Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 274843, 21 pages
doi:10.1155/2011/274843

Research Article

Exponential Stability in Hyperbolic Thermoelastic Diffusion Problem with Second Sound

Moncef Aouadi

Département de Mathématique, Institut Supérieur des Sciences Appliquées et de Technologie de Mateur, Route de Tabarka, Mateur 7030, Tunisia

Correspondence should be addressed to Moncef Aouadi, moncef_aouadi@yahoo.fr

Received 4 May 2011; Accepted 28 June 2011

Academic Editor: Sabri Arik

We consider a thermoelastic diffusion problem in one space dimension with second sound. The thermal and diffusion disturbances are modeled by Cattaneo's law for heat and diffusion equations to remove the physical paradox of infinite propagation speed in the classical theory within Fourier's law. The system of equations in this case is a coupling of three hyperbolic equations. It poses some new analytical and mathematical difficulties. The exponential stability of the slightly damped and totally hyperbolic system is proved. Comparison with classical theory is given.

1. Introduction

The classical model for the propagation of heat turns into the well-known Fourier's law

$$q + k\nabla\theta = 0, \tag{1.1}$$

where θ is temperature (difference to a fixed constant reference temperature), q is the heat conduction vector, and k is the coefficient of thermal conductivity. The model using classic Fourier's law inhibits the physical paradox of infinite propagation speed of signals. To eliminate this paradox, a generalized thermoelasticity theory has been developed subsequently. The development of this theory was accelerated by the advent of the second sound effects observed experimentally in materials at a very low temperature. In heat transfer problems involving very short time intervals and/or very high heat fluxes, it has been revealed that the inclusion of the second sound effects to the original theory yields results which are realistic and very much different from those obtained with classic Fourier's law.

The first theory was developed by Lord and Shulman [1]. In this theory, a modified law of heat conduction, the Cattaneo's law,

$$\tau_0 q_t + q + k\nabla\theta = 0 \tag{1.2}$$

replaces the classic Fourier's law. The heat equation associated with this a hyperbolic one and, hence, automatically eliminates the paradox of infinite speeds. The positive parameter τ_0 is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature.

The development of high technologies in the years before, during, and after the second world war pronouncedly affected the investigations in which the fields of temperature and diffusion in solids cannot be neglected. The problems connected with the diffusion of matter in thermoelastic bodies and the interaction of mechanodiffusion processes have become the subject of research by many authors. At elevated and low temperatures, the processes of heat and mass transfer play a decisive role in many satellite problems, returning space vehicles, and landing on water or land. These days, oil companies are interested in the process of thermodiffusion for more efficient extraction of oil from oil deposits.

Nowacki [2] developed the classic theory of thermoelastic diffusion under Fourier's law. Sherief et al. [3] derived the theory of thermoelastic diffusion with second sound. Aouadi [4] derived the theory of micropolar thermoelastic diffusion under Cattaneo's law. Recently, Aouadi [5–7] derived the general equations of motion and constitutive equations of the linear thermoelastic diffusion theory in the context of different media, with uniqueness and existence theorems.

In recent years, a relevant task has been developed to obtain exponential stability of solutions in thermoelastic theories. The classical theory was first considered by Dafermos [8] and Slemrod [9] and it has been studied in the book of Jiang and Racke [10] and the contribution of Lebeau and Zuazua [11]. One should mention a paper by Sherief [12], where the stability of the null solution also in higher dimension is proved. We mention also the work of Tarabek [13] who studied even one dimensional nonlinear systems and obtained the strong convergence of derivatives of solutions to zero. Racke [14] proved the exponential decay of linear and nonlinear thermoelastic systems with second sound in one dimension for various boundary conditions. Messaoudi and Said-Houari [15] proved the exponential stability in one-dimensional nonlinear thermoelasticity with second sound. Soufyane [16] established an exponential and polynomial decay results of porous thermoelasticity including a memory term.

Recently, Aouadi and Soufyane [17] proved the polynomial and exponential stability for one-dimensional problem in thermoelastic diffusion theory under Fourier's law. To the author's knowledge, no work has been done regarding the exponential stability of the thermoelastic diffusion theory with second sound though similar research in thermoelasticity has been popular in recent years. This paper will devote to study the exponential stability of the solution of the one-dimensional thermoelastic diffusion theory under Cattaneo's law. The model that we consider is interesting not only because we take into account the thermal-diffusion effect, but also because Cattaneo's law is physically more realistic than Fourier's law. In this case, the governing equations corresponds to the coupling of three hyperbolic equations. This question is new in thermoelastic theories and poses new analytical and mathematical difficulties. This kind of coupling has not been considered previously, and we have a few results concerning the existence, uniqueness, and exponential decay. For this reason,

the exponential decay of the solution is very interesting and also very difficult. We obtain the exponential decay by the multiplier method and constructing generalized Lyapunov functional.

The remaining part of this paper is organized as follows: in Section 2, we give basic equations, and for completeness, we discuss the well-posedness of the initial boundary value problem in a semigroup setting. In Section 3, we derive the various energy estimates, and we state the exponential decay of the solution. In Section 4, we provide arguments for showing that the two systems, either $\tau_0 > 0,\ \tau > 0$ or $\tau_0 = \tau = 0$, are close to each other, in the sense of energy estimates, of order τ_0^2 and τ^2.

2. Basic Equations and Preliminaries

The governing equations for an isotropic, homogenous thermoelastic diffusion solid are as follows (see [3]):

(i) the equation of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{2.1}$$

(ii) the stress-strain-temperature-diffusion relation

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}\left(\lambda e_{kk} - \beta_1 \theta - \beta_2 C\right), \tag{2.2}$$

(iii) the displacement-strain relation

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2.3}$$

(iv) the energy equation

$$q_{i,i} = -\rho T_0 \dot{S}, \tag{2.4}$$

(v) the Cattaneo's law for temperature

$$-k\theta_{,i} = q_i + \tau_0 \dot{q}_i, \tag{2.5}$$

(vi) the entropy-strain-temperature-diffusion relation

$$\rho T_0 S = \beta_1 T_0 e_{kk} + \rho c_E \theta + a T_0 C, \tag{2.6}$$

(vii) the equation of conservation of mass

$$\eta_{i,i} = -\dot{C}, \tag{2.7}$$

(viii) the Cattaneo's law for chemical potential

$$-\hbar P_{,i} = \eta_i + \tau \dot{\eta}_i, \tag{2.8}$$

(ix) the chemical-strain-temperature-diffusion relation

$$P = -\beta_2 e_{kk} - a\theta + bC, \tag{2.9}$$

where $\beta_1 = (3\lambda + 2\mu)\alpha_t$ and $\beta_2 = (3\lambda + 2\mu)\alpha_c$, α_t, and α_c are, respectively, the coefficients of linear thermal and diffusion expansion and λ and μ are Lamé's constants. $\theta = T - T_0$ is small temperature increment, T is the absolute temperature of the medium, and T_0 is the reference uniform temperature of the body chosen such that $|\theta/T_0| \ll 1$. q_i is the heat conduction vector, k is the coefficient of thermal conductivity, and c_E is the specific heat at constant strain. σ_{ij} are the components of the stress tensor, u_i are the components of the displacement vector, e_{ij} are the components of the strain tensor, S is the entropy per unit mass, P is the chemical potential per unit mass, C is the concentration of the diffusive material in the elastic body, $\hbar$ is the diffusion coefficient, η_i denotes the flow of the diffusing mass vector, "a" is a measure of thermodiffusion effect, "b" is a measure of diffusive effect, and ρ is the mass density. τ_0 is the thermal relaxation time, which will ensure that the heat conduction equation will predict finite speeds of heat propagation. τ is the diffusion relaxation time, which will ensure that the equation satisfied by the concentration will also predict finite speeds of propagation of matter from one medium to the other.

