Research Article

Periodic Solutions for a Class of $n$-th Order Functional Differential Equations

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We study the existence of periodic solutions for $n$-th order functional differential equations:

$$x^{(n)}(t) = \sum_{i=0}^{n-1} b_i [x^{(i)}(t)]^k + f(x(t - \tau(t))) + p(t),$$

where $b_i, i = 0, 1, \ldots, n - 1$ are constants, $k$ is a positive odd, $f \in C^1(R, R)$ for $\forall x \in R$, $p \in C(R, R)$ with $p(t + T) = p(t)$.

In recent years, there are many papers studying the existence of periodic solutions of first-, second- or third-order differential equations [1–12]. For example, in [5], Zhang and Wang studied the following differential equations:

$$x''(t) + ax^{2k-1}(t) + bx^{2k-1}(t) + cx^{2k-1}(t) + g(t, x(t - \tau_1), x'(t - \tau_2)) = p(t).$$

1. Introduction

In this paper, we are concerned with the existence of periodic solutions of the following $n$-th order functional differential equations:

$$x^{(n)}(t) = \sum_{i=0}^{n-1} b_i [x^{(i)}(t)]^k + f(x(t - \tau(t))) + p(t),$$

(1.1)
The authors established the existence of periodic solutions of (1.2) under some conditions on $a, b, c$, and $2k - 1$.

In [13–24], periodic solutions for $n, 2n,$ and $2n + 1$ th order differential equations were discussed. For example, in [22, 24], Pan et al. studied the existence of periodic solutions of higher order differential equations of the form

$$x^{(n)}(t) = \sum_{i=1}^{n-1} b_i x^{(i)}(t) + f(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) + p(t).$$

(1.3)

The authors obtained the results based on the damping terms $x^{(i)}(t)$ and the delay $\tau_i(t)$.

In present paper, by using Mawhin’s continuation theorem, we will establish some theorems on the existence of periodic solutions of (1.1). The results are related to not only $b_i$ and $f(t, x)$ but also the positive odd $k$. In addition, we give an example to illustrate our new results.

2. Some Lemmas

We investigate the theorems based on the following lemmas.

**Lemma 2.1** (see [17]). Let $m_1 > 1$, $\alpha \in [0, +\infty)$ be constants, $s \in C(R, R)$ with $s(t + T) = s(t)$, and $s(t) \in [-\alpha, \alpha]$, for all $t \in [0, T]$. Then for $\forall x \in C^1(R, R)$ with $x(t + T) = x(t)$, one has

$$\int_0^T |x(t) - x(t - s(t))|^{m_1} dt \leq 2\alpha^{m_1} \int_0^T |x'(t)|^{m_1} dt. \quad (2.1)$$

**Lemma 2.2.** Let $k \geq 1$, $\alpha \in [0, +\infty)$ be constants, $s \in C(R, R)$ with $s(t + T) = s(t)$, and $s(t) \in [-\alpha, \alpha]$, for all $t \in [0, T]$. Then for $\forall x \in C^1(R, R)$ with $x(t + T) = x(t)$, one has

$$\int_0^T \left| x^k(t) - x^k(t - s(t)) \right|^{(k+1)/k} dt \leq 2\alpha^{(k+1)/k} k^{1/k} \left[ (k - 1) \int_0^T |x(t)|^{k+1} dt + \int_0^T |x'(t)|^{k+1} dt \right]. \quad (2.2)$$

**Proof.** Let $F(t) = x^k(t)$. By Lemma 2.2, one has

$$\int_0^T \left| x^k(t) - x^k(t - s(t)) \right|^{(k+1)/k} dt = \int_0^T |F(t) - F(t - s(t))|^{(k+1)/k} dt \leq 2\alpha^{(k+1)/k} \int_0^T |F(t)|^{(k+1)/k} dt$$

$$= 2\alpha^{(k+1)/k} \int_0^T |kx^{k-1}(t)x'(t)|^{(k+1)/k} dt$$

$$= 2\alpha^{(k+1)/k} k^{(k+1)/k} \int_0^T |x(t)|^{(k(k-1)+1)/k} |x'(t)|^{(k+1)/k} dt.$$. 

(2.3)
By inequality

\[
xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \quad x \geq 0, \ y \geq 0, \ \frac{1}{p} + \frac{1}{q} = 1, \tag{2.4}
\]

one has

\[
|x(t)|^{(k-1)(k+1)/k} |x'(t)|^{(k+1)/k} \leq \frac{(k-1)|x(t)|^{k+1}}{k} + \frac{|x'(t)|^{k+1}}{k}. \tag{2.5}
\]

Thus we obtain

\[
\int_0^T |x^k(t) - x^k(t - s(t))|^{(k+1)/k} dt \leq 2\alpha^{(k+1)/k} k^{1/k} \left( (k-1) \int_0^T |x(t)|^{k+1} dt + \int_0^T |x'(t)|^{k+1} dt \right). \tag{2.6}
\]

\[\square\]

**Lemma 2.3.** If \( k \geq 1 \) is an integer, \( x \in C^n(R, R) \), and \( x(t + T) = x(t) \), then

\[
\left( \int_0^T |x'(t)|^k dt \right)^{1/k} \leq T \left( \int_0^T |x''(t)|^k dt \right)^{1/k} \leq \cdots \leq T^{n-1} \left( \int_0^T |x^{(n)}(t)|^k dt \right)^{1/k}. \tag{2.7}
\]

The proof of Lemma 2.3 is easy, here we omit it.

We first introduce Mawhin's continuation theorem.

Let \( X \) and \( Y \) be Banach spaces, \( L : D(L) \subset X \rightarrow Y \) are a Fredholm operator of index zero, here \( D(L) \) denotes the domain of \( L \). \( P : X \rightarrow X \), \( Q : Y \rightarrow Y \) be projectors such that

\[
\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L, \quad X = \text{Ker } L \oplus \text{Ker } P, \quad Y = \text{Im } L \oplus \text{Im } Q. \tag{2.8}
\]

It follows that

\[
L|_{D(L) \cap \text{Ker } P} : D(L) \cap \text{Ker } P \rightarrow \text{Im } L \tag{2.9}
\]

is invertible, we denote the inverse of that map by \( K_p \). Let \( \Omega \) be an open bounded subset of \( X \), \( D(L) \cap \overline{\Omega} \neq \emptyset \), the map \( N : X \rightarrow Y \) will be called \( L \)-compact in \( \overline{\Omega} \), if \( QN(\overline{\Omega}) \) is bounded and \( K_p(I - Q)N : \overline{\Omega} \rightarrow X \) is compact.

