Research Article

Existence of Positive Periodic Solutions for Periodic Neutral Lotka-Volterra System with Distributed Delays and Impulses

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By using a fixed-point theorem of strict-set-contraction, we investigate the existence of positive periodic solutions for a class of the following impulsive neutral Lotka-Volterra system with distributed delays: $x'_i(t) = x_i(t)[r_i(t) - \sum_{j=1}^n a_{ij}(t)x_j(t) - \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi)x_j(t+\xi)d\xi - \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi)x'_j(t+\xi)d\xi]$, $\Delta x_i(t_k) = -I_{ik}(x_i(t_k))$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots$. Some verifiable criteria are established easily.

1. Introduction

It is well known that natural environments are physically highly variable, and in response, birth rates, death rates, and other vital rates of populations vary greatly in time. Theoretical evidence suggests that many population and community patterns represent intricate interactions between biology and variation in the physical environment (see [1–4]). Thus, the focus in theoretical models of population and community dynamics must be not only on how populations depend on their own population densities or the population densities of other organisms, but also on how populations change in response to the physical environment. It is reasonable to study the models of population with periodic coefficients. In addition to the theoretical and practical significance, the Lotka-Volterra model is one of the famous models for dynamics of population; therefore it has been studied extensively [5–9]. In view of the above effects, by applying a fixed-point theorem of strict-set-contraction, Li [10] established criteria to guarantee the existence of positive periodic solutions of the following neutral Lotka-Volterra system with distributed delays:

$$x'_i(t) = x_i(t) \left[a_i(t) - \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 K_{ij}(\theta) x_j(t+\theta) d\theta \right.$$

$$\left. - \sum_{j=1}^n c_{ij}(t) \int_{-\hat{\tau}_{ij}}^0 \widehat{K}_{ij}(\theta) x'_j(t+\theta) d\theta \right],$$

$$i = 1, 2, \dots, n, \quad (1)$$

where $a_i, b_{ij}, c_{ij} \in C(R, R^+)$ ($i, j = 1, 2, \dots, n$) are ω -periodic functions and $T_{ij}, \hat{T}_{ij} \in (0, +\infty)$ ($i, j = 1, 2, \dots, n$) and $K_{ij}, \widehat{K}_{ij} \in (R, R^+)$ satisfying $\int_{-T_{ij}}^0 K_{ij}(\theta) d\theta = 1$, $\int_{-\hat{T}_{ij}}^0 \widehat{K}_{ij}(\theta) d\theta = 1$, $i, j = 1, 2, \dots, n$. On the other side, birth of many species is an annual birth pulse or harvesting. To have a more accurate description of many mathematical ecology systems, we need to consider the use of impulsive differential equations [11–13]. Some qualitative properties such as oscillation, periodicity, asymptotic behavior, and stability properties have been investigated extensively by many authors over the past few years [14–18]. However, to our knowledge, there are few published papers discussing the existence of periodic solutions for neutral Lotka-Volterra system with distributed delays and impulses. In this paper, we are concerned with the following neutral Lotka-Volterra system with distributed delays and impulses:

$$\begin{aligned}
x_i'(t) = x_i(t) & \left[r_i(t) - \sum_{j=1}^n a_{ij}(t) x_j(t) \right. \\
& - \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \\
& \left. - \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x_j'(t+\xi) d\xi \right], \\
& i = 1, 2, \dots, n, \quad t \neq t_k,
\end{aligned} \quad (2)$$

$$\Delta x_i(t_k) = -I_{ik}(x_i(t_k)), \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots,$$

where $r_i, a_{ij}, b_{ij}, c_{ij} \in C(R, R^+)$ ($i, j = 1, 2, \dots, n$) are ω -periodic functions and $\tau_{ij}, \sigma_{ij} \in (0, +\infty)$ ($i, j = 1, 2, \dots, n$) and $f_{ij}, g_{ij} \in C(R, R^+)$ satisfying $\int_{-\tau_{ij}}^0 f_{ij}(\xi) d\xi = 1$, $\int_{-\sigma_{ij}}^0 g_{ij}(\xi) d\xi = 1$, $i, j = 1, 2, \dots, n$. Moreover, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ (here $x(t_k^+)$ represents the right limit of $x(t)$ at the point t_k), $I_{ik} \in C(R^+, R^+)$; that is, x changes decreasingly suddenly at times t_k ; $\omega > 0$ is a constant, $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$. We assume that there exists an integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $I_{i(k+q)} = I_{ik}$, where $0 < t_1 < t_2 < \dots < t_q < \omega$.

The main purpose of this paper is by using a fixed-point theorem of strict-set-contraction [19, 20] to establish new criteria to guarantee the existence of positive periodic solutions of the system (2).