We will now formulate a different alternative form that will be useful in proving uniqueness in the next section. In this new formulation, we will use the chemical potential as a state variable instead of the concentration. From (2.9), we obtain

$$C = \gamma_2 e_{kk} + nP + d\theta. \tag{2.10}$$

The alternative form can be written by substituting (2.10) into (2.1)–(2.8),

$$\begin{gathered}
\sigma_{ij,j} = \rho \ddot{u}_i, \\
\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}\left(\lambda_0 e_{kk} - \gamma_1 \theta - \gamma_2 P\right), \\
q_{i,i} = -\rho T_0 \dot{S}, \\
-k\theta_{,i} = q_i + \tau_0 \dot{q}_i, \\
\rho S = c\theta + \gamma_1 e_{kk} + dP, \\
\eta_{i,i} = -\dot{C}, \\
-\hbar P_{,i} = \eta_i + \tau \dot{\eta}_i,
\end{gathered} \tag{2.11}$$

where

$$\gamma_1 = \beta_1 + \frac{a\beta_2}{b}, \qquad \gamma_2 = \frac{\beta_2}{b}, \qquad \lambda_0 = \lambda - \frac{\beta_2^2}{b}, \qquad c = \frac{\rho c_E}{T_0} + \frac{a^2}{b}, \qquad d = \frac{a}{b}, \qquad n = \frac{1}{b} \tag{2.12}$$

are physical positive constants satisfying the following condition:

$$cn - d^2 > 0. \tag{2.13}$$

Note that this condition implies that

$$c\theta^2 + 2d\theta P + nP^2 > 0. \tag{2.14}$$

Condition (2.13) is needed to stabilize the thermoelastic diffusion system (see [18] for more information on this).

We assume throughout this paper that the condition (2.13) is satisfied.

For the sake of simplicity, we assume that $\rho = 1$, and we study the exponential stability in one-dimension space. If $u = u(x,t)$, $\theta = \theta(x,t)$, and $P = P(x,t)$ describe the displacement, relative temperature and chemical potential, respectively, our equations take the form

$$\begin{aligned}
u_{tt} - \alpha u_{xx} + \gamma_1\theta_x + \gamma_2 P_x &= 0, \quad \text{in }]0,\ell[\times \mathbb{R}^+,\\
c\theta_t + dP_t + q_x + \gamma_1 u_{xt} &= 0, \quad \text{in }]0,\ell[\times \mathbb{R}^+,\\
\tau_0 q_t + q + k\theta_x &= 0, \quad \text{in }]0,\ell[\times \mathbb{R}^+,\\
d\theta_t + nP_t + \eta_x + \gamma_2 u_{xt} &= 0, \quad \text{in }]0,\ell[\times \mathbb{R}^+,\\
\tau\eta_t + \eta + \hbar\eta_x &= 0, \quad \text{in }]0,\ell[\times \mathbb{R}^+,
\end{aligned} \tag{2.15}$$

where $\alpha = \lambda_0 + 2\mu > 0$. The system is subjected to the following initial conditions:

$$\begin{aligned}
u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in]0,\ell[,\\
\theta(x,0) &= \theta_0(x), \quad q(x,0) = q_0(x), \quad x \in]0,\ell[,\\
P(x,0) &= P_0(x), \quad \eta(x,0) = \eta_0(x), \quad x \in]0,\ell[,
\end{aligned} \tag{2.16}$$

and boundary conditions

$$u(0,t) = u(\ell,t) = 0, \quad q(0,t) = q(\ell,t) = 0, \quad \eta(0,t) = \eta(\ell,t) = 0, \quad t \geq 0. \tag{2.17}$$

For the sake of simplicity, we present a short direct discussion of the the well-posedness for the linear initial boundary value $(2.15)_1$–(2.17). We transform the system $(2.15)_1$–(2.17) into a first-order system of evolution type, finally applying semigroup theory. For a solution (u, θ, q, P, η), let $\mathcal{U}$ be defined as

$$\mathcal{U} = \begin{pmatrix} \alpha u_x \\ u_t \\ \theta \\ q \\ P \\ \eta \end{pmatrix}, \qquad \mathcal{U}(0,\cdot) = \mathcal{U}_0 = \begin{pmatrix} \alpha u_{0,x} \\ u_1 \\ \theta_0 \\ q_0 \\ P_0 \\ \eta_0 \end{pmatrix}. \tag{2.18}$$

The initial-boundary value problem $(2.15)_1$–(2.17) is equivalent to problem

$$\frac{d\mathcal{U}}{dt} + Q^{-1}\mathcal{M}\mathcal{U} = 0, \qquad \mathcal{U}(0) = \mathcal{U}_0, \tag{2.19}$$

where

$$Q = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 \\ 0 & 0 & 0 & \frac{\tau_0}{k} & 0 & 0 \\ 0 & 0 & d & 0 & n & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\tau}{\hbar} \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} 0 & -\partial_x & 0 & 0 & 0 & 0 \\ -\partial_x & 0 & \gamma_1\partial_x & 0 & \gamma_2\partial_x & 0 \\ 0 & \gamma_1\partial_x & 0 & \partial_x & 0 & 0 \\ 0 & 0 & \partial_x & \frac{1}{k} & 0 & 0 \\ 0 & \gamma_2\partial_x & 0 & 0 & 0 & \partial_x \\ 0 & 0 & 0 & 0 & \partial_x & \frac{1}{\hbar} \end{pmatrix}. \tag{2.20}$$

We consider the Hilbert space $\pounds = \mathcal{U} \in (L^2(0,\ell))^6$ with inner product

$$\langle U, W\rangle_{\pounds} = \langle U, QW\rangle_{L^2}. \tag{2.21}$$

Let $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \subset \pounds \to \pounds$ such that

$$\mathcal{A}\mathcal{U} = Q^{-1}\mathcal{M}\mathcal{U}. \tag{2.22}$$

The domain of $\mathcal{A}$ is

$$\mathfrak{D}(\mathcal{A}) = \left\{ \mathcal{U} = \left(U^1, U^2, U^3, U^4, U^5, U^6\right)^T \in \pounds / U^2, U^4, U^6 \in H_0^1(0,\ell),\ U^1, U^3, U^5 \in H^1(0,\ell) \right\}, \tag{2.23}$$

that is,

$$\frac{d\mathcal{U}}{dt} + \mathcal{A}\mathcal{U} = 0, \qquad \mathcal{U}(0) = \mathcal{U}_0 \in \mathfrak{D}(\mathcal{A}). \tag{2.24}$$

On the other hand, if $\mathcal{U}$ satisfies (2.24) for $\mathcal{U}_0$ defined in (2.18), then

$$u(\cdot,t) = u_0(\cdot) + \int_0^t U^2(\cdot,s)ds, \qquad \theta = U^3, \qquad q = U^4, \qquad P = U^5, \qquad \eta = U^6 \tag{2.25}$$

satisfy $(2.15)_1$–(2.17); that is, (2.24) and $(2.15)_1$–(2.17) are equivalent (in the chosen spaces). The well-posedness is now a corollary of the following lemma characterizing $\mathcal{A}$ as a generator of a C_0-semigroup of contractions.

Lemma 2.1. (i)$\mathfrak{D}(\mathcal{A})$ *is dense in* £, *and the operator* $-\mathcal{A}$ *is dissipative.*
(ii)$\mathcal{A}$ *is closed.*
(iii)

$$\mathfrak{D}(\mathcal{A}^*) = \mathfrak{D}(\mathcal{A}), \qquad \mathcal{A}^*W = Q^{-1}\begin{pmatrix} 0 & \partial_x & 0 & 0 & 0 & 0 \\ \partial_x & 0 & -\gamma_1\partial_x & 0 & -\gamma_2\partial_x & 0 \\ 0 & -\gamma_1\partial_x & 0 & -\partial_x & 0 & 0 \\ 0 & 0 & -\partial_x & \dfrac{1}{k} & 0 & 0 \\ 0 & -\gamma_2\partial_x & 0 & 0 & 0 & -\partial_x \\ 0 & 0 & 0 & 0 & -\partial_x & \dfrac{1}{\hbar} \end{pmatrix} W. \tag{2.26}$$

Proof. (i) The density of $\mathfrak{D}(\mathcal{A})$ in £ is obvious, and we have

$$\mathrm{Re}\langle -AU, U\rangle_{£} = -\frac{1}{k}\int_0^\ell q^2 dx - \frac{1}{\hbar}\int_0^\ell \eta^2 dx \le 0. \tag{2.27}$$

Then $-\mathcal{A}$ is dissipative.

