for every j, $u_j \to u$ in $L^1(\Omega)$ as $j \to \infty$. Then*

$$\|Du\|(\mathbb{R}^n) \leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j}. \tag{5.27}$$

Proof. Since $\partial\Omega$ is Lipschitz, the functions u_j are in particular in $BV(\mathbb{R}^n)$. Let us denote by p_j' the exponent conjugate to p_j; by (2.2), Hölder's inequality and Young's inequality we have

$$\begin{aligned}
\|Du\|(\mathbb{R}^n) &\leq \liminf_{j\to\infty} \|Du_j\|(\mathbb{R}^n) = \liminf_{j\to\infty} \int_\Omega |\nabla u_j| \\
&\leq \liminf_{j\to\infty} \left(\int_\Omega |\nabla u_j|^{p_j} \right)^{1/p_j} |\Omega|^{1/p_j'} \\
&\leq \liminf_{j\to\infty} \left[\left(\int_\Omega |\nabla u_j|^{p_j} \right) + |\Omega| \cdot \frac{p_j^{-p_j'/p_j}}{p_j'} \right] \\
&\leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j} + \limsup_{j\to\infty} |\Omega| \cdot \frac{p_j^{-p_j'/p_j}}{p_j'} = \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j}.
\end{aligned} \tag{5.28}$$

□

Lemma 5.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $p_j \to 1$ as $j \to \infty$ ($p_j \geq 1$), $0 < \|u_j\|_{L^\infty(\Omega)} \leq c$ for every j ($c > 0$), $u \in L^1(\Omega)$, and $u_j \to u$ in $L^1(\Omega)$ as $j \to \infty$. Then*

$$\lim_{j\to\infty} \int_\Omega |u_j|^{p_j} = \int_\Omega |u|. \tag{5.29}$$

Proof. Let us denote by p_j' the exponent conjugate to p_j. By Hölder's inequality and Young's inequality we have

$$\begin{aligned}
\int_\Omega |u| &= \lim_{j\to\infty} \int_\Omega |u_j| \leq \liminf_{j\to\infty} \left(\int_\Omega |u_j|^{p_j} \right)^{1/p_j} |\Omega|^{1/p_j'} \\
&\leq \liminf_{j\to\infty} \left[\left(\int_\Omega |u_j|^{p_j} \right) + |\Omega| \cdot \frac{p_j^{-p_j'/p_j}}{p_j'} \right] \\
&= \liminf_{j\to\infty} \int_\Omega |u_j|^{p_j} + \limsup_{j\to\infty} |\Omega| \cdot \frac{p_j^{-p_j'/p_j}}{p_j'} = \liminf_{j\to\infty} \int_\Omega |u_j|^{p_j}.
\end{aligned} \tag{5.30}$$

On the other hand from $0 < \|u_j\|_{L^\infty(\Omega)} \leq c$ and $p_j \geq 1$ we have

$$\int_\Omega \frac{|u_j|}{\|u_j\|_\infty} \geq \int_\Omega \left(\frac{|u_j|}{\|u_j\|_\infty} \right)^{p_j}, \tag{5.31}$$

so that

$$\int_\Omega |u| = \lim_{j\to\infty}\int_\Omega |u_j| \geq \limsup_{j\to\infty}\|u_j\|_\infty^{1-p_j}\cdot\int_\Omega |u_j|^{p_j} \geq \limsup_{j\to\infty}\int_\Omega |u_j|^{p_j}. \tag{5.32}$$

The last equation and (5.30) end the proof. □

Lemma 5.10. *Let $e_{2,p}$ be a second eigenfunction of the p-Laplacian. Then*

$$\|e_{2,p}\|_\infty \leq 4^n\cdot\lambda_2(p;\Omega)^{n/p}\cdot\|e_{2,p}\|_1. \tag{5.33}$$

Proof. The proof can be found in [21]. □

Theorem 5.11. *Let $e_{2,p}$ be second eigenfunctions of the p-Laplacian such that $||e_{2,p}||_p = 1$. Then (after possibly passing to a subsequence) $e_{2,p}$ converge, as $p \to 1$, in $L^1(\Omega)$ and hence pointwise a.e. to a function $u \in BV(\Omega)$ such that $||u||_1 = 1$ and $||Du||(\mathbb{R}^n) \leq h_2(\Omega)$. Moreover, u cannot be strictly positive or strictly negative.*

Proof. From Theorem 5.5, Lemma 5.10, and Hölder's inequality, $e_{2,p}$ are uniformly bounded in $L^\infty(\Omega)$. Moreover, we have

$$\|De_{2,p}\|(\mathbb{R}^n) = \int_\Omega |\nabla e_{2,p}| \leq \left(\int_\Omega |\nabla e_{2,p}|^p\right)^{1/p}|\Omega|^{1/p'} = \lambda_2(p;\Omega)^{1/p}\cdot|\Omega|^{1/p'}, \tag{5.34}$$

where p' is the exponent conjugate to p. Since $\lambda_2(p;\Omega) \to h_2(\Omega)$, the functions are uniformly bounded in $BV(\Omega)$; hence there exists a subsequence converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$. From Lemma 5.8 we have

$$\|Du\|(\mathbb{R}^n) \leq \liminf_{p\to 1}\lambda_2(p;\Omega) = h_2(\Omega). \tag{5.35}$$

Finally, Lemma 5.9 yields $||u||_1 = 1$.