For convenience, we introduce the notation

$$\begin{aligned}
f^M &= \max_{t \in [0, \omega]} \{|f(t)|\}, \quad f^L = \min_{t \in [0, \omega]} \{|f(t)|\}, \\
\gamma_i &= \limsup_{u \rightarrow 0} \sum_{t \leq t_k \leq t+\omega} \frac{I_{ik}(u)}{u}, \quad i = 1, 2, \dots, n, \\
\delta_i &= e^{-\int_0^\omega r_i(t) dt}, \quad i = 1, 2, \dots, n, \\
\alpha_{ij} &= \int_0^\omega [\delta_j a_{ij}(t) - b_{ij}(t) - c_{ij}(t)] dt,
\end{aligned} \quad (3)$$

$$\beta_{ij} = \int_0^\omega [a_{ij}(t) + b_{ij}(t) + c_{ij}(t)] dt, \quad i, j = 1, 2, \dots, n,$$

where f is a continuous ω -periodic function.

Throughout this paper, we assume that

$$\begin{aligned}
(A_1) \quad & \delta_i = e^{-\int_0^\omega r_i(t) dt} < 1, \quad i = 1, 2, \dots, n; \\
(A_2) \quad & \delta_j a_{ij}(t) - b_{ij}(t) - c_{ij}(t) \geq 0, \quad i, j = 1, 2, \dots, n; \\
(A_3) \quad & (1 + r_i^L)(\delta_i^2/(1 - \delta_i))\alpha_{ij} \geq \max_{t \in [0, \omega]} \{a_{ij}(t) + b_{ij}(t) + c_{ij}(t)\}, \quad i, j = 1, 2, \dots, n; \\
(A_4) \quad & (r_i^M - 1)(1/\delta_i(1 - \delta_i))\beta_{ij} \leq \min_{t \in [0, \omega]} \{\delta_j a_{ij}(t) - b_{ij}(t) - c_{ij}(t)\}, \quad i, j = 1, 2, \dots, n; \\
(A_5) \quad & \max_{1 \leq i \leq n} \{\sum_{j=1}^n c_{ij}^M\} \leq \min_{1 \leq i \leq n} \{(\delta_i^2/(1 - \delta_i))\} \\
& \min_{1 \leq j \leq n} \{\alpha_{ij}\}.
\end{aligned}$$

The paper is organized as follows. In Section 2, we give some definitions and lemmas to prove the main results of this paper. In Section 3, by using a fixed-point theorem of strict-set-contraction, we established some criteria to guarantee the existence of at least one positive periodic solution of system (2). Finally, in Section 4, we give an example to show the validity of our result.

2. Preliminaries

In order to obtain the existence of a periodic solution of system (2), we first introduce some definitions and lemmas.

Definition 1 (see [13]). A function $x_i : R \rightarrow (0, +\infty)$ is said to be a positive solution of (2), if the following conditions are satisfied:

- (a) $x_i(t)$ is absolutely continuous on each (t_k, t_{k+1}) ;
- (b) for each $k \in Z_+$, $x_i(t_k^+)$ and $x_i(t_k^-)$ exist and $x_i(t_k^-) = x_i(t_k)$;
- (c) $x_i(t)$ satisfies the first equation of (2) for almost everywhere in R and $x_i(t_k)$ satisfies the second equation of (2) at impulsive point t_k , $k \in Z_+$.

Definition 2 (see [21]). Let X be a real Banach space and E a closed, nonempty subset of X . E is a cone provided

- (i) $\alpha x + \beta y \in E$ for all $x, y \in E$ and all $\alpha, \beta \geq 0$,
- (ii) $x, -x \in E$ imply $x = 0$.

Definition 3 (see [21]). Let A be a bounded subset in X . Define

$$\begin{aligned}
\alpha_X(A) &= \inf \left\{ \delta > 0 : \text{there is a finite number} \right. \\
& \quad \text{of subsets } A_i \subset A \text{ such that} \\
& \quad \left. A = \bigcup_i A_i \text{ and } \text{diam}(A_i) \leq \delta \right\},
\end{aligned} \quad (4)$$

where $\text{diam}(A_i)$ denotes the diameter of the set A_i , obviously, $0 \leq \alpha_X(A) < \infty$. So $\alpha_X(A)$ is called the Kuratowski measure of noncompactness of X .

Definition 4 (see [21]). Let X, Y be two Banach spaces and $D \subset X$, a continuous and bounded map $T : D \rightarrow Y$, is called k -set contractive if, for any bounded set $S \subset D$, we have

$$\alpha_Y(T(S)) \leq k\alpha_X(S); \quad (5)$$

ϕ is called strict-set-contractive if it is k -set-contractive for some $0 \leq k < 1$.

Definition 5 (see [22]). The set $F \in PC_\omega$ is said to be quasi-equicontinuous in $[0, \omega]$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in F$, $k \in N^+$, $t_1, t_2 \in (t_{k-1}, t_k) \cap [0, \omega]$, and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.

Lemma 6 (see [22]). *The set $F \subset PC_\omega$ is relatively compact if and only if*

(1) F is bounded; that is, $\|x\| \leq M$, for each $x \in F$, and some $M > 0$;

(2) F is quasi-equicontinuous in $[0, \omega]$.