(ii) Let $(\mathcal{U}_n)_n \subset \mathfrak{D}(\mathcal{A})$, $\mathcal{U}_n \to \mathcal{U} \in$ £, and $\mathcal{A}\mathcal{U}_n \to \mathcal{F} \in$ £, as $n \to \infty$. Then,

$$\forall \Phi \in £ : \langle A\mathcal{U}_n, \Phi\rangle_{£} \longrightarrow \langle F, \Phi\rangle_{£}. \tag{2.28}$$

Choosing successively

(1) $\Phi = (\Phi^1, 0, 0, 0, 0, 0)^T$, $\Phi^1 \in H^1(0,\ell)$,

(2) $\Phi = (0, 0, \Phi^3, 0, 0, 0)^T$, $\Phi^3 \in H^1(0,\ell)$,

(3) $\Phi = (0, 0, 0, 0, \Phi^5, 0)^T$, $\Phi^5 \in H^1(0,\ell)$,

(4) $\Phi = (0, 0, 0, \Phi^4, 0, 0)^T$, $\Phi^4 \in C_0^\infty(0,\ell)$,

(5) $\Phi = (0,0,0,0,0,\Phi^6)^T$, $\Phi^6 \in C_0^\infty(0,\ell)$,

(6) $\Phi = (0,\Phi^2,0,0,0)^T$, $\Phi^2 \in C_0^\infty(0,\ell)$,

we obtain

(1) $U^2 \in H_0^1(0,\ell)$ and $-\partial_x U^2 = [Q\mathcal{F}]^1$ (first component),

(2) $U^4 \in H_0^1(0,\ell)$ and $\gamma_1 \partial_x U^2 + \partial_x U^4 = [Q\mathcal{F}]^3$,

(3) $U^6 \in H_0^1(0,\ell)$ and $\gamma_2 \partial_x U^2 + \partial_x U^6 = [Q\mathcal{F}]^5$,

(4) $U^3 \in H^1(0,\ell)$ and $\partial_x U^3 + 1/k U^4 = [Q\mathcal{F}]^4$,

(5) $U^5 \in H^1(0,\ell)$ and $\partial_x U^5 + 1/\hbar U^6 = [Q\mathcal{F}]^6$,

(6) $U^1 \in H^1(0,\ell)$ and $-\partial_x U^1 + \gamma_1 \partial_x U^3 + \gamma_1 \partial_x U^5 = [Q\mathcal{F}]^2$.

(iii)

$$\mathcal{W} \in \mathfrak{D}(\mathcal{A}^*) \Longleftrightarrow \exists \mathcal{F} \in \pounds, \quad \forall \Phi \in \mathfrak{D}(\mathcal{A}) : \langle A\Phi, \mathcal{W}\rangle_{\pounds} \longrightarrow \langle \Phi, \mathcal{F}\rangle_{\pounds}. \tag{2.29}$$

Choosing Φ appropriately as in the proof of (ii), the conclusion follows. □

With the Hille-Yosida theorem (see [19]) C_0-semigroups, we can state the following result.

Theorem 2.2. (i) *The operator $-\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of linear contractions $T(t) = e^{-t\mathcal{A}}$ over the space $\pounds$ for $t \geq 0$.*

(ii) *For any $\mathcal{U}_0 \in \mathfrak{D}(\mathcal{A})$, there exists a unique solution $\mathcal{U}(t) \in C^1([0,\infty);\pounds) \cap C^0([0,\infty);\mathfrak{D}(\mathcal{A}))$ to* (2.24) *given by $\mathcal{U}(t) = e^{-t\mathcal{A}}\mathcal{U}_0$.*

(iii) *If $\mathcal{U}_0 \in \mathfrak{D}(\mathcal{A}^n)$, $n \in \mathbb{N}$, then $\mathcal{U}(t) \in C^0([0,\infty);\mathfrak{D}(\mathcal{A}^n))$ and* (2.24) *yields higher regularity in t.*

Moreover, we will use the Young inequality

$$\pm ab \leq \frac{a^2}{\delta} + \frac{\delta}{4} b^2, \quad \forall a,b \in \mathbb{R},\ \delta > 0. \tag{2.30}$$

The differential of $(2.15)_2$ and $(2.15)_4$ together with boundary conditions (2.17) yields

$$\int_0^\ell \theta(x,t)dx = \int_0^\ell \theta_0(x)dx, \quad \int_0^\ell P(x,t)dx = \int_0^\ell P_0(x)dx, \quad t \geq 0. \tag{2.31}$$

Then, $\overline{\theta}$ and $\overline{P}$ defined by

$$\overline{\theta}(x,t) := \theta(x,t) - \frac{1}{\ell}\int_0^\ell \theta_0(x)dx, \qquad \overline{P}(x,t) := P(x,t) - \frac{1}{\ell}\int_0^\ell P_0(x)dx \tag{2.32}$$

satisfy with u, q, and η the same differential equations $(2.15)_1$–(2.17) as (u, θ, q, P, η), but additionally, we have the Poincaré inequality

$$\int_0^\ell v^2 dx \leq \frac{\ell^2}{\pi^2} \int_0^\ell v_x^2 dx, \tag{2.33}$$

for $v = \overline{\theta}(\cdot, t)$ as well as for $v = \overline{P}$, $v = u$, $v = q$ or $v = \eta$.

In the sequel, we will work with $\overline{\theta}$ and $\overline{P}$ but still write θ and P for simplicity until we will have proved Theorem 3.2.

From $(2.15)_3$ and $(2.15)_5$, we conclude

$$\begin{aligned} \int_0^\ell \theta_x^2 dx &\leq \frac{2\tau_0^2}{k^2} \int_0^\ell q_t^2 dx + \frac{2}{k^2} \int_0^\ell q^2 dx, \\ \int_0^\ell P_x^2 dx &\leq \frac{2\tau^2}{\hbar^2} \int_0^\ell \eta_t^2 dx + \frac{2}{\hbar^2} \int_0^\ell \eta^2 dx. \end{aligned} \tag{2.34}$$

Finally, for the sake of simplicity, we will employ the same symbols C for different constants, even in the same formula. In particular, we will denote by the same symbol C_i different constants due to the use of Poincaré's inequality on the interval $[0, \ell]$.

3. Exponential Stability

Let (u, θ, q, P, η) be a solution to problem $(2.15)_1$–(2.17). Multiplying $(2.15)_1$ by u_t, $(2.15)_2$ by θ, $(2.15)_3$ by q, $(2.15)_4$ by P, and $(2.15)_5$ by η and integrating from 0 to ℓ, we get

$$\frac{d}{dt}\mathcal{E}_1(t) = -\frac{1}{k} \int_0^\ell q^2 dx - \frac{1}{\hbar} \int_0^\ell \eta^2 dx, \tag{3.1}$$

where

$$\mathcal{E}_1(t) = \frac{1}{2} \int_0^\ell \left(u_t^2 + \alpha u_x^2 + c\theta^2 + nP^2 + 2d\theta P + \frac{\tau_0}{k} q^2 + \frac{\tau}{\hbar} \eta^2 \right) dx. \tag{3.2}$$

Differentiating (2.15) with respect to t, we get in the same manner

$$\frac{d}{dt}\mathcal{E}_2(t) = -\frac{1}{k} \int_0^\ell q_t^2 dx - \frac{1}{\hbar} \int_0^\ell \eta_t^2 dx, \tag{3.3}$$

where

$$\mathcal{E}_2(t) = \frac{1}{2} \int_0^\ell \left(u_{tt}^2 + \alpha u_{xt}^2 + c\theta_t^2 + nP_t^2 + 2d\theta_t P_t + \frac{\tau_0}{k} q_t^2 + \frac{\tau}{\hbar} \eta_t^2 \right) dx. \tag{3.4}$$