**Lemma 2.4** (see [25]). Let \( L \) be a Fredholm operator of index zero and let \( N \) be \( L \)-compact on \( \overline{\Omega} \). Assume that the following conditions are satisfied:

(i) \( Lx \neq \lambda Nx \), for all \( x \in \partial \Omega \cap D(L), \ \lambda \in (0, 1); \)

(ii) \( QNx \neq 0 \), for all \( x \in \partial \Omega \cap \text{Ker } L; \)

(iii) \( \text{deg} \{ QNx, \Omega \cap \text{Ker } L, 0 \} \neq 0, \)

then the equation \( Lx = Nx \) has at least one solution in \( \overline{\Omega} \cap D(L) \).
Now, we define \( Y = \{ x \in C(R, R) \mid x(t + T) = x(t) \} \) with the norm \( |x|_{\infty} = \max_{t \in [0, T]} |x(t)| \) and \( X = \{ x \in C^{n-1}(R, R) \mid x(t + T) = x(t) \} \) with norm \( \|x\| = \max\{|x|_{\infty}, |x'|_{\infty}, \ldots, |x^{(n-1)}|_{\infty}\} \). It is easy to see that \( X, Y \) are two Banach spaces. We also define the operators \( L \) and \( N \) as follows:

\[
L : D(L) \subset X \rightarrow Y, \quad Lx = x^{(n)}, \quad D(L) = \{ x \mid x \in C^n(R, R), \ x(t + T) = x(t) \},
\]

\[
N : X \rightarrow Y, \quad Nx = \sum_{i=1}^{n-1} b_i \left[ x^{(i)}(t) \right]^k - f(t, x(t - \tau(t))) + p(t).
\] (2.10)

It is easy to see that (1.1) can be converted to the abstract equation \( Lx = Nx \). Moreover, from the definition of \( L \), we see that \( \ker L = R, \ \dim(\ker L) = 1, \ \Im L = \{ y \mid y \in Y, \int_0^T y(s) \, ds = 0 \} \) is closed, and \( \dim(Y \setminus \Im L) = 1 \), one has \( \text{codim}(\Im L) = \dim(\ker L) \). So \( L \) is a Fredholm operator with index zero. Let

\[
P : X \rightarrow \ker L, \quad Px = x(0), \quad Q : Y \rightarrow Y \setminus \Im L, \quad Qy = \frac{1}{T} \int_0^T y(t) \, dt,
\] (2.11)

and let

\[
L|_{D(L) \cap \ker P} : D(L) \cap \ker P \rightarrow \Im L.
\] (2.12)

Then \( L|_{D(L) \cap \ker P} \) has a unique continuous inverse \( K_0 \). One can easily find that \( N \) is \( L \)-compact in \( \overline{\Omega} \), where \( \overline{\Omega} \) is an open bounded subset of \( X \).

### 3. Main Result

**Theorem 3.1.** Suppose \( n = 2m + 1, \ m > 0 \) an integer and the following conditions hold:

\( (H_1) \) The function \( f \) satisfies

\[
\lim_{x \rightarrow \infty} \frac{|f(t, x)|}{x^k} \leq \gamma,
\]

where \( \gamma \geq 0 \).

\( (H_2) \)

\[
|b_0| > \gamma + \theta_2.
\]

\( (H_3) \) There is a positive integer \( 0 < s \leq m \) such that

\[
b_{2s} \neq 0, \quad \text{if } s = m,
\]

\[
b_{2s} \neq 0, \ b_{2s+i} = 0, \quad i = 1, 2, \ldots, 2m - 2s, \quad \text{if } 0 < s < m.
\] (3.4)
Consider the equation

\[ A_2(2s,k) + \theta_1 T^{(2s-1)k} + \frac{(\gamma + \theta_2)(A_1(2s,k) + \theta_1 T^{(2s-1)k})}{|b_0| - \gamma - \theta_2} + k|b_0| T^{2s} \left[ \frac{A_1(2s,k) + \theta_1 T^{(2s-1)k}}{|b_0| - \gamma - \theta_2} \right]^{(k-1)/k} < |b_2|, \quad 1 < s \leq m, \quad (3.5) \]

\[ \theta_1 T^k + \frac{(\gamma + \theta_2)(A_1(2s,k) + \theta_1 T^k)}{|b_0| - \gamma - \theta_2} + k|b_0| T^2 \left[ \frac{A_1(2s,k) + \theta_1 T^k}{|b_0| - \gamma - \theta_2} \right]^{(k-1)/k} < |b_2|, \quad 1 < s \leq m, \quad (3.5) \]

where \( A_1(s,k) = \sum_{i=1}^{s} |b_i| T^{(s-i)k} \), \( A_2(s,k) = \sum_{i=1}^{s-2} |b_i| T^{(s-i)k} \), \( \theta_1 = 2^{k/(k+1)} \beta |\tau(t)|^{\alpha k^{1/(k+1)}} \), \( \theta_2 = 2^{k/(k+1)} \beta |\tau(t)|^{\alpha k^{1/(k+1)}} \). Then (1.1) has at least one \( T \)-periodic solution.