Lemma 7. $x_i(t)$ is an ω -periodic solution of (2) is equivalent to $x_i(t)$ is an ω -periodic solution of the following equation:

$$\begin{aligned} x_i(t) = & \int_t^{t+\omega} G_i(t, s) x_i(s) \\ & \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\ & \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \\ & + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)), \end{aligned} \quad (6)$$

where

$$G_i(t, s) = \frac{e^{-\int_0^s r_i(\theta) d\theta}}{1 - e^{-\int_0^\omega r_i(\theta) d\theta}}, \quad s \in [t, t+\omega], \quad i = 1, 2, \dots, n. \quad (7)$$

Proof. Assume that $x_i(t) \in X$, $i = 1, 2, \dots, n$, is a periodic solution of (2). Then, we have

$$\begin{aligned} & \frac{d}{dt} \left[x_i(t) e^{-\int_0^t r_i(\theta) d\theta} \right] \\ & = e^{-\int_0^t r_i(\theta) d\theta} x_i(t) \\ & \times \left[-\sum_{j=1}^n a_{ij}(t) x_j(t) - \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \right. \\ & \quad \left. - \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \right], \\ & \quad t \neq t_k, \quad i = 1, 2, \dots, n. \end{aligned} \quad (8)$$

Integrating the previous equation with $[t, t+\omega]$, we can have

$$\begin{aligned} & x_i(s) e^{-\int_0^s r_i(\theta) d\theta} \Big|_t^{t_{m_1}+\omega} + x_i(s) e^{-\int_0^s r_i(\theta) d\theta} \Big|_{t_{m_1}+\omega}^{t_{m_2}+\omega} \\ & + \dots + x_i(s) e^{-\int_0^s r_i(\theta) d\theta} \Big|_{t_{m_q}+\omega}^{t+\omega} \\ & = \int_t^{t+\omega} e^{-\int_0^s r_i(\theta) d\theta} x_i(s) \\ & \times \left[-\sum_{j=1}^n a_{ij}(s) x_j(s) - \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\ & \quad \left. - \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds, \end{aligned} \quad (9)$$

where $t_{m_k} + \omega \in (t, t+\omega)$, $m_k \in \{1, 2, \dots, q\}$, $k = 1, 2, \dots, q$, $n \in \mathbb{Z}_+$. Therefore,

$$\begin{aligned} & x_i(t) e^{-\int_0^t r_i(\theta) d\theta} \left[1 - e^{-\int_t^{t+\omega} r_i(\theta) d\theta} \right] \\ & + \sum_{t \leq t_k < t+\omega} \Delta x_i(t_{m_k}) e^{-\int_0^{t_{m_k}+\omega} r_i(\theta) d\theta} \\ & = \int_t^{t+\omega} e^{-\int_0^s r_i(\theta) d\theta} x_i(s) \\ & \times \left[\sum_{j=1}^n a_{ij}(t) x_j(t) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\ & \quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds, \end{aligned} \quad (10)$$

which can be transformed into

$$\begin{aligned} x_i(t) = & \int_t^{t+\omega} G_i(t, s) x_i(s) \\ & \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\ & \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \\ & + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)), \quad i = 1, 2, \dots, n. \end{aligned} \quad (11)$$

Thus, x_i is a periodic solution of (6). If $x_i(t) \in E$, $i = 1, 2, \dots, n$, is a periodic solution of (6), for any $t = t_k$, from (6) we have

$$\begin{aligned}
 x'(t) &= \frac{d}{dt} \left\{ \int_t^{t+\omega} G_i(t, s) x_i(s) \right. \\
 &\quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) \right. \\
 &\quad \left. + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \Big\} \\
 &= \left[G_i(t, t+\omega) x_i(t+\omega) \right. \\
 &\quad \times \left(\sum_{j=1}^n a_{ij}(t+\omega) x_j(t+\omega) \right. \\
 &\quad + \sum_{j=1}^n b_{ij}(t+\omega) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\omega+\xi) d\xi \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(t+\omega) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\omega+\xi) d\xi \right) \\
 &\quad - G_i(t, t) x_i(t) \\
 &\quad \times \left(\sum_{j=1}^n a_{ij}(t) x_j(t) + \sum_{j=1}^n b_{ij}(t) \right. \\
 &\quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \right) \Big] \\
 &\quad + r_i(t) x_i(t) \\
 &= x_i(t) \left[r_i(t) - \sum_{j=1}^n a_{ij}(t) x_j(t) \right. \\
 &\quad - \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t-\xi) d\xi \\
 &\quad \left. - \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \right]. \tag{12}
 \end{aligned}$$