Let us define the functionals

$$\begin{aligned}\mathcal{F}(t) &= -\int_0^\ell \left(\frac{1}{\alpha}u_{xt}u_x + \frac{3}{\alpha^2\gamma_1}qu_{tt} + \frac{3}{\alpha}q\theta_x + \frac{3\gamma_2}{\alpha\gamma_1}qP_x + \frac{3c}{\alpha\gamma_1}\theta_x u_t + \frac{3d}{\alpha\gamma_1}P_x u_t\right)dx,\\ \mathcal{G}(t) &= -\int_0^\ell \left(\frac{1}{\alpha}u_{xt}u_x + \frac{3}{\alpha^2\gamma_2}\eta u_{tt} + \frac{3}{\alpha}\eta P_x + \frac{3\gamma_1}{\alpha\gamma_2}\eta\theta_x + \frac{3d}{\alpha\gamma_2}P_x u_t + \frac{3n}{\alpha\gamma_2}\theta_x u_t\right)dx.\end{aligned} \tag{3.5}$$

Lemma 3.1. *Let (u,θ,q,P,η) be a solution to problem* $(2.15)_1$–(2.17). *Then, one has*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) \le{}& \frac{-1}{36\alpha^2}\int_0^\ell \left(u_{tt}^2 + \frac{\alpha\pi^2}{\ell^2}u_t^2 + \theta^2 + P^2\right)dx - \frac{1}{\alpha}\int_0^\ell u_{xt}^2dx + \frac{27}{\gamma_1^2\alpha^2}\int_0^\ell q_t^2dx\\ &+ \left(\frac{27c^2}{\gamma_1^2} + \frac{3c\gamma_2}{2\alpha\gamma_1} + \frac{5\gamma_1^2}{6\alpha^2} + \frac{\ell^2}{36\alpha^2\pi^2} + \frac{3c}{2\alpha}\right)\int_0^\ell \theta_x^2dx\\ &+ \left(\frac{27d^2}{\gamma_2^2} + \frac{3c\gamma_2}{2\alpha\gamma_1} + \frac{5\gamma_2^2}{6\alpha^2} + \frac{\ell^2}{36\alpha^2\pi^2} + \frac{3c}{2\alpha}\right)\int_0^\ell P_x^2dx,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) \le{}& \frac{-1}{36\alpha^2}\int_0^\ell \left(u_{tt}^2 + \frac{\alpha\pi^2}{\ell^2}u_t^2 + \theta^2 + P^2\right)dx - \frac{1}{\alpha}\int_0^\ell u_{xt}^2dx + \frac{27}{\gamma_2^2\alpha^2}\int_0^\ell \eta_t^2dx\\ &+ \left(\frac{27n^2}{\gamma_2^2} + \frac{3n\gamma_1}{2\alpha\gamma_2} + \frac{5\gamma_2^2}{6\alpha^2} + \frac{\ell^2}{36\alpha^2\pi^2} + \frac{3n}{2\alpha}\right)\int_0^\ell P_x^2dx\\ &+ \left(\frac{27d^2}{\gamma_1^2} + \frac{3n\gamma_1}{2\alpha\gamma_2} + \frac{5\gamma_1^2}{6\alpha^2} + \frac{\ell^2}{36\alpha^2\pi^2} + \frac{3n}{2\alpha}\right)\int_0^\ell \theta_x^2dx.\end{aligned} \tag{3.7}$$

Proof. We will only prove (3.6) and (3.7) can be obtained analogously. Multiplying $(2.15)_1$ by u_{xx}/α and using the Young inequality, we get

$$\begin{aligned}\int_0^\ell u_{xx}^2dx &= -\frac{1}{\alpha}\int_0^\ell u_{ttx}u_xdx + \frac{\gamma_1}{\alpha}\int_0^\ell \theta_x u_{xx}dx + \frac{\gamma_2}{\alpha}\int_0^\ell P_x u_{xx}dx\\ &\le -\frac{1}{\alpha}\frac{d}{dt}\int_0^\ell u_{xt}u_xdx + \frac{1}{\alpha}\int_0^\ell u_{xt}^2dx + \frac{3\gamma_1^2}{4\alpha^2}\int_0^\ell \theta_x^2dx + \frac{3\gamma_2^2}{4\alpha^2}\int_0^\ell P_x^2dx + \frac{2}{3}\int_0^\ell u_{xx}^2dx,\end{aligned} \tag{3.8}$$

which implies

$$\frac{1}{\alpha}\frac{d}{dt}\int_0^\ell u_{xt}u_xdx \le -\frac{1}{3}\int_0^\ell u_{xx}^2dx + \frac{1}{\alpha}\int_0^\ell u_{xt}^2dx + \frac{3\gamma_1^2}{4\alpha^2}\int_0^\ell \theta_x^2dx + \frac{3\gamma_2^2}{4\alpha^2}\int_0^\ell P_x^2dx. \tag{3.9}$$

Multiplying $(2.15)_2$ by $3u_{xt}/(\alpha\gamma_1)$ and using the Young inequality, we get

$$\begin{aligned}\frac{3}{\alpha}\int_0^\ell u_{xt}^2dx &= \frac{3}{\alpha\gamma_1}\int_0^\ell qu_{xxt}dx + \frac{3c}{\alpha\gamma_1}\int_0^\ell \theta_{xt}u_tdx + \frac{3d}{\alpha\gamma_1}\int_0^\ell P_{xt}u_tdx \\ &= \frac{3}{\alpha\gamma_1}\frac{d}{dt}\int_0^\ell qu_{xx}dx - \frac{3}{\alpha\gamma_1}\int_0^\ell q_tu_{xx}dx + \frac{3c}{\alpha\gamma_1}\frac{d}{dt}\int_0^\ell \theta_xu_tdx - \frac{3c}{\alpha\gamma_1}\int_0^\ell \theta_xu_{tt}dx \\ &\quad + \frac{3d}{\alpha\gamma_1}\frac{d}{dt}\int_0^\ell P_xu_tdx - \frac{3d}{\alpha\gamma_1}\int_0^\ell P_xu_{tt}dx.\end{aligned} \tag{3.10}$$

Substituting $(2.15)_1$ in the above equation, yields

$$\begin{aligned}\frac{3}{\alpha}\int_0^\ell u_{xt}^2dx &= \frac{3}{\alpha^2\gamma_1}\frac{d}{dt}\int_0^\ell qu_{tt}dx + \frac{3}{\alpha}\frac{d}{dt}\int_0^\ell q\theta_xdx + \frac{3\gamma_2}{\alpha\gamma_1}\frac{d}{dt}\int_0^\ell qP_xdx - \frac{3}{\alpha\gamma_1}\int_0^\ell q_tu_{xx}dx \\ &\quad + \frac{3c}{\alpha\gamma_1}\frac{d}{dt}\int_0^\ell \theta_xu_tdx - \frac{3c}{\gamma_1}\int_0^\ell \theta_xu_{xx}dx + \frac{3c}{\alpha}\int_0^\ell \theta_x^2dx + \frac{3c\gamma_2}{\alpha\gamma_1}\int_0^\ell \theta_xP_xdx \\ &\quad + \frac{3d}{\alpha\gamma_1}\frac{d}{dt}\int_0^\ell P_xu_tdx - \frac{3d}{\gamma_1}\int_0^\ell P_xu_{xx}dx + \frac{3c}{\alpha}\int_0^\ell \theta_xP_xdx + \frac{3c\gamma_2}{\alpha\gamma_1}\int_0^\ell P_x^2dx.\end{aligned} \tag{3.11}$$