Proof. Consider the equation

\[ Lx = \lambda Nx, \quad \lambda \in (0,1), \quad (3.6) \]

where \( L \) and \( N \) are defined by (2.10). Let

\[ \Omega_1 = \{ x \in D(L) \mid \text{Ker} L, Lx = \lambda Nx \text{ for some } \lambda \in (0,1) \}. \quad (3.7) \]

For \( x \in \Omega_1 \), one has

\[ x^{(n)}(t) = \lambda \sum_{i=0}^{2^n} b_i \left[ x^{(i)}(t) \right]^k + \lambda f(t, x(t - \tau(t))) + \lambda p(t), \quad \lambda \in (0,1). \quad (3.8) \]

Multiplying both sides of (3.8) by \( x(t) \), and integrating them on \([0,T]\), one has for \( \lambda \in (0,1) \)

\[ \int_0^T x^{(n)}(t)x(t)dt = \lambda \sum_{i=0}^{2^n} b_i \int_0^T \left[ x^{(i)}(t) \right]^k x(t)dt + \lambda \int_0^T f(t, x(t - \tau(t)))x(t)dt + \lambda \int_0^T p(t)x(t)dt. \quad (3.9) \]

Since for any positive integer \( i \),

\[ \int_0^T x^{(2i-1)}(t)x(t)dt = 0, \quad (3.10) \]
and in view of $n = 2m + 1$ and $k$ is odd, it follows from (3.3) and (3.9) that

$$
|b_0| \int_0^T |x(t)|^{k+1} dt
$$

$$
\leq \sum_{i=1}^{2s} |b_i| \int_0^T |x^{(i)}(t)|^k |x(t)| dt + \int_0^T |f(t, x(t - \tau(t)))| |x(t)| dt + \int_0^T |p(t)| |x(t)| dt
$$

$$
\leq \sum_{i=1}^{2s} |b_i| \int_0^T |x^{(i)}(t)|^k |x(t)| dt + \int_0^T |f(t, x(t))| |x(t)| dt
$$

$$
+ \int_0^T |f(t, x) - f(t, x(t - \tau(t)))| |x(t)| dt + \int_0^T |p(t)| |x(t)| dt.
$$

(3.11)

By using Hölder inequality and Lemma 2.1, from (3.11), we obtain

$$
|b_0| \int_0^T |x(t)|^{k+1} dt
$$

$$
\leq \left( \int_0^T |x(t)|^{k+1} dt \right)^{1/(k+1)} \left[ \sum_{i=1}^{2s} |b_i| \left( \int_0^T |x^{(i)}(t)|^{k+1} dt \right)^{k/(k+1)}
$$

$$
+ \left( \int_0^T |f(t, x(t))|^{(k+1)/k} dt \right)^{k/(k+1)}
$$

$$
+ \left( \int_0^T |f(t, x) - f(t, x(t - \tau(t)))|^{(k+1)/k} dt \right)^{k/(k+1)}
$$

$$
+ \left( \int_0^T |p(t)|^{(k+1)/k} dt \right)^{k/(k+1)} \right].
$$

$$
\leq \left( \int_0^T |x(t)|^{k+1} dt \right)^{1/(k+1)} \left[ \sum_{i=1}^{2s} |b_i| T^{(2s-i)k} \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{k/(k+1)}
$$

$$
+ \left( \int_0^T |f(t, x(t))|^{(k+1)/k} dt \right)^{k/(k+1)}
$$

$$
+ \left( \int_0^T |f(t, x) - f(t, x(t - \tau(t)))|^{(k+1)/k} dt \right)^{k/(k+1)}
$$

$$
+ |p(t)| \infty T^{k/(k+1)} \right].
$$

(3.12)
So

$$|b_0| \left( \int_0^T |x(t)|^{k+1} dt \right)^{k/(k+1)} \leq A_1(2s, k) \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{k/(k+1)} + \left( \int_0^T |f(t, x(t))|^{(k+1)/k} dt \right)^{k/(k+1)}$$

$$+ \left( \int_0^T |f(t, x(t)) - f(t, x(t - \tau(t)))|^{(k+1)/k} dt \right)^{k/(k+1)} + \nu_1,$$

where $\nu_1$ is a positive constant. Choosing a constant $\varepsilon > 0$ such that

$$\gamma + \varepsilon + \theta_2 < |b_0|,$$

$$A_2(2s, k) + \theta_1 T^{(2s-1)k} + \frac{(\gamma + \varepsilon + \theta_2)(A_1(2s, k) + \theta_1 T^{(2s-1)k})}{|b_0| - (\gamma + \varepsilon) - \theta_2} + k|b_0| T^{2s} \left[ \frac{A_1(2s, k) + \theta_1 T^{(2s-1)k}}{|b_0| - (\gamma + \varepsilon) - \theta_2} \right]^{(k-1)/k} < |b_2|,$$

$$\text{if } 1 < s \leq m,$$

$$\theta_1 T^k + \frac{(\gamma + \varepsilon + \theta_2)(A_1(2s, k) + \theta_1 T^k)}{|b_0| - (\gamma + \varepsilon) - \theta_2} + k|b_0| T^2 \left[ \frac{A_1(2s, k) + \theta_1 T^k}{|b_0| - (\gamma + \varepsilon) - \theta_2} \right]^{(k-1)/k} < |b_2|,$$

$$\text{if } s = 1,$$

(3.15)

for the above constant $\varepsilon > 0$, we see from (3.1) that there is a constant $\delta > 0$ such that

$$|f(t, x(t))| < (\gamma + \varepsilon)|x(t)|^k, \quad \text{for } |x(t)| > \delta, \ t \in [0, T].$$

(3.16)

Denote

$$\Delta_1 = \{ t \in [0, T] : |x(t)| \leq \delta \}, \quad \Delta_2 = \{ t \in [0, T] : |x(t)| > \delta \}.$$

(3.17)

Since

$$\int_0^T |f(t, x(t))|^{(k+1)/k} dt \leq \int_{\Delta_1} |f(t, x(t))|^{(k+1)/k} dt + \int_{\Delta_2} |f(t, x(t))|^{(k+1)/k} dt$$

$$\leq (f_\delta)^{(k+1)/k} T + (\gamma + \varepsilon)^{(k+1)/k} \int_0^T |x(t)|^{k+1} dt$$

$$= (f_\delta)^{(k+1)/k} T + (\gamma + \varepsilon)^{(k+1)/k} \int_0^T |x(t)|^{k+1} dt,$$

(3.18)
using inequality

\[(a + b)^l \leq a^l + b^l \quad \text{for} \quad a \geq 0, \quad b \geq 0, \quad 0 \leq l \leq 1, \quad (3.19)\]

it follows from (3.18) that

\[\left( \int_0^T |f(t, x(t))|^{(k+1)/k} \, dt \right)^{k/(k+1)} \leq f_\delta T^{k/(k+1)} + (Y + \varepsilon) \left( \int_0^T |x(t)|^{k+1} \, dt \right)^{k/(k+1)}. \quad (3.20)\]