For any $t = t_j$, $j \in Z_+$, we have from (6) that

$$\begin{aligned}
 &x_i(t_j^+) - x_i(t_j) \\
 &= \int_{t_j}^{t_j+\omega} [G_i(t_j^+, s) - G_i(t_j, s)] x_i(s) \\
 &\quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \\
 &\quad + \sum_{t_j^+ \leq t_k < t_j+\omega} G_i(t_j^+, t_k) I_{ik}(x_i(t_k)) \\
 &\quad - \sum_{t_j \leq t_k < t_j+\omega} G_i(t_j, t_k) I_{ik}(x_i(t_k)) \\
 &= -I_{ik}(x(t_k)). \tag{13}
 \end{aligned}$$

Hence $x_i(t)$ is a positive ω -periodic solution of (2). Thus we complete the proof of Lemma 7. $\square$

Lemma 8 (see [20, 23]). *Let E be a cone of the real Banach space X and $E_{r,R} = \{x \in E : r \leq \|x\| \leq R\}$ with $0 < r < R$. Assume that $A : E_{r,R} \rightarrow E$ is strict-set-contractive such that one of the following two conditions is satisfied:*

- (a) $Ax \not\leq x$, for all $x \in E$, $\|x\| = r$ and $Ax \not\geq x$, for all $x \in E$, $\|x\| = R$;
- (b) $Ax \not\geq x$, for all $x \in E$, $\|x\| = r$ and $Ax \not\leq x$, for all $x \in E$, $\|x\| = R$.

Then A has at least one fixed point in $E_{r,R}$.

In order to apply Lemma 7 to system (1), one sets

$$\begin{aligned}
 PC(R) &= \{x = (x_1, x_2, \dots, x_n)^T : \\
 &\quad R \longrightarrow R \mid x \in C((t_k, t_{k+1}), R), \\
 &\quad \exists x(t_k^-) = x(t_k), x(t_k^+), k \in N\}, \\
 PC^1(R) &= \{x = (x_1, x_2, \dots, x_n)^T : \\
 &\quad R \longrightarrow R \mid x \in C^1((t_k, t_{k+1}), R), \\
 &\quad \exists x'(t_k^-) = x'(t_k), x(t_k^+), k \in N, t \in R\}. \tag{14}
 \end{aligned}$$

Define

$$X = \{x = (x_1, x_2, \dots, x_n)^T : x \in PC(R) \mid x(t+\omega) = x(t)\} \tag{15}$$

with the norm defined by $\|x\|_0 = \sum_{i=1}^n |x_i|_0$, where $|x_i|_0 = \sup_{t \in [0, \omega]} \{|x_i(t)|\}$, $i = 1, 2, \dots, n$, and

$$Y = \{x = (x_1, x_2, \dots, x_n)^T : x \in PC^1(R) \mid x(t + \omega) = x(t), t \in R\} \quad (16)$$

with the norm defined by $\|x\|_1 = \sum_{i=1}^n |x_i|_1$, where $|x_i|_1 = \max\{|x_i|_0, |x'_i|_0\}$. Then X and Y are both Banach spaces. Define the cone E in Y by

$$E = \{x = (x_1, x_2, \dots, x_n)^T : x \in PC^1(R) \mid x_i(t) = \delta_j |x_i(t)|_1, t \in [0, \omega], i, j = 1, 2, \dots, n\}. \quad (17)$$

Let the map T be defined by

$$(Tx)(t) = ((T_1x)(t), (T_2x)(t), \dots, (T_nx)(t))^T, \quad (18)$$

where $x \in E, t \in R$,

$$\begin{aligned} (T_i x)(t) &= \int_t^{t+\omega} G_i(t, s) x_i(s) \\ &\times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\ &\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \\ &\quad + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)), \end{aligned} \quad (19)$$

where

$$G_i(t, s) = \frac{e^{-\int_t^s r_i(\theta) d\theta}}{1 - e^{-\int_0^\omega r_i(\theta) d\theta}}, \quad s \in [t, t + \omega], i = 1, 2, \dots, n. \quad (20)$$

It is obvious to see that $G_i(t + \omega, s + \omega) = G_i(t, s)$, $\partial G_i(t, s) / \partial t = r_i(t) G_i(t, s)$, $G_i(t, t + \omega) - G_i(t, t) = -1$, and

$$\frac{\delta_i}{1 - \delta_i} \leq G_i(t, s) \leq \frac{1}{1 - \delta_i}, \quad s \in [t, t + \omega], i = 1, 2, \dots, n. \quad (21)$$

In what follows, we will give some lemmas concerning E and A defined by (17) and (18), respectively.

Lemma 9. Assume that $(A_1) - (A_3)$ hold.

(i) If $\max\{r_i^M, i = 1, 2, \dots, n\} \leq 1$, then $T : E \rightarrow E$ is well defined.

(ii) If (A_4) holds and $\max\{r_i^M, i = 1, 2, \dots, n\} > 1$, then $T : E \rightarrow E$ is well defined.