Using the estimates

$$\begin{aligned}-\frac{3}{\alpha\gamma_1}\int_0^\ell q_tu_{xx}dx &\le \frac{1}{12}\int_0^\ell u_{xx}^2dx + \frac{27}{\gamma_1^2\alpha^2}\int_0^\ell q_t^2dx, \\ -\frac{3c}{\gamma_1}\int_0^\ell \theta_xu_{xx}dx &\le \frac{1}{12}\int_0^\ell u_{xx}^2dx + \frac{27c^2}{\gamma_1^2}\int_0^\ell \theta_x^2dx, \\ \frac{3c\gamma_2}{\alpha\gamma_1}\int_0^\ell \theta_xP_xdx &\le \frac{3c\gamma_2}{2\alpha\gamma_1}\int_0^\ell \theta_x^2dx + \frac{3c\gamma_2}{2\alpha\gamma_1}\int_0^\ell P_x^2dx, \\ -\frac{3d}{\gamma_2}\int_0^\ell P_xu_{xx}dx &\le \frac{1}{12}\int_0^\ell u_{xx}^2dx + \frac{27d^2}{\gamma_2^2}\int_0^\ell P_x^2dx, \\ \frac{3c}{\alpha}\int_0^\ell \theta_xP_xdx &\le \frac{3c}{2\alpha}\int_0^\ell \theta_x^2dx + \frac{3c}{2\alpha}\int_0^\ell P_x^2dx,\end{aligned} \tag{3.12}$$

$$\frac{3}{\alpha}\int_0^\ell u_{xt}^2dx \le \frac{d}{dt}\int_0^\ell\left(\frac{3}{\alpha^2\gamma_1}qu_{tt}+\frac{3}{\alpha}q\theta_x+\frac{3\gamma_2}{\alpha\gamma_1}qP_x+\frac{3c}{\alpha\gamma_1}\theta_xu_t+\frac{3d}{\alpha\gamma_1}P_xu_t\right)dx+\frac{1}{4}\int_0^\ell u_{xx}^2dx$$
$$+\frac{27}{\gamma_1^2\alpha^2}\int_0^\ell q_t^2dx+\frac{3c}{\alpha}\left(\frac{9c\alpha}{\gamma_1^2}+\frac{\gamma_2}{2\gamma_1}+\frac{1}{2}\right)\int_0^\ell\theta_x^2dx+\frac{3n}{\alpha}\left(\frac{9c\alpha}{\gamma_2^2}+\frac{\gamma_1}{2\gamma_2}+\frac{1}{2}\right)\int_0^\ell\theta_x^2dx. \tag{3.13}$$

Combining (3.9) and (3.13), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) &\le -\frac{1}{12}\int_0^\ell u_{xx}^2dx-\frac{2}{\alpha}\int_0^\ell u_{xt}^2dx+\frac{27}{\gamma_1^2\alpha^2}\int_0^\ell q_t^2dx\\ &+\frac{3c}{\alpha}\left(\frac{9c\alpha}{\gamma_1^2}+\frac{\gamma_2}{2\gamma_1}+\frac{\gamma_1^2}{4c\alpha}+\frac{1}{2}\right)\int_0^\ell\theta_x^2dx\\ &+\frac{3c}{\alpha}\left(\frac{9d^2\alpha}{\gamma_2^2c}+\frac{\gamma_2}{2\gamma_1}+\frac{\gamma_2^2}{4c\alpha}+\frac{1}{2}\right)\int_0^\ell P_x^2dx.\end{aligned} \tag{3.14}$$

Now, we conclude from (2.15) that

$$\begin{aligned}&\int_0^\ell\left(u_{tt}^2+\frac{\alpha\pi^2}{\ell^2}u_t^2+\theta^2+P^2\right)dx\\ &\quad=\int_0^\ell\left(\alpha^2u_{xx}^2+\gamma_1^2\theta_x^2+\gamma_2^2P_x^2-2\alpha\gamma_1u_{xx}\theta_x-2\alpha\gamma_2u_{xx}P_x+2\gamma_1\gamma_2\theta_xP_x+\frac{\alpha\pi^2}{\ell^2}u_t^2+\theta^2+P^2\right)dx\\ &\quad\le 3\alpha^2\int_0^\ell u_{xx}^2dx+\left(3\gamma_1^2+\frac{\ell^2}{\pi^2}\right)\int_0^\ell\theta_x^2dx+\left(3\gamma_2^2+\frac{\ell^2}{\pi^2}\right)\int_0^\ell P_x^2dx+\alpha\int_0^\ell u_{xt}^2dx\end{aligned} \tag{3.15}$$

whence

$$\begin{aligned}-\int_0^\ell u_{xx}^2dx &\le \frac{-1}{3\alpha^2}\int_0^\ell\left(u_{tt}^2+\frac{\alpha^2\pi^2}{\ell^2}u_t^2+\theta^2+P^2\right)dx\\ &+\frac{1}{\alpha^2}\left(\gamma_1^2+\frac{\ell^2}{3\pi^2}\right)\int_0^\ell\theta_x^2dx+\frac{1}{\alpha^2}\left(\gamma_2^2+\frac{\ell^2}{3\pi^2}\right)\int_0^\ell P_x^2dx+\frac{1}{3\alpha}\int_0^\ell u_{xt}^2dx.\end{aligned} \tag{3.16}$$

Combining (3.14) and (3.16), we get our conclusion follows. □

Multiplying $(2.15)_2$ by θ_t and $(2.15)_4$ by P_t, and summing the results, yields

$$-\frac{d}{dt}\int_0^\ell (q\theta_x + \eta P_x)dx = -\int_0^\ell \left(c\theta_t^2 + nP_t^2 + 2d\theta_t P_t\right)dx - \int_0^\ell q_t\theta_x dx \\ - \gamma_1 \int_0^\ell u_{xt}\theta_t dx - \int_0^\ell \eta_t P_x dx - \gamma_2 \int_0^\ell u_{xt}P_t dx. \tag{3.17}$$

Using the estimates

$$\begin{aligned} \gamma_1 \int_0^\ell u_{xt}\theta_t dx &\le \frac{c}{4}\int_0^\ell \theta_t^2 dx + \frac{\gamma_1^2}{c}\int_0^\ell u_{xt}^2 dx, \\ \gamma_2 \int_0^\ell u_{xt}P_t dx &\le \frac{\delta}{4}\int_0^\ell P_t^2 dx + \frac{\gamma_2^2}{\delta}\int_0^\ell u_{xt}^2 dx, \\ -2d\int_0^\ell \theta_t P_t dx &\le \frac{c}{4}\int_0^\ell \theta_t^2 dx + \frac{4d^2}{c}\int_0^\ell P_t^2 dx, \\ -\frac{d}{dt}\int_0^\ell (q\theta_x + \eta P_x)dx &\le -\frac{c}{2}\int_0^\ell \theta_t^2 dx - \left(n - \frac{\delta}{4} - \frac{4d^2}{c}\right)\int_0^\ell P_t^2 dx \\ &\quad + \frac{1}{2}\int_0^\ell q_t^2 dx + \frac{1}{2}\int_0^\ell \eta_t^2 dx + \frac{1}{2}\int_0^\ell \theta_x^2 dx \\ &\quad + \frac{1}{2}\int_0^\ell P_x^2 dx + \left(\frac{\gamma_1^2}{c} + \frac{\gamma_2^2}{\delta}\right)\int_0^\ell u_{xt}^2 dx. \end{aligned} \tag{3.18}$$

Choosing δ such as

$$n - \frac{\delta}{4} - \frac{4d^2}{c} > \frac{n}{2} \tag{3.19}$$

yields

$$-\frac{d}{dt}\int_0^\ell (q\theta_x + \eta P_x)dx \le -\frac{c}{2}\int_0^\ell \theta_t^2 dx - \frac{n}{2}\int_0^\ell P_t^2 dx + \frac{1}{2}\int_0^\ell q_t^2 dx + \frac{1}{2}\int_0^\ell \eta_t^2 dx \\ + \frac{1}{2}\int_0^\ell \theta_x^2 dx + \frac{1}{2}\int_0^\ell P_x^2 dx + \left(\frac{\gamma_1^2}{c} + \frac{\gamma_2^2}{\delta}\right)\int_0^\ell u_{xt}^2 dx. \tag{3.20}$$

Now, we will show the main result of this section.