From (3.2) and by Lemma 2.2, one has

\[\left( \int_0^T |f(t, x(t)) - f(t, x(t - \tau(t)))|^{(k+1)/k} \, dt \right)^{k/(k+1)} \leq \beta \left[ \int_0^T |x^k(t) - x^k(t - \tau(t))|^{(k+1)/k} \, dt \right]^{k/(k+1)} \]

\[\leq 2^{k/(k+1)} \beta |\tau(t)|_\infty k^{1/(k+1)} \left[ (k - 1) \int_0^T |x(t)|^{k+1} \, dt + \int_0^T |x'(t)|^{k+1} \, dt \right]^{k/(k+1)} \]

\[\leq 2^{k/(k+1)} \beta |\tau(t)|_\infty k^{1/(k+1)} \left[ (k - 1)^{k/(k+1)} \left( \int_0^T |x(t)|^{k+1} \, dt \right)^{k/(k+1)} \right. \]

\[+ \left. \left( \int_0^T |x'(t)|^{k+1} \, dt \right)^{k/(k+1)} \right] \]

\[\leq 2^{k/(k+1)} \beta |\tau(t)|_\infty k^{1/(k+1)} (k - 1)^{k/(k+1)} \left( \int_0^T |x(t)|^{k+1} \, dt \right)^{k/(k+1)} \]

\[+ 2^{k/(k+1)} \beta |\tau(t)|_\infty k^{1/(k+1)} T^{(2\varepsilon - 1)k} \left( \int_0^T |x(2\varepsilon)(t)|^{k+1} \, dt \right)^{k/(k+1)} \]

\[= \theta_2 \left( \int_0^T |x(t)|^{k+1} \, dt \right)^{k/(k+1)} + \theta_1 T^{(2\varepsilon - 1)k} \left( \int_0^T |x(2\varepsilon)(t)|^{k+1} \, dt \right)^{k/(k+1)}. \quad (3.21)\]
Substituting the above formula into (3.13), one has

$$\left[|b_0| - (\gamma + \varepsilon) - \theta_2\right]\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)}$$

(3.22)

$$\leq \left[A_1(2s, k) + \theta_1 T(2s-1)^k\right]\left(\int_0^T |x^{(2s)}(t)|^{k+1} dt\right)^{k/(k+1)} + \mu_2,$$

where \(\mu_2\) is a positive constant. That is

$$\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)} \leq \frac{A_1(2s, k) + \theta_1 T(2s-1)^k}{|b_0| - (\gamma + \varepsilon) - \theta_2}\left(\int_0^T |x^{(2s)}(t)|^{k+1} dt\right)^{k/(k+1)} + \mu_3,$$  (3.23)

where \(\mu_3\) is a positive constant.

On the other hand, multiplying both sides of (3.8) by \(x^{(2s)}(t)\), and integrating on \([0, T]\), one has

$$\int_0^T x^{(n)}(t)x^{(2s)}(t)dt$$

(3.24)

$$= \sum_{i=0}^{2s} b_i \int_0^T \left[x^{(i)}(t)\right]^k x^{(2s)}(t)dt + \int_0^T f(t, x(t - \tau(t)))x^{(2s)}(t)dt + \int_0^T p(t)x^{(2s)}(t)dt.$$

If \(1 < s \leq m\), since

$$\int_0^T x^{(2m+1)}(t)x^{(2s)}(t)dt = 0, \quad \int_0^T \left[x^{(2s-1)}(t)\right]^k x^{(2s)}(t)dt = 0,$$  (3.25)

$$\int_0^T [x(t)]^k x^{(2s)}(t)dt = -k \int_0^T [x(t)]^{k-1} x^{(2s-1)}(t)x'(t)dt,$$  (3.26)
by using Hölder inequality and Lemma 2.1, from (3.23), one has

\[ |b_{2s}| \int_0^T \left| x^{(2s)}(t) \right|^{k+1} dt \]

\[ \leq \int_0^T \left| x^{(2s)}(t) \right| \left( \sum_{i=1}^{2s-2} |b_i| \left| x^{(i)}(t) \right|^k + |f(t, x(t - \tau(t)))| + |p(t)| \right) dt \]

\[ + k|b_0| \int_0^T |x(t)|^{k-1} \left| x'(t) \right| dt \]

\[ \leq \left( \int_0^T \left| x^{(2s)}(t) \right|^{k+1} dt \right)^{1/(k+1)} \left( \sum_{i=1}^{2s-2} |b_i| T^{(2s-i)k} \left( \int_0^T \left| x^{(2s)}(t) \right|^{k+1} dt \right)^{k/(k+1)} \right. \]

\[ + \left. \left( \int_0^T \left| f(t, x(t)) \right|^{(k+1)/k} dt \right)^{k/(k+1)} \right) \]

\[ + \left( \int_0^T \left| f(t, x(t)) - f(t, x(t - \tau)) \right|^{(k+1)/k} dt \right)^{k/(k+1)} \]

\[ + |p(t)|_\infty T^{k/(k+1)} \]

\[ + k|b_0| |x'(t)|_\infty \int_0^T \left| x(t) \right|^{k-1} \left| x^{(2s-1)}(t) \right| dt. \]

Since \( x(0) = x(T) \), there exists \( \xi \in [0, T] \) such that \( x'(\xi) = 0 \). So for \( t \in [0, T] \)

\[ x'(t) = x'(\xi) + \int_\xi^t x''(\sigma) d\sigma. \]  

(3.28)

Using Hölder inequality and Lemma 2.1, one has

\[ \left| x'(t) \right|_\infty \leq \int_0^T \left| x''(t) \right| dt \leq T^{k/(k+1)} \left( \int_0^T \left| x''(t) \right|^{k+1} dt \right)^{1/(k+1)} \]

\[ \leq T^{2s-1-(1/(k+1))} \left( \int_0^T \left| x^{(2s)}(t) \right|^{k+1} dt \right)^{1/(k+1)} \].