Proof. For any $x \in E$, it is clear that $Tx \in PC^1(R)$. From (18), for $t \in [0, \omega]$, we have

$$\begin{aligned} (T_i x)(t + \omega) &= \int_{t+\omega}^{t+2\omega} G_i(t + \omega, s) x_i(s) \\ &\times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s + \xi) d\xi \right. \\ &\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s + \xi) d\xi \right] ds \\ &\quad + \sum_{t+\omega \leq t_k < t+2\omega} G_i(t + \omega, t_k) I_{ik}(x_i(t_k)) \\ &= \int_t^{t+\omega} G_i(t + \omega, u + \omega) x_i(u + \omega) \\ &\times \left[\sum_{j=1}^n a_{ij}(u + \omega) x_j(u + \omega) + \sum_{j=1}^n b_{ij}(u + \omega) \right. \\ &\quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(u + \omega + \xi) d\xi \\ &\quad \left. + \sum_{j=1}^n c_{ij}(u + \omega) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(u + \omega + \xi) d\xi \right] ds \\ &\quad + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)) \\ &= \int_t^{t+\omega} G_i(t, u) x_i(u) \\ &\times \left[\sum_{j=1}^n a_{ij}(u) x_j(u) + \sum_{j=1}^n b_{ij}(u) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(u + \xi) d\xi \right. \\ &\quad \left. + \sum_{j=1}^n c_{ij}(u) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(u + \xi) d\xi \right] ds \\ &\quad + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)) \\ &= (T_i x)(t), \quad i = 1, 2, \dots, n. \end{aligned} \quad (22)$$

That is, $(T_i x)(t + \omega) = (T_i x)(t)$, $t \in [0, \omega]$. So $Tx \in Y$. In view of (A_2) , for $x \in E$, $t \in [0, \omega]$, we have

$$\begin{aligned}
& \sum_{j=1}^n a_{ij}(t) x_j(t) + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \\
& + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \\
& \geq \sum_{j=1}^n a_{ij}(t) \delta_j |x_j|_1 - \sum_{j=1}^n b_{ij}(t) |x_j|_1 - \sum_{j=1}^n c_{ij}(t) |x_j|_1 \\
& = \sum_{j=1}^n [a_{ij}(t) \delta_j - b_{ij}(t) - c_{ij}(t)] |x'_j|_1 \geq 0, \quad i=1, 2, \dots, n.
\end{aligned} \tag{23}$$

Therefore, for $x \in E$, $t \in [0, \omega]$, we find

$$\begin{aligned}
& |T_i x|_0 \\
& \leq \frac{1}{1-\delta_i} \left\{ \int_0^\omega x_i(s) \right. \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \right. \\
& \quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \\
& \quad \left. \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \right. \\
& \quad \left. + \sum_{t \leq t_k < t+\omega} I_{ik}(x_i(t_k)) \right\}, \\
& (T_i x)(t) \\
& \geq \frac{\delta_i}{1-\delta_i} \left\{ \int_t^{t+\omega} x_i(s) \right. \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) \right. \\
& \quad + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \\
& \quad \left. \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \right. \\
& \quad \left. + \sum_{t \leq t_k < t+\omega} I_{ik}(x_i(t_k)) \right\}
\end{aligned} \tag{24}$$

$$\begin{aligned}
& = \frac{\delta_i}{1-\delta_i} \\
& \quad \times \left\{ \int_0^\omega x_i(s) \right. \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) \right. \\
& \quad + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \\
& \quad \left. \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \right. \\
& \quad \left. + \sum_{t \leq t_k < t+\omega} I_{ik}(x_i(t_k)) \right\} \\
& \geq \delta_i |T_i x|_0, \quad i=1, 2, \dots, n.
\end{aligned} \tag{25}$$

Now, we show that $(T_i x)(t) \geq \delta_i |T_i x|_1$, $i=1, 2, \dots, n$, $t \in [0, \omega]$. From (18), we obtain

$$\begin{aligned}
& (T_i x)'(t) \\
& = G_i(t, t+\omega) x_i(t+\omega) \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(t+\omega) x_j(t+\omega) \right. \\
& \quad + \sum_{j=1}^n b_{ij}(t+\omega) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\omega+\xi) d\xi \\
& \quad \left. + \sum_{j=1}^n c_{ij}(t+\omega) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\omega+\xi) d\xi \right] \\
& - G_i(t, t) x_i(t) \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(t) x_j(t) + \sum_{j=1}^n b_{ij}(t) \right. \\
& \quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \\
& \quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \right] \\
& + r_i(t) (T_i x)(t) \\
& = r_i(t) (T_i x)(t) - x_i(t)
\end{aligned}$$

$$\begin{aligned}
& \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \right. \\
& \quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x_j'(t+\xi) d\xi \right], \\
& \quad i = 1, 2, \dots, n.
\end{aligned} \tag{26}$$

It follows from (23) and (25) that if $(T_i x)'(t) \geq 0$, $i = 1, 2, \dots, n$, then

$$\begin{aligned}
(T_i x)'(t) & \leq r_i(t) (T_i x)(t) \leq r_i^M (T_i x)(t) \leq (T_i x)(t), \\
& \quad i = 1, 2, \dots, n.
\end{aligned} \tag{27}$$