Theorem 3.2. *Let (u,θ,q,P,η) be a solution to problem* $(2.15)_1$–(2.17). *Then, the associated energy of first and second order*

$$\begin{aligned}\mathcal{E}(t) &= \mathcal{E}_1(t) + \mathcal{E}_2(t)\\ &= \frac{1}{2}\sum_{j=1}^{2}\int_0^{\ell}\Big(\left(\partial_t^{j-1}u_t\right)^2 + \alpha\left(\partial_t^{j-1}u_x\right)^2 + c\left(\partial_t^{j-1}\theta\right)^2\\ &\qquad + n\left(\partial_t^{j-1}P\right)^2 + 2d\partial_t^{j-1}\theta\partial_t^{j-1}P + \frac{\tau_0}{k}\left(\partial_t^{j-1}q\right)^2 + \frac{\tau}{\hbar}\left(\partial_t^{j-1}\eta\right)^2\Big)(x,t)\,dx\end{aligned} \tag{3.21}$$

decays exponentially; that is,

$$\exists c_0 > 0,\ \exists C_0 > 0,\ \forall t \geq 0\ \ \mathcal{E}(t) \leq C_0\mathcal{E}(0)e^{-c_0 t}. \tag{3.22}$$

Bounds for c_0 and C_0 can be given explicitly in terms of the coefficient $\alpha,\gamma_1,\gamma_2,c,n,d,k,\hbar,\tau_0,\tau$, and ℓ.

Proof. Now, we define the desired Lyapunov functional $\mathcal{N}(t)$. For $\varepsilon > 0$, to be determined later on, let

$$\mathcal{N}(t) = \frac{1}{\varepsilon}\mathcal{E}(t) + \mathcal{F}(t) + \mathcal{G}(t) - \varepsilon\int_0^{\ell}\left(q\theta_x + \eta P_x\right)dx. \tag{3.23}$$

Then, we conclude from (3.1)–(3.6) and (3.14) that

$$\begin{aligned}\frac{d}{dt}\mathcal{N}(t) \leq{}& -\frac{1}{\varepsilon k}\int_0^{\ell}q^2dx - \frac{1}{\varepsilon\hbar}\int_0^{\ell}\eta^2dx - \frac{1}{18\alpha^2}\int_0^{\ell}\left(u_{tt}^2 + \frac{\alpha\pi^2}{\ell^2}u_t^2 + \theta^2 + P^2\right)dx\\ &- \left(\frac{2}{\alpha} - \frac{\varepsilon\gamma_2^2}{n} - \frac{\varepsilon\gamma_1^2}{\delta}\right)\int_0^{\ell}u_{xt}^2dx - \left(\frac{1}{\varepsilon k} - \frac{27}{\gamma_1^2\alpha^2} - \frac{\varepsilon}{2}\right)\int_0^{\ell}q_t^2dx\\ &- \left(\frac{1}{\varepsilon\hbar} - \frac{27}{\gamma_2^2\alpha^2} - \frac{\varepsilon}{2}\right)\int_0^{\ell}\eta_t^2dx - \frac{\varepsilon}{2}\int_0^{\ell}\left(c\theta_t^2 + nP_t^2\right)dx\\ &+ \left(\varpi + \frac{\varepsilon}{2}\right)\int_0^{\ell}\theta_x^2dx + \left(\varrho + \frac{\varepsilon}{2}\right)\int_0^{\ell}P_x^2dx,\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}\varpi &= \frac{27\left(c^2+d^2\right)}{\gamma_1^2} + \frac{3c\gamma_2}{2\alpha\gamma_1} + \frac{3n\gamma_1}{2\alpha\gamma_2} + \frac{5\gamma_1^2}{3\alpha^2} + \frac{\ell^2}{18\alpha^2\pi^2} + \frac{3}{2\alpha}(c+n),\\ \varrho &= \frac{27\left(n^2+d^2\right)}{\gamma_1^2} + \frac{3c\gamma_2}{2\alpha\gamma_1} + \frac{3n\gamma_1}{2\alpha\gamma_2} + \frac{5\gamma_2^2}{3\alpha^2} + \frac{\ell^2}{18\alpha^2\pi^2} + \frac{3}{2\alpha}(c+n).\end{aligned} \tag{3.25}$$

Using (2.34), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{N}(t) \le{}& -\frac{1}{k}\left(\frac{1}{\varepsilon}-\frac{2}{k}\left(\varpi+\frac{\varepsilon}{2}\right)\right)\int_0^\ell q^2dx-\frac{1}{\hbar}\left(\frac{1}{\varepsilon}-\frac{2}{\hbar}\left(\varrho+\frac{\varepsilon}{2}\right)\right)\int_0^\ell \eta^2dx\\ &-\frac{1}{18\alpha^2}\int_0^\ell\left(u_{tt}^2+\frac{\alpha\pi^2}{\ell^2}u_t^2+\theta^2+P^2\right)dx-\left(\frac{2}{\alpha}-\frac{\varepsilon\gamma_2^2}{n}-\frac{\varepsilon\gamma_1^2}{\delta}\right)\int_0^\ell u_{xt}^2dx\\ &-\left(\frac{1}{\varepsilon k}-\frac{27}{\gamma_1^2\alpha^2}-\frac{\varepsilon}{2}-\frac{2\tau_0^2}{k^2}\left(\varpi+\frac{\varepsilon}{2}\right)\right)\int_0^\ell q_t^2dx\\ &-\left(\frac{1}{\varepsilon\hbar}-\frac{27}{\gamma_2^2\alpha^2}-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}-\frac{2\tau_0^2}{\hbar^2}\left(\varrho+\frac{\varepsilon}{2}\right)\right)\int_0^\ell \eta_t^2dx-\frac{\varepsilon}{2}\int_0^\ell\left(c\theta_t^2+nP_t^2\right)dx.\end{aligned} \tag{3.26}$$

Using (2.13) and choosing $0<\varepsilon<1$ such that all terms on the right-hand side of (3.27) become negative,

$$\begin{aligned}&\frac{2}{\alpha\left(\left(\gamma_1^2/c\right)+\left(\gamma_2^2/\delta\right)\right)}\\ &\quad<\varepsilon<\min\left\{\frac{k}{1+2\varpi},\frac{\hbar}{1+2\varrho},\frac{1}{k\left((1/2)+\left(\tau_0^2/k^2\right)(1+2\varpi)+\left(27/\gamma_1^2\alpha^2\right)\right)},\right.\\ &\qquad\left.\frac{1}{\hbar\left((1/2)+\left(\tau^2/\hbar^2\right)(1+2\varrho)+\left(27/\gamma_2^2\alpha^2\right)\right)}\right\}.\end{aligned} \tag{3.27}$$

Choosing ε as in (3.27), we obtain from

$$\frac{d}{dt}\mathcal{N}(t)\le -c_1\int_0^\ell\left(u_{tt}^2+u_t^2+u_{xt}^2+\theta^2+P^2+\theta_t^2+P_t^2+q_t^2+\eta_t^2+q^2+\eta^2\right)dx, \tag{3.28}$$

where

$$c_1=\frac{1}{2}\min\left\{\frac{1}{\varepsilon k},\frac{1}{\varepsilon\hbar},\frac{1}{9\alpha^2},\frac{2}{\alpha},\varepsilon\right\}, \tag{3.29}$$

which implies

$$\frac{d}{dt}\mathcal{N}(t)\le -c_2\mathcal{E}(t), \tag{3.30}$$

with

$$c_2=\frac{c_1}{2}\min\left\{1,\alpha,c,d,n,\frac{\tau_0}{k},\frac{\tau}{\hbar}\right\}. \tag{3.31}$$