Using inequality

\[ \left( \frac{1}{T} \int_0^T \left| t(t) \right|^{r} \right)^{1/r} \leq \left( \frac{1}{T} \int_0^T \left| x(t) \right|^{r} \right)^{1/l} \]

for \( 0 \leq r \leq l, \forall x \in R. \)  

(3.30)
and applying Hölder inequality and by Lemma 2.1, we obtain

\[
\int_0^T |x(t)|^{k-1} |x^{(2s-1)}(t)| dt \leq \left( \int_0^T |x(t)|^k dt \right)^{(k-1)/k} \left( \int_0^T |x^{(2s-1)}(t)|^k dt \right)^{1/k}
\]

\[
\leq T^{1/(k+1)} \left( \int_0^T |x(t)|^{k+1} dt \right)^{(k-1)/(k+1)} \left( \int_0^T |x^{(2s-1)}(t)|^{k+1} dt \right)^{1/(k+1)}
\]

\[
\leq T^{1+1/(k+1)} \left( \int_0^T |x(t)|^{k+1} dt \right)^{(k-1)/(k+1)} \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{1/(k+1)}.
\]

(3.31)

Substituting the above formula, (3.20), (3.27), and (3.30) into (3.26), one has

\[
|b_{2s}| \int_0^T |x^{(2s)}(t)|^{k+1} dt
\]

\[
\leq \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{1/(k+1)} \left\{ A_2(2s, k) + \theta_1 T^{(2s-1)k} \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{k/(k+1)} + \left[ (\gamma + \varepsilon) + \theta_2 \right] \left( \int_0^T |x(t)|^{k+1} dt \right)^{k/(k+1)} + (|p(t)|_{\infty} + f_\delta) T^{k/(k+1)}\right\}
\]

\[
+ k|b_0||T^{2s} \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{2/(k+1)} \left( \int_0^T |x(t)|^{k+1} dt \right)^{(k-1)/(k+1)}.
\]

(3.32)

Then, one has

\[
\left| |b_{2s}| - A_2(2s, k) - \theta_1 T^{(2s-1)k} \right| \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{k/(k+1)}
\]

\[
\leq k|b_0||T^{2s} \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{1/(k+1)} \left( \int_0^T |x(t)|^{k+1} dt \right)^{(k-1)/(k+1)}
\]

\[
+ \left[ (\gamma + \varepsilon) + \theta_2 \right] \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{k/(k+1)} + u_4.
\]

(3.33)
where \( u_4 \) is a positive constant. Using inequality
\[
(a + b)^l \leq a^l + b^l \quad \text{for } a \geq 0, \ b \geq 0, \ 0 \leq l \leq 1,
\]
(3.34)

it follows from (3.33) that
\[
\left( \int_0^T |x(t)|^{k+1} dt \right)^{(k-1)/(k+1)} \leq \left[ \frac{A_1(2s, k) + \theta_1 T (2s - 1) k}{|b_0| - (\gamma + \varepsilon) - \theta_2} \right]^{(k-1)/k} \left( A_1(2s, k) + \theta_1 T (2s - 1) k \right)^{(k-1)/(k+1)}
\]
(3.35)

where \( u_5 \) is a positive constant. Substituting the above formula and (3.33) into (3.33), one has
\[
\left\{ \begin{array}{l}
|b_2| - A_2(2s, k) - \theta_1 T (2s - 1) k - \frac{(\gamma + \varepsilon + \theta_2) (A_1(2s, k) + \theta_1 T (2s - 1) k)}{|b_0| - (\gamma + \varepsilon) - \theta_2} \\
-k |b_0| T^{2s} \left[ \frac{A_1(2s, k) + \theta_1 T (2s - 1) k}{|b_0| - (\gamma + \varepsilon) - \theta_2} \right]^{(k-1)/k} \left( \int_0^T |x^{(2s)}(t)| dt \right)^{k/(k+1)}
\end{array} \right.
\]
(3.36)

\[
\leq u_5 k |b_0| T^{2s} \left( \int_0^T |x^{(2s)}(t)| dt \right)^{1/(k+1)} + u_6,
\]

where \( u_6 \) is a positive constant.

If \( s = 1 \), since \( \int_0^T [x'(t)] x''(t) dt = 0, \int_0^T [x(t)] x''(t) dt = -k \int_0^T [x(t)]^{k-1} [x'(t)]^2 dt \), from (3.24), one has
\[
b_2 \int_0^T [x''(t)]^{k+1} dt = -k b_0 \int_0^T [x(t)]^{k-1} [x'(t)]^2 dt + \int_0^T f(t, x(t - \tau)) x''(t) dt + \int_0^T p(t) x''(t) dt.
\]
(3.37)

Applying the above method, one has
\[
\left\{ \begin{array}{l}
|b_2| - \theta_1 T^k - \frac{(\gamma + \varepsilon + \theta_2) (A_1(2, k) + \theta_1 T^k)}{|b_0| - (\gamma + \varepsilon) - \theta_2} - k |b_0| T^2 \left[ \frac{A_1(2, k) + \theta_1 T^k}{|b_0| - (\gamma + \varepsilon) - \theta_2} \right]^{(k-1)/k}
\end{array} \right.
\]
\[
\times \left( \int_0^T |x''(t)|^{k+1} dt \right)^{k/(k+1)} \leq u_7 k |b_0| T^2 \left( \int_0^T |x''(t)|^{k+1} dt \right)^{1/(k+1)} + u_8,
\]
(3.38)
where \( u_I, u_s \) is a positive constant. Hence there is a constant \( M_1, M_2 > 0 \) such that

\[
\int_0^T |x^{(2s)}(t)|^{k+1} dt \leq M_1, \quad (3.39)
\]
\[
\int_0^T |x(t)|^{k+1} dt \leq M_2. \quad (3.40)
\]