On the other hand, from (25) and (A_3) , if $(T_i x)'(t) < 0$, $i = 1, 2, \dots, n$, then

$$\begin{aligned}
& - (T_i x)'(t) \\
& = -r_i(t) (T_i x)(t) + x_i(t) \\
& \times \left[\sum_{j=1}^n a_{ij}(t) x_j(t) + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \right. \\
& \quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x_j'(t+\xi) d\xi \right] \\
& \leq |x_i|_1 \sum_{j=1}^n [a_{ij}(t) + b_{ij}(t) + c_{ij}(t)] |x_i|_1 - r_i^L (T_i x)(t) \\
& \leq (1 + r_i^L) \frac{\delta_i^2}{1 - \delta_i} |x_i|_1 \\
& \times \sum_{j=1}^n \left\{ \int_0^\omega [\delta_j a_{ij}(s) - b_{ij}(s) - c_{ij}(s)] |x_i|_1 ds \right\} \\
& - r_i^L (T_i x)(t) \\
& = (1 + r_i^L) \int_t^{t+\omega} \frac{\delta_i}{1 - \delta_i} \delta_i |x_i|_1 \\
& \times \sum_{j=1}^n [\delta_j a_{ij}(s) - b_{ij}(s) - c_{ij}(s)] |x_i|_1 ds - r_i^L (T_i x)(t) \\
& \leq (1 + r_i^L) \left\{ \int_t^{t+\omega} \int_t^{t+\omega} G_i(t, s) x_i(s) \right. \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \right. \\
& \quad \times \left. \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right.
\end{aligned}$$

It follows from (27) and (28) that $|(T_i x)'|_0 \leq |T_i x|_0$, $i = 1, 2, \dots, n$. So $|T_i x|_1 = |T_i x|_0$, $i = 1, 2, \dots, n$. By (25), we have $(T_i x)(t) \geq \delta_i |T_i x|_1$, $i = 1, 2, \dots, n$. Hence, $Tx \in E$. This completes the proof of (i). In view of the proof of (ii), we only need to prove that $(T_i x)'(t) \geq 0$, $i = 1, 2, \dots, n$, implies

$$(T_i x)'(t) \leq (T_i x)(t), \quad i = 1, 2, \dots, n. \tag{29}$$

From (23), (26), (A_2) , and (A_4) , we have

$$\begin{aligned}
& (T_i x)'(t) \\
& = r_i(t) (T_i x)(t) - x_i(t) \\
& \times \left[\sum_{j=1}^n a_{ij}(t) x_j(t) - \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \right. \\
& \quad \left. - \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x_j'(t+\xi) d\xi \right] \\
& \leq r_i(t) (T_i x)(t) - \delta_i |x_i|_1 \\
& \times \sum_{j=1}^n [\delta_j a_{ij}(t) - b_{ij}(t) - c_{ij}(t)] |x_j|_1 \\
& \leq r_i^M (T_i x)(t) - \delta_i |x_i|_1 \frac{r_i^M - 1}{\delta_i (1 - \delta_i)} \\
& \times \sum_{j=1}^n \left\{ \int_0^\omega [a_{ij}(s) + b_{ij}(s) + c_{ij}(s)] |x_j|_1 ds \right\} \\
& = r_i^M (T_i x)(t) - (r_i^M - 1) \int_t^{t+\omega} \frac{1}{1 - \delta_i} |x_i|_1 \\
& \times \sum_{j=1}^n [a_{ij}(s) + b_{ij}(s) + c_{ij}(s)] |x_j|_1 ds \\
& \leq r_i^M (T_i x)(t) - (r_i^M - 1)
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \int_t^{t+\omega} G_i(t, s) x_i(s) \right. \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \right. \\
& \quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \\
& \quad \left. \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \right. \\
& \quad \left. + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)) \right\} \\
& = r_i^M (T_i x)(t) - (r_i^M - 1) (T_i x)(t) = (T_i x)(t), \\
& \quad i = 1, 2, \dots, n.
\end{aligned} \tag{30}$$

The proof of (ii) is complete. Thus we complete the proof of Lemma 9. $\square$

Lemma 10. Assume that $(A_1)-(A_3)$ hold and $R \max_{1 \leq j \leq n} \{\sum_{i=1}^n b_{ij}^M\} < 1$.

(i) If $\max\{r_i^M, i = 1, 2, \dots, n\} \leq 1$, then $T : E \cap \overline{\Omega_R} \rightarrow E$ is strict-set-contractive.

(ii) If (A_4) holds and $\min\{r_i^M, i = 1, 2, \dots, n\} > 1$, then $T : E \cap \overline{\Omega_R} \rightarrow E$ is strict-set-contractive, where $\Omega_R = \{x \in Y : |x|_1 < R\}$.