On the other hand, we have

$$\exists \varepsilon_2 > 0, \quad \exists C_1,\ C_2 > 0, \quad \forall \varepsilon \le \varepsilon_2, \quad \forall t > 0 : C_1 \mathcal{E}(t) \le \mathcal{N}(t) \le C_2 \mathcal{E}(t), \tag{3.32}$$

where $C_1,\ C_2$ are determined as follows. Let

$$\mathcal{H}(t) = \mathcal{N}(t) - \frac{1}{\varepsilon}\mathcal{E}(t), \tag{3.33}$$

then

$$|\mathcal{H}(t)| \le C_1 \mathcal{E}(t), \tag{3.34}$$

with

$$\begin{aligned} C_1 = \max\Bigg\{ & \frac{3(c+d)}{\alpha\gamma_1} + \frac{3(n+d)}{\alpha\gamma_2}, \frac{3}{\alpha^2}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right), \frac{1}{\alpha^2}, \frac{3k}{\tau_0}\left(\frac{1}{\alpha^2\gamma_1} + \frac{1}{\alpha} + \frac{\gamma_2}{\alpha\gamma_1} + \frac{2}{\alpha k^2}\left(1 + \frac{c}{\gamma_1}\right)\right), \\ & \frac{3\hbar}{\tau}\left(\frac{1}{\alpha^2\gamma_2} + \frac{1}{\alpha} + \frac{\gamma_1}{\alpha\gamma_2} + \frac{2}{\alpha\hbar^2}\left(1 + \frac{n}{\gamma_2}\right), \frac{6\tau_0}{\alpha k}\left(1 + \frac{c}{\gamma_1}\right), \frac{6\tau}{\alpha\hbar}\left(1 + \frac{n}{\gamma_2}\right)\right)\Bigg\}. \end{aligned} \tag{3.35}$$

Choosing

$$\varepsilon \le \varepsilon_2 = \frac{1}{2C_1}, \tag{3.36}$$

we have

$$\begin{gathered} C_2 = \frac{1}{\varepsilon} + C_1, \\ \varepsilon = \min\{\varepsilon_1, \varepsilon_2\}. \end{gathered} \tag{3.37}$$

Moreover, from (3.30) and (3.32), we derive

$$\frac{d}{dt}\mathcal{N}(t) \le -c_0 \mathcal{N}(t), \tag{3.38}$$

with

$$c_0 = \frac{c_2}{C_2}, \tag{3.39}$$

hence

$$\mathcal{N}(t) \le e^{-c_0 t}\mathcal{N}(0). \tag{3.40}$$

Applying (3.32) again, we have proved

$$\mathcal{E}(t) \leq C_0 \mathcal{E}(0) e^{-c_0 t}, \tag{3.41}$$

with

$$C_0 = \frac{C_2}{C_1}, \tag{3.42}$$

and it holds. □

The copper material was chosen for purposes of numerical evaluations. The physical constants given by Table 1 are found in [20].

Successively we can approximately compute $\varepsilon_1, C_1, \varepsilon_2, \varepsilon, c_1, c_2, C_0$, and c_0 from the previous equations, getting finally

$$c_0 \approx 2.68 \times 10^{-56}, \tag{3.43}$$

which indicates a slow decay of the energy in the beginning but does not mean that solutions do not decay.

Remark 3.3. In particular, we can get

$$d_0 = O(\tau_0, \tau) \quad \text{as } (\tau_0, \tau) \longrightarrow (0,0). \tag{3.44}$$

Although the estimate for τ_0 and τ are very coarse and might be far from being sharp, it indicates a slow decay of the energy in usually measured time periods. The above relation of course does not imply that solutions to the limiting case $(\tau_0, \tau) = (0,0)$ do not decay. Instead, the decay rate of the thermodiffusion system provides a better rate; that is,

$$\begin{aligned} c\theta_t + dP_t - k\theta_{xx} &= 0, \quad \text{in }]0,\ell[\times \mathbb{R}^+, \\ d\theta_t + nP_t - \hbar P_{xx} &= 0, \quad \text{in }]0,\ell[\times \mathbb{R}^+, \end{aligned} \tag{3.45}$$

with initial conditions

$$\theta(x,0) = \theta_0(x), \quad P(x,0) = P_0(x), \quad x \in]0,\ell[, \tag{3.46}$$

and boundary conditions

$$\theta(0,t) = \theta(\ell,t) = 0, \quad P(0,t) = P(\ell,t) = 0, \quad t \geq 0. \tag{3.47}$$

In this case, we have

$$\frac{d}{dt}E(t) = -k\int_0^\ell \theta^2 dx - \hbar \int_0^\ell P^2 dx, \tag{3.48}$$

Table 1: Values of the constants.

$\mu = 3.86 \times 10^{10}\,\mathrm{kg/(m\,s^3)}$	$\lambda = 7.76 \times 10^{10}\,\mathrm{kg/(m\,s^3)}$	$\rho = 8954\,\mathrm{kg/m^3}$
$c_E = 383.1\,\mathrm{J/(Kg\,K)}$	$\alpha_t = 1.78 \times 10^{-5}\,\mathrm{K^{-1}}$	$k = 386\,\mathrm{W/(m\,K)}$
$\alpha_c = 1.98 \times 10^{-4}\,\mathrm{m^3/kg}$	$\hbar = 0.85 \times 10^{-8}\,\mathrm{kg\,s/m^3}$	$T_0 = 293\,\mathrm{K}$
$a = 1.2 \times 10^4\,\mathrm{m^2/(s^2\,K)}$	$b = 0.9 \times 10^6\,\mathrm{m^5/(s^2\,Kg)}$	$\tau_0 = 10^{-12}\,\mathrm{s}$
$\tau = 10^{-11}\,\mathrm{s}$	$\ell = 6 \times 10^{-4}\,\mathrm{m}$	

where

$$E(t) = \frac{1}{2}\int_0^{\ell}\left(c\theta^2 + nP^2 + 2d\theta P + \frac{\tau_0}{k}q^2 + \frac{\tau}{\hbar}\eta^2\right)dx. \tag{3.49}$$

Using the Poincaré inequality, we get

$$E(t) \le e^{-\nu t}E(0), \tag{3.50}$$

with

$$\nu = 2\frac{\pi^2}{\ell^2}\left(\frac{k}{c} + \frac{\hbar}{n}\right) \approx 2.43 \times 10^{-3}. \tag{3.51}$$

4. The Limit Case $(\tau_0, \tau) \to (0,0)$

We will show that the energy of the difference of the solution (u, θ, P, q, η) to $(2.15)_1$–(2.17) and the the solution $(\tilde{u}, \tilde{\theta}, \tilde{P}, \tilde{q}, \tilde{\eta})$ to the corresponding system with $(\tau_0, \tau) = (0,0)$ (see [17]) vanishes of order $\tau_0^2 + \tau^2$ as $(\tau_0, \tau) \to (0,0)$, provided the values at $t = 0$ coincide. For this purpose, let $(U, \Theta, \phi, \Phi, \varphi)$ denote the difference

$$U = u - \tilde{u}, \qquad \Theta = \theta - \tilde{\theta}, \qquad \phi = q - \tilde{q}, \qquad \Phi = P - \tilde{P}, \qquad \varphi = \eta - \tilde{\eta}, \tag{4.1}$$

then $(U, \Theta, \phi, \Phi, \varphi)$ satisfies

$$\begin{gathered} U_{tt} - \alpha U_{xx} + \gamma_1\Theta_x + \gamma_2\Phi_x = 0, \\ c\Theta_t + d\Phi_t + \phi_x + \gamma_1 U_{xt} = 0, \\ \tau_0\phi_t + \phi + k\Theta_x = \tau_0 k\tilde{\theta}_{xt}, \\ d\Theta_t + n\Phi_t + \varphi_x + \gamma_2 U_{xt} = 0, \\ \tau\varphi_t + \eta + \hbar\varphi_x = \tau\hbar\tilde{P}_{xt}, \end{gathered} \tag{4.2}$$

$$\begin{gathered} U(x,0) = 0, \qquad U_t(x,0) = 0, \qquad \Theta(x,0) = 0, \qquad \Phi(x,0) = 0, \\ \phi(x,0) = 0, \qquad \varphi(x,0) = 0, \qquad x \in]0, \ell[, \end{gathered} \tag{4.3}$$

$$U(0,t) = U(\ell,t) = 0, \qquad \Theta(0,t) = \Theta(\ell,t) = 0, \qquad \Phi(0,t) = \Phi(\ell,t) = 0. \tag{4.4}$$