From (3.5), using Hölder inequality and Lemma 2.1, one has

\[
\begin{align*}
\int_0^T |x^{(n)}(t)| dt &\leq \sum_{i=0}^{2s} |b_i| \int_0^T |x^{(i)}(t)|^k dt + \int_0^T |f(t, x(t))| dt \\
&\quad + \int_0^T |f(t, x(t)) - f(t, x(t - \tau(t)))| dt + \int_0^T |p(t)| dt \\
&\leq \sum_{i=1}^{2s} |b_i| T^{(2s-i)k+1/(k+1)} + \theta_1 T^{(2s-1)k+1/(k+1)} \left( \int_0^T |x^{(2s)}(t)|^{k+1} dt \right)^{k/(k+1)} \\
&\quad + \left[ |b_0| + (\gamma + \varepsilon) + \theta_2 \right] T^{1/(k+1)} \left( \int_0^T |x(t)|^{k+1} dt \right)^{k/(k+1)} + (|p(t)|_\infty + f_0)T \\
&\leq \sum_{i=1}^{2s} |b_i| T^{(2s-i)k+1/(k+1)} + \theta_1 T^{(2s-1)k+1/(k+1)} (M_1)^{k/(k+1)} \\
&\quad + |b_0| + (\gamma + \varepsilon) + \theta_2 (M_2)^{k/(k+1)} + (|p(t)|_\infty + f_0)T = M, \quad (3.41)
\end{align*}
\]

where \( M \) is a positive constant. We claim that

\[
|\ x^{(i)}(t) | \leq T^{n-i-1} \int_0^T |x^{(n)}(t)| dt, \quad i = 1, 2, \ldots, n-1. \quad (3.42)
\]

In fact, noting that \( x^{(n-2)}(0) = x^{(n-2)}(T) \), there must be a constant \( \xi_1 \in [0, T] \) such that \( x^{(n-1)}(\xi_1) = 0 \), we obtain

\[
|\ x^{(n-1)}(t) | = |x^{(n-1)}(\xi_1) + \int_{\xi_1}^t x^{(n)}(s) ds| \leq |x^{(n-1)}(\xi_1)| + \int_0^T |x^{(n)}(t)| dt = \int_0^T |x^{(n)}(t)| dt. \quad (3.43)
\]

Similarly, since \( x^{(n-3)}(0) = x^{(n-3)}(T) \), there must be a constant \( \xi_2 \in [0, T] \) such that \( x^{(n-2)}(\xi_2) = 0 \), from (3.43) we get

\[
|\ x^{(n-2)}(t) | = |x^{(n-2)}(\xi_2) + \int_{\xi_2}^t x^{(n-1)}(s) ds| \leq \int_0^T |x^{(n-1)}(t)| dt \leq T \int_0^T |x^{(n)}(t)| dt. \quad (3.44)
\]
By induction, we conclude that (3.42) holds. Furthermore, one has

\[
\left| x^{(i)}(t) \right|_{\infty} \leq T^{n-i-1} \int_0^T \left| x^{(m)}(t) \right| dt \leq T^{n-i-1} M, \quad i = 1, 2, \ldots, n - 1.
\] (3.45)

It follows from (3.39) that there exists a \( \xi \in [0, T] \) such that \( |x(\xi)| \leq M_2^{1/(k+1)} \). Applying Lemma 2.1, we get

\[
|x(t)|_{\infty} \leq x(\xi) + \int_{\xi}^t x'(t)dt \leq M_2^{1/(k+1)}
\]

\[+ T^{k/(k+1)} \left( \int_0^T \left| x'(t) \right|^{k+1} dt \right)^{1/(k+1)},\]

\[
\leq M_2^{1/(k+1)} + T^{2s-1+1/(k+1)} \left( \int_0^T \left| x^{(2s)}(t) \right|^{k+1} dt \right)^{1/(k+1)}, \]

\[= M_2^{1/(k+1)} + T^{2s-1+1/(k+1)} M_1^{1/(k+1)}.\] (3.46)

It follows that there is a constant \( A > 0 \) such that \( \|x\| \leq A. \) Thus \( \Omega_1 \) is bounded.

Let \( \Omega_2 = \{ x \in \text{Ker} L, QNx = 0 \}. \) Suppose \( x \in \Omega_2 \), then \( x(t) = d \in R \) and satisfies

\[
QN x = \frac{1}{T} \int_0^T \left[ -b_0 d^k - f(t, d) + p(t) \right] dt = 0.
\] (3.47)

We will prove that there exists a constant \( B > 0 \) such that \( |d| \leq B. \) If \( |d| \leq \delta \), taking \( \delta = B \), we get \( |d| \leq B. \) If \( |d| > \delta \), from (3.47), one has

\[
|b_0||d|^k = \left| \frac{1}{T} \int_0^T \left[ -f(t, d) + p(t) \right] dt \right|
\]

\[\leq \frac{1}{T} \int_0^T \left| f(t, d) \right| dt + \left| p(t) \right|_{\infty} \leq (y + \varepsilon)|d|^k + \left| p(t) \right|_{\infty}.
\] (3.48)

Thus

\[
|d| \leq \left[ \frac{\left| p(t) \right|_{\infty}}{|b_0| - (y + \varepsilon)} \right]^{1/k}.
\] (3.49)

Taking \( \left[ \frac{\left| p(t) \right|_{\infty}}{|b_0| - (y + \varepsilon)} \right]^{1/k} = B \), one has \( |d| \leq B \), which implies \( \Omega_2 \) is bounded. Let \( \Omega \) be a nonempty open bounded subset of \( X \) such that \( \Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \). We can easily see that \( L \) is a Fredholm operator of index zero and \( N \) is \( L \)-compact on \( \overline{\Omega} \). Then by the above argument, we
have

(i) $Lx \neq \lambda Nx$, for all $x \in \partial \Omega \cap D(L), \lambda \in (0,1),$

(ii) $QNx \neq 0$, for all $x \in \partial \Omega \cap \text{Ker } L$.

At last we will prove that condition (iii) of Lemma 2.4 is satisfied. We take

$$H : (\Omega \cap \text{Ker } L) \times [0,1] \rightarrow \text{Ker } L,$$

$$H(d,\mu) = \mu d + \frac{1-\mu}{T} \int_0^T \left[ -b_0 d^k - f(t,d) + p(t) \right] dt. \quad (3.50)$$

From assumptions $(H_1)$ and $(H_2)$, we can easily obtain $H(d,\mu) \neq 0$, for all $(d,\mu) \in \partial \Omega \cap \text{Ker } L \times [0,1]$, which results in

$$\deg\{QN,\Omega \cap \text{Ker } L,0\} = \deg\{H(.,0),\Omega \cap \text{Ker } L,0\} = \deg\{H(.,1),\Omega \cap \text{Ker } L,0\} \neq 0. \quad (3.51)$$

Hence, by using Lemma 2.2, we know that (1.1) has at least one $T$-periodic solution.