The fact that u cannot be strictly positive or strictly negative is a consequence of Corollary 5.7. □

6. Nonradiality of the Second Eigenfunction in a Planar Disc

In this section we will apply the previously found results to the particular case where the domain $\Omega \subset \mathbb{R}^2$ is a disc, in order to establish whether a second eigenfunction can be radial or not. Let us recall that the existence of radial eigenfunctions was shown in [22]; in this case, one has to solve the ordinary differential equation

$$\begin{aligned} -\left(r|u'|^{p-2}u'\right)' &= \lambda r|u|^{p-2}u \quad \text{in } (0,R),\\ u'(0) &= 0,\\ u(R) &= 0, \end{aligned} \tag{6.1}$$

The question seems to be still an open problem in its full generality, except for the case $p = 2$ (see [23]) where the answer is negative. In the following theorem it is shown that the answer is also negative if p is sufficiently close to 1.

Theorem 6.1. *Let $\Omega = B_1(0) \subset \mathbb{R}^2$ be the unit disc. Then, for p close to 1, the second eigenfunction of the p-Laplacian in Ω cannot be radial.*

Proof. From the results in Section 4 and Theorem 5.5 we have

$$\lim_{p\to 1}\lambda_2(p;\Omega) = 3.15429\ldots. \tag{6.2}$$

Then there exists $p_0 > 1$ such that

$$\lambda_2(p;\Omega) \le 3.5 \tag{6.3}$$

for $1 < p < p_0$. Let us suppose that there exists $p \in (1, p_0)$ such that a second eigenfunction of the p-Laplacian is radial; this implies that its nodal domains are a disc $B_r(0)$ of radius r $(0 < r < 1)$, compactly contained in Ω, and an annulus $A = \Omega \setminus B_r(0)$. If we suppose w.l.o.g. $p < 1.1$, Cheeger's inequality allows us to state that

$$\begin{aligned}
\lambda_2(p;\Omega) &\ge \left(\frac{h_1(B_r(0))}{p}\right)^p = \left(\frac{2}{rp}\right)^p \ge \left(\frac{1.818}{r}\right)^p \ge \frac{1.818}{r},\\
\lambda_2(p;\Omega) &\ge \left(\frac{h_1(A)}{p}\right)^p = \left(\frac{2}{(1-r)p}\right)^p \ge \left(\frac{1.818}{1-r}\right)^p \ge \frac{1.818}{1-r}.
\end{aligned} \tag{6.4}$$

Indeed, since the Cheeger set of A is A itself (see Remark 2.8), one has

$$h_1(A) = \frac{\mathrm{Per}(A)}{V(A)} = \frac{2\pi(1+r)}{\pi(1-r^2)} = \frac{2}{1-r}. \tag{6.5}$$

Then we have the following compatibility conditions:

$$\begin{aligned}
\frac{1.818}{r} &\le 3.5 \Longrightarrow r \ge 0.519,\\
\frac{1.818}{1-r} &\le 3.5 \Longrightarrow 1-r \ge 0.519 \Longrightarrow r \le 0.481
\end{aligned} \tag{6.6}$$

which are incompatible. Hence we obtain the claim. □

The following image was obtained by an implementation of the numerical method described in [24].

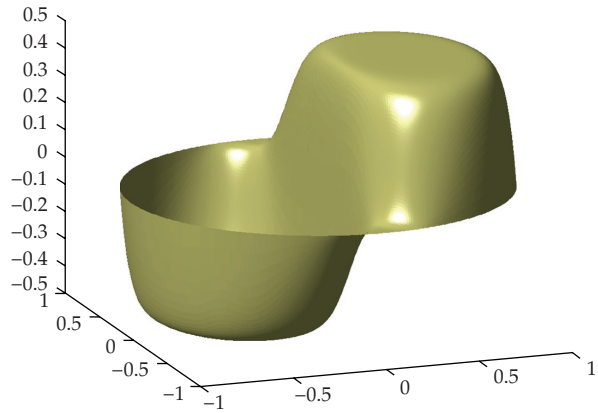

Figure 5: Second eigenfunction in a planar disc for $p = 1.1$. The value of $\lambda_2(p;\Omega)$ is about 4.199 (picture courtesy of Jiří Horák).

Acknowledgments

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