Here, we assumed the compatibility conditions

$$q_0 = -k\theta_{0,x}, \qquad \eta_0 = -\hbar P_{0,x}. \tag{4.5}$$

If $E(t)$ denotes the energy of first order for $(U,\Theta,\phi,\Phi,\psi)$; that is,

$$\frac{d}{dt}E(t) = -\frac{1}{k}\int_0^\ell \phi^2 dx + \tau_0 \int_0^\ell \widetilde{\theta}_{xt}\phi dx - \frac{1}{\hbar}\int_0^\ell \psi^2 dx + \tau \int_0^\ell \widetilde{P}_{xt}\psi dx, \tag{4.6}$$

where

$$E(t) = \frac{1}{2}\int_0^\ell \left(U_t^2 + \alpha U_x^2 + c\Theta^2 + n\Phi^2 + 2d\Theta\Phi + \frac{\tau_0}{k}\phi^2 + \frac{\tau}{\hbar}\psi^2\right)dx. \tag{4.7}$$

Using the Young inequality, we obtain

$$\frac{d}{dt}E(t) \le -\frac{1}{2k}\int_0^\ell \phi^2 dx + \frac{\tau_0^2 k}{2}\int_0^\ell \widetilde{\theta}_{xt}^2 dx - \frac{1}{2\hbar}\int_0^\ell \psi^2 dx + \frac{\tau^2\hbar}{2}\int_0^\ell \widetilde{P}_{xt}^2 dx. \tag{4.8}$$

Using initial condition (4.3) yields

$$E(t) \le \frac{\tau_0^2 k}{2}\int_0^t\int_0^\ell \widetilde{\theta}_{xt}^2(x,s)dx\,ds + \frac{\tau^2\hbar}{2}\int_0^t\int_0^\ell \widetilde{P}_{xt}^2(x,s)dx\,ds, \tag{4.9}$$

from where we get for $T > 0$ fixed, $t \in [0,T]$

$$E(t) \le \frac{\tau_0^2 k}{2}\int_0^T\int_0^\ell \widetilde{\theta}_{xt}^2(x,s)dx\,ds + \frac{\tau^2\hbar}{2}\int_0^T\int_0^\ell \widetilde{P}_{xt}^2(x,s)dx\,ds. \tag{4.10}$$

Moreover, since

$$\int_0^\infty\int_0^\ell \widetilde{\theta}_{xt}^2(x,s)dx\,ds < \infty, \qquad \int_0^\infty\int_0^\ell \widetilde{P}_{xt}^2(x,s)dx\,ds < \infty, \tag{4.11}$$

because of the exponential decay of the solution corresponding to the problem when $(\tau_0,\tau) = (0,0)$ (see [17]), we obtain a uniform bound on the right-hand side,

$$\exists C > 0, \quad \forall t \ge 0 : E(t) \le C\left(\tau_0^2 + \tau^2\right), \tag{4.12}$$

where

$$C = \frac{1}{2}\min\left\{k\int_0^T\int_0^\ell \widetilde{\theta}_{xt}^2(x,s)dx\,ds, \hbar\int_0^T\int_0^\ell \widetilde{P}_{xt}^2(x,s)dx\,ds\right\}. \tag{4.13}$$

Then we have

$$E(t) \le \mathcal{O}\left(\tau_0^2+\tau^2\right) \quad \text{as } (\tau_0,\tau)\longrightarrow(0,0), \tag{4.14}$$

also

$$\frac{E(t)}{\tau_0^2+\tau^2}\longrightarrow 0 \quad \text{as } t\longrightarrow 0. \tag{4.15}$$

5. Concluding Remarks

(1) By comparison of the approximate value of $c_0 \approx 2.68\times 10^{-56}$ with the value $c_0 \approx 1.75\times 10^{-13}$ of the problem corresponding to $(\tau_0,\tau) = (0,0)$ computed in [17], we remark the second value is significantly larger than the first. This confirms that thermoelastic models with second sound are physically more realistic than those given in the classic context.

(2) By comparison of the approximate value of $c_0 \approx 2.68\times 10^{-56}$ with the value $\nu \approx 2.43\times 10^{-3}$ of the thermodiffusion problem, we conclude that the slow decay of the elastic part is responsible for the low bounds on the decay rates obtained in this paper and in [17].

(3) Finally, we remark that in [17], the exponential decay of the solution was proved by means of the first energy only, while in our case, it is necessary to use second-order derivatives because of the more complicated system with second sound.

References

[1] H. W. Lord and Y. Shulman, "A generalized dynamical theory of thermoelasticity," *Journal of the Mechanics and Physics of Solids*, vol. 15, no. 5, pp. 299–309, 1967.

[2] W. Nowacki, "Dynamical problems of thermoelastic diffusion in solids I," *Bulletin de l'Académie Polonaise des Sciences Serie des Sciences Techniques*, vol. 22, p. 55–64; 129–135; 257–266, 1974.

[3] H. H. Sherief, F. A. Hamza, and H. A. Saleh, "The theory of generalized thermoelastic diffusion," *International Journal of Engineering Science*, vol. 42, no. 5-6, pp. 591–608, 2004.

[4] M. Aouadi, "Theory of generalized micropolar thermoelastic diffusion under Lord-Shulman model," *Journal of Thermal Stresses*, vol. 32, no. 9, pp. 923–942, 2009.

[5] M. Aouadi, "The coupled theory of micropolar thermoelastic diffusion," *Acta Mechanica*, vol. 208, no. 3-4, pp. 181–203, 2009.

[6] M. Aouadi, "A theory of thermoelastic diffusion materials with voids," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 61, no. 2, pp. 357–379, 2010.

[7] M. Aouadi, "Qualitative results in the theory of thermoelastic diffusion mixtures," *Journal of Thermal Stresses*, vol. 33, no. 6, pp. 595–615, 2010.

[8] C. M. Dafermos, "Contraction semigroups and trends to equilibrium in continuum mechanics," in *Applications of Methods of Functional Analysis to Problems in Mechanics*, P. Germain and B. Nayroles, Eds., vol. 503 of *Springer Lectures Notes in Mathematics*, pp. 295–306, Springer, Berlin, Germany, 1976.

[9] M. Slemrod, "Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional nonlinear thermoelasticity," *Archive for Rational Mechanics and Analysis*, vol. 76, no. 2, pp. 97–133, 1981.
[10] S. Jiang and R. Racke, *Evolution Equations in Thermoelasticity*, vol. 112 of *Monographs and Surveys in Pure and Applied Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2000.
[11] G. Lebeau and E. Zuazua, "Decay rates for the three-dimensional linear system of thermoelasticity," *Archive for Rational Mechanics and Analysis*, vol. 148, no. 3, pp. 179–231, 1999.
[12] H. H. Sherief, "On uniqueness and stability in generalized thermoelasticity," *Quarterly of Applied Mathematics*, vol. 44, no. 4, pp. 773–778, 1987.
[13] M. A. Tarabek, "On the existence of smooth solutions in one-dimensional nonlinear thermoelasticity with second sound," *Quarterly of Applied Mathematics*, vol. 50, no. 4, pp. 727–742, 1992.
[14] R. Racke, "Thermoelasticity with second sound—exponential stability in linear and non-linear 1-d," *Mathematical Methods in the Applied Sciences*, vol. 25, no. 5, pp. 409–441, 2002.
[15] S. A. Messaoudi and B. Said-Houari, "Exponential stability in one-dimensional non-linear thermoelasticity with second sound," *Mathematical Methods in the Applied Sciences*, vol. 28, no. 2, pp. 205–232, 2005.
[16] A. Soufyane, "Energy decay for porous-thermo-elasticity systems of memory type," *Applicable Analysis*, vol. 87, no. 4, pp. 451–464, 2008.
[17] M. Aouadi and A. Soufyane, "Polynomial and exponential stability for one-dimensional problem in thermoelastic diffusion theory," *Applicable Analysis*, vol. 89, no. 6, pp. 935–948, 2010.
[18] M. Aouadi, "Generalized theory of thermoelastic diffusion for anisotropic media," *Journal of Thermal Stresses*, vol. 31, no. 3, pp. 270–285, 2008.
[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.
[20] L. Thomas, *Fundamentals of Heat Transfer*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1980.