**Theorem 3.2.** Suppose $n = 4m + 1$, $m > 0$ an integer and conditions $(H_1), (H_2)$ hold. If

$(H_5)$ there is a positive integer $0 < s \leq m$ such that

$$b_{4s-3} \neq 0, \quad b_{4s-3+i} = 0, \quad i = 1,2,\ldots,4m - 4s + 3, \quad (3.52)$$

$(H_6)$

$$A_2(4s - 3,k) + \theta_1 T^{(4s-4)k} + \frac{(\gamma + \theta_2)(A_1(4s - 3,k) + \theta_1 T^{(4s-4)k})}{|b_0| - \gamma - \theta_2}$$

$$+ k|b_0| T^{4s-3} \left[ \frac{A_1(4s - 3,k) + \theta_1 T^{4s-4}}{|b_0| - \gamma - \theta_2} \right]^{(k-1)/k} < |b_{4s-3}|, \quad \text{if } 1 < s \leq m, \quad (3.53)$$

$$\theta_1 + \frac{(\gamma + \theta_2)(A_1(1,k) + \theta_1)}{|b_0| - \gamma - \theta_2} < |b_1|, \quad \text{if } s = 1,$$

then (1.1) has at least one $T$-periodic solution.

**Proof.** From the proof of Theorem 3.1, one has

$$\left( \int_0^T |x(t)|^{k+1} dt \right)^{k/(k+1)} \leq \frac{A_1(4s - 3,k) + \theta_1 T^{(4s-4)k}}{|b_0| - (\gamma + \epsilon) - \theta_2} \left( \int_0^T |x(t)|^{k+1} dt \right)^{k/(k+1)} + u_9, \quad (3.54)$$
where $u_0$ is a positive constant. Multiplying both sides of (3.8) by $x^{(4s-3)}(t)$, and integrating on $[0,T]$, one has

$$
\int_0^T x^{(n)}(t)x^{(4s-3)}(t)\,dt = -\lambda \sum_{i=0}^{4s-3} b_i \int_0^T [x^{(i)}(t)]^k x^{(4s-3)}(t)\,dt \\
- \lambda \int_0^T f(t,x(t-\tau))x^{(4s-3)}(t)\,dt + \lambda \int_0^T p(t)x^{(4s-3)}(t)\,dt.
$$

(3.55)

Since

$$
\int_0^T x^{(4m+1)}(t)x^{(4s-3)}(t)\,dt = (-1)^{2m-2s+2} \int_0^T [x^{(2m+2s-1)}(t)]^2 \,dt,
$$

(3.56)

then it follows from (3.55) and (3.56) that

$$
b_{4s-3} \int_0^T \left| x^{(4s-3)}(t) \right|^{k+1} \,dt \leq -\sum_{i=0}^{4s-4} b_i \int_0^T \left[ x^{(i)}(t) \right]^k x^{(4s-3)}(t)\,dt \\
- \int_0^T f(t,x(t-\tau))x^{(4s-3)}(t)\,dt + \int_0^T p(t)x^{(4s-3)}(t)\,dt.
$$

(3.57)

By using the same way as in the proof of Theorem 3.1, the following theorems can be proved in case $1 < s \leq m$ or $s = 1$.

**Theorem 3.3.** Suppose $n = 4m + 1$, $m > 0$ for a positive integer and conditions $(H_1), (H_2)$ hold. If

$$(H_7) \text{ there is a positive integer } 0 < s \leq m \text{ such that }$$

$$b_{4s-1} \neq 0, \quad b_{4s-1+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 1,$$

(3.58)

$$(H_8)$$

$$A_2(4s-1,k) + \theta_1 T^{(4s-2)k} + \frac{(\gamma + \theta_2) (A_1(4s-1,k) + \theta_1 T^{(4s-2)k})}{|b_0| - \gamma - \theta_2}$$

$$+ k|b_0| T^{4s-1} \left[ \frac{A_1(4s-1,k) + \theta_1 T^{(4s-2)k}}{|b_0| - \gamma - \theta_2} \right]^{(k-1)/k} < |b_{4s-1}|,$$

(3.59)

then (1.1) has at least one $T$-periodic solution.

**Theorem 3.4.** Suppose $n = 4m + 3$, $m \geq 0$ an integer and conditions $(H_1), (H_2)$ hold. If

$$(H_9) \text{ there is a positive integer } 0 \leq s \leq m \text{ such that }$$

$$b_{4s+1} \neq 0, \quad b_{4s+1+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 1,$$

(3.60)
(H_{10})

\[ A_2(4s + 1, k) + \theta_1 T^{4sk} + \frac{(\gamma + \vartheta_2)(A_1(4s + 1, k) + \theta_1 T^{4sk})}{|b_0| - \gamma - \vartheta_2} \]

\[ + k|b_0|T^{4s+1} \left[ \frac{A_1(4s + 1, k) + \theta_1 T^{4sk}}{|b_0| - \gamma - \vartheta_2} \right]^{(k-1)/k} < |b_{4s+1}|, \quad \text{if} \quad 0 < s \leq m, \]

\[ \theta_1 + \frac{(\gamma + \vartheta_2)(A_1(1, k) + \vartheta_1)}{|b_0| - \gamma - \vartheta_2} < |b_1|, \quad \text{if} \quad s = 0, \]

(3.61)

then (1.1) has at least one T-periodic solution.

**Theorem 3.5.** Suppose \( n = 4m + 3, m > 0 \) an integer and conditions \((H_1), (H_2)\) hold. If

\((H_{11})\) there is a positive integer \( 0 < s \leq m \) such that

\[ b_{4s-1} \neq 0, \quad b_{4s-1+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 3, \]

(3.62)

\((H_{12})\)

\[ A_2(4s - 1, k) + \theta_1 T^{(4s-2)k} + \frac{(\gamma + \vartheta_2)(A_1(4s - 1, k) + \theta_1 T^{(4s-2)k})}{|b_0| - \gamma - \vartheta_2} \]

\[ + k|b_0|T^{4s-1} \left[ \frac{A_1(4s - 1, k) + \theta_1 T^{(4s-2)k}}{|b_0| - \gamma - \vartheta_2} \right]^{(k-1)/k} < |b_{4s-1}|, \]

(3.63)

then (1.1) has at least one T-periodic solution.