Proof. We only need to prove (i), since the proof of (ii) is similar. It is easy to see that T is continuous and bounded. Now we prove that $\alpha_Y(T(S)) \leq R \max_{1 \leq j \leq n} \{\sum_{i=1}^n b_{ij}^M\} \alpha_Y(S)$ for any bounded set $S \in \overline{\Omega_R}$. Let $\eta = \alpha_Y(S)$. Then, for any positive number $\epsilon < R \max_{1 \leq j \leq n} \{\sum_{i=1}^n b_{ij}^M\} \eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with $\text{diam}(S_i) \leq \eta + \epsilon$. Therefore

$$|x - y|_1 \leq \eta + \epsilon, \quad \text{for any } x, y \in S_i. \tag{31}$$

As S and S_i are precompact in X , it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i = \bigcup_j S_{ij}$ and

$$|x - y|_0 \leq \epsilon, \quad \text{for any } x, y \in S_{ij}. \tag{32}$$

In addition, for any $x \in S$ and $t \in [0, \omega]$, we have

$$\begin{aligned}
& (T_i x)(t) \\
& = \int_t^{t+\omega} G_i(t, s) x_i(s) \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s+\xi) d\xi \right. \\
& \quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(s+\xi) d\xi \right] ds \\
& \quad + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)) \\
& \leq \frac{R^2}{1 - \delta_i} \int_0^\omega \left[\sum_{j=1}^n a_{ij}(s) + \sum_{j=1}^n b_{ij}(s) + \sum_{j=1}^n c_{ij}(s) \right] ds \\
& \quad + \frac{1}{1 - \delta_i} \sum_{t \leq t_k < t+\omega} I_{ik}(x_i(t_k)) := B_i, \quad i = 1, 2, \dots, n,
\end{aligned} \tag{33}$$

$$\begin{aligned}
& |(T_i x)'(t)| \\
& = \left| r_i(t) (T_i x)(t) - x_i(t) \right. \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(t) x_j(t) + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \right. \\
& \quad \left. \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \right] \right| \\
& \leq r_i^M B_i + R^2 \sum_{j=1}^n (a_{ij}^M + b_{ij}^M + c_{ij}^M), \quad i = 1, 2, \dots, n.
\end{aligned} \tag{34}$$

Hence,

$$\begin{aligned}
& \|(Tx)\|_0 \leq \sum_{i=1}^n B_i, \\
& \|(Tx)'\|_0 \leq \sum_{i=1}^n \left[B_i r_i^M + R^2 \sum_{j=1}^n (a_{ij}^M + b_{ij}^M + c_{ij}^M) \right].
\end{aligned} \tag{35}$$

Applying the Arzela-Ascoli theorem, we know that $T(S)$ is precompact in X . Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$|Tx - Ty|_0 \leq \epsilon, \quad \text{for any } x, y \in S_{ijk}. \tag{36}$$

From (23), (26), (31)–(34), and (A_2) , for any $x, y \in S_{ijk}$, we have

$$\begin{aligned}
 & |(T_i x)' - (T_i y)'|_0 \\
 &= \max_{t \in [0, \omega]} \left\{ \left| r_i(t) (T_i x)(t) - r_i(t) (T_i y)(t) - x_i(t) \right. \right. \\
 &\quad \times \left[\sum_{j=1}^n a_{ij}(t) x_j(t) \right. \\
 &\quad \left. + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \right. \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \right] \\
 &\quad + y_i(t) \left[\sum_{j=1}^n a_{ij}(t) y_j(t) \right. \\
 &\quad \left. + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) y_j(t+\xi) d\xi \right. \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) y'_j(t+\xi) d\xi \right] \Bigg\} \\
 &\leq \max_{t \in [0, \omega]} \{ |r_i(t)| [(T_i x)(t) - (T_i y)(t)] \} \\
 &\quad + \max_{t \in [0, \omega]} \left\{ \left| x_i(t) \left[\left(\sum_{j=1}^n a_{ij}(t) x_j(t) \right. \right. \right. \right. \\
 &\quad \left. + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(t+\xi) d\xi \right. \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x'_j(t+\xi) d\xi \right) \\
 &\quad - \left(\sum_{j=1}^n a_{ij}(t) y_j(t) + \sum_{j=1}^n b_{ij}(t) \right. \\
 &\quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) y_j(t+\xi) d\xi + \sum_{j=1}^n c_{ij}(t) \\
 &\quad \times \int_{-\sigma_{ij}}^0 g_{ij}(\xi) y'_j(t+\xi) d\xi \Bigg) \Bigg\} \\
 &\quad + \max_{t \in [0, \omega]} \left\{ \left| [x_i(t) - y_i(t)] \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 &\times \left(\sum_{j=1}^n a_{ij}(t) y_j(t) + \sum_{j=1}^n b_{ij}(t) \right. \\
 &\quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) y_j(t+\xi) d\xi \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) y'_j(t+\xi) d\xi \right) \Bigg\} \\
 &\leq r_i^M |(T_i x) - (T_i y)|_0 \\
 &\quad + R \max_{t \in [0, \omega]} \left\{ \sum_{j=1}^n \left[a_{ij}(t) |x_j(t) - y_j(t)| + b_{ij}(t) \right. \right. \\
 &\quad \times \int_{-\tau_{ij}}^0 f_{ij}(\xi) \\
 &\quad \times |x_i(t+\xi) - y_i(t+\xi)| d\xi + c_{ij}(t) \\
 &\quad \times \int_{-\sigma_{ij}}^0 g_{ij}(\xi) \\
 &\quad \times |x'_j(t+\xi) - y'_j(t+\xi)| d\xi \Bigg] \Bigg\} \\
 &\quad + \max_{t \in [0, \omega]} \left\{ \left| [x_i(t) - y_i(t)] \right. \right. \\
 &\quad \times \left(\sum_{j=1}^n a_{ij}(t) y_j(t) \right. \\
 &\quad \left. + \sum_{j=1}^n b_{ij}(t) \int_{-\tau_{ij}}^0 f_{ij}(\xi) y_j(t+\xi) d\xi \right. \\
 &\quad \left. + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) y'_j(t+\xi) d\xi \right) \Bigg\} \\
 &\leq r_i^M \epsilon + R \epsilon \sum_{j=1}^n a_{ij}^M + R \epsilon \sum_{j=1}^n b_{ij}^M + |x_i|_0 (\eta + \epsilon) \\
 &\quad \times \sum_{j=1}^n c_{ij}^M + R \epsilon \sum_{j=1}^n (a_{ij}^M + b_{ij}^M + c_{ij}^M) \\
 &= |x_i|_0 \sum_{j=1}^n c_{ij}^M \eta + \tilde{B}_i \epsilon,
 \end{aligned} \tag{37}$$