**Theorem 3.6.** Suppose \( n = 4m, m > 0 \) an integer and conditions \((H_1)\) hold. If

\((H_{13})\)

\[ b_0 > \gamma + \vartheta_2, \]

(3.64)

\((H_{14})\) there is a positive integer \( 0 < s \leq 2m \) such that

\[ b_{2s-1} \neq 0, \quad \text{if} \quad s = 2m, \]

\[ b_{2s-1} \neq 0, \quad b_{2s-1+i} = 0, \quad i = 1, 2, \ldots, 4m - 2s, \quad \text{if} \quad 0 < s < 2m, \]

(3.65)
Suppose Theorem 3.8.

Theorem 3.7. Suppose \( n = 4m + 2, m > 0 \) an integer and conditions \((H_1)\) hold. If

\[(H_{16})\]

\[-b_0 > \gamma + \theta_2,\]  \hspace{1cm} (3.67)

\[(H_{17})\] there is a positive integer \( 0 < s \leq 2m + 1 \) such that

\[(H_{18})\]

\[-kb_0T^{2s-1} \left[ \frac{A_1(2s - 1, k) + \theta_1T^{(2s-2)k}}{b_0 - \gamma - \theta_2} \right]^{(k-1)/k} < b_{2s-1}, \quad \text{if } 1 < s \leq 2m + 1,\]  \hspace{1cm} (3.69)

then (1.1) has at least one \( T \)-periodic solution.

Theorem 3.8. Suppose \( n = 4m, m > 0 \) is an integer, and conditions \((H_1), (H_{13})\) hold. If

\[(H_{19})\] there is a positive integer \( 0 < s \leq m \) such that

\[(H_{20})\]

\[b_{4s-2} \neq 0, \quad b_{4s-2+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 1,\]  \hspace{1cm} (3.70)
Theorem 3.9. Suppose $(H_{20})$

\[
A_2(4s - 2, k) + \theta_1 T^{(4s-3)k} + \frac{(\gamma + \theta_2)(A_1(4s - 2, k) + \theta_1 T^{(4s-3)k})}{b_0 - \gamma - \theta_2} 
+ k b_0 T^{4s-2} \left[ \frac{A_1(4s - 2, k) + \theta_1 T^{(4s-3)k}}{b_0 - \gamma - \theta_2} \right]^{(k-1)/k} < |b_{4s-2}|, \quad \text{if } 1 < s \leq m, \quad (3.71)
\]

\[
(\gamma + \theta_2)(A_1(2, k) + \theta_1 T^k) \frac{b_0 - \gamma - \theta_2}{b_0 - \gamma - \theta_2} + k b_0 T^2 \left[ \frac{A_1(2, k) + \theta_1 T^k}{b_0 - \gamma - \theta_2} \right]^{(k-1)/k} < |b_2|, \quad \text{if } s = 1,
\]

then (1.1) has at least one $T$-periodic solution.

Theorem 3.10. Suppose $n = 4m$, $m > 1$ an integer and conditions $(H_1), (H_{15})$ hold. If

$(H_{21})$ there is a positive integer $1 < s \leq m$ such that

\[
b_{4s-4} \neq 0, \quad b_{4s-4+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 3, \quad (3.72)
\]

then (1.1) has at least one $T$-periodic solution.

Theorem 3.10. Suppose $n = 4m + 2$, $m \geq 1$ an integer and conditions $(H_1), (H_{16})$ hold. If

$(H_{23})$ there is a positive integer $1 \leq s \leq m$ such that

\[
b_{4s} \neq 0, \quad b_{4s+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 1, \quad (3.74)
\]

then (1.1) has at least one $T$-periodic solution.
Theorem 3.11. Suppose \( n = 4m + 2, m \geq 1 \) is an integer, and conditions (\( H_1 \)), (\( H_{16} \)) hold. If 

\[(H_{25}) \text{there is a positive integer } 1 \leq s \leq m \text{ such that}\]

\[
b_{4s-2} \neq 0, \quad b_{4s-2+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 3,
\]

\[(H_{26})\]

\[
A_2(4s - 2, k) + \theta_1 T^{(4s-3)k} \frac{(y + \theta_2)(A_1(4s - 2, k) + \theta_1 T^{(4s-3)k})}{-b_0 - y - \theta_2} < |b_{4s-2}|, \quad \text{if } 1 < s \leq m,
\]

\[
\theta_1 T^k + \frac{(y + \theta_2)(A_1(2, k) + \theta_1 T^k)}{-b_0 - y - \theta_2} - kb_0 T^2 \left[ \frac{A_1(2, k) + \theta_1 T^k}{-b_0 - y - \theta_2} \right]^{(k-1)/k} < |b_2|, \quad \text{if } s = 1,
\]

then (1.1) has at least one \( T \)-periodic solution.

The proofs of Theorem 3.3–3.11 are similar to that of Theorem 3.1.

Example 3.12. Consider the following equation:

\[
x^{(5)}(t) + 300\left[x''(t)\right]^3 + \frac{1}{50}\left[x'(t)\right]^3 + \frac{1}{100}\left[x(t)\right]^3 + \frac{1}{300}(\sin t)\left[x(t - \frac{\pi}{10})\right]^3 = \cos t,
\]

where \( n = 5, k = 3, b_4 = b_3 = 0, b_2 = 300, b_1 = 1/50, b_0 = 1/100, f(t, x) = 1/300(\sin t)\chi^3, p(t) = \cos t, \tau(t) = \pi/10. \) Thus, \( T = 2\pi, \gamma = 1/300, A_1(2, k) = |b_1|(2\pi)^3 + |b_2| = 1/50 \times (2\pi)^3 + 200. \) Obviously assumptions (\( H_1 \))–(\( H_3 \)) hold and

\[
\theta_1 T^k + \frac{(y + \theta_2)(A_1(2, k) + \theta_1 T^k)}{|b_0| - y - \theta_2} + k|b_0|(2\pi)^2 \left[ \frac{A_1(2, k) + \theta_1 T^k}{|b_0| - y - \theta_2} \right]^{(k-1)/k} < |b_2|.
\]

By Theorem 3.1, we know that (3.78) has at least one \( 2\pi \)-periodic solution.

References


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