where

$$\tilde{B}_i = r_i^M + 2R \sum_{j=1}^n (a_{ij}^M + b_{ij}^M + c_{ij}^M), \quad i = 1, 2, \dots, n. \tag{38}$$

From (36) and (37) we obtain

$$\begin{aligned} \|Tx - Ty\|_1 &\leq \left(\sum_{i=1}^n |x_i|_0 \sum_{j=1}^n c_{ij}^M \right) \eta + \epsilon \sum_{i=1}^n \tilde{B}_i \\ &\leq R \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n c_{ij}^M \right\} \eta + \epsilon \sum_{i=1}^n \tilde{B}_i, \quad \text{for any } x, y \in S_{ijk}. \end{aligned} \quad (39)$$

As ϵ is arbitrarily small, it follows that

$$\alpha_Y(T(S)) \leq R \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^n c_{ij}^M \right\} \alpha_Y(S). \quad (40)$$

Therefore, T is strict-set-contractive. The proof of Lemma 10 is complete. $\square$

3. Main Results

Our main result of this paper is as follows.

Theorem 11. Assume that (A_1) – (A_3) , (A_5) hold.

- (i) If $\max\{r_i^M, i = 1, 2, \dots, n\} \leq 1$, then system (2) has at least one positive ω -periodic solution.
- (ii) If (A_4) holds and $\min\{r_i^M, i = 1, 2, \dots, n\} > 1$, then system (2) has at least one positive ω -periodic solution.

Proof. We only need to prove (i), since the proof of (ii) is similar. Let

$$\begin{aligned} R &= \left(\min_{i \in [1, n]} \left\{ \frac{\delta_i^2}{1 - \delta_i} \min_{j \in [1, n]} \{\alpha_{ij}\} \right\} \right)^{-1}, \\ 0 &< r < \min_{i \in [1, n]} \left\{ \frac{\delta_i(1 - \delta_i) - \gamma_i}{\max_{j \in [1, n]} \{\beta_{ij}\}} \right\}. \end{aligned} \quad (41)$$

Then it is easy to see that $0 < r < R$. From Lemmas 9 and 10, we know that T is strict-set-contractive on $E_{r, R}$. In view of (26), we see that if there exists $x^* \in E$ such that $Tx^* = x^*$, then x^* is one positive ω -periodic solution of system (2). Now, we will prove that condition (ii) of Lemma 7 holds. First, we prove that $Tx \not\geq x$, for all $x \in E$, $\|x\|_1 < r$. Otherwise, there exists $x \in E$, $\|x\|_1 < r$, such that $Tx \geq x$. So $\|x\|_0 > 0$ and $Tx - x \geq 0$, which implies that

$$\begin{aligned} T_i x(t) - x_i(t) &\geq \delta_i |T_i x - x_i|_1 \geq 0, \\ \text{for any } t \in [0, \omega], \quad i &= 1, 2, \dots, n. \end{aligned} \quad (42)$$

Moreover, for $t \in [0, \omega]$, we have

$$\begin{aligned} (T_i x)(t) &= \int_t^{t+\omega} G_i(t, s) x_i(s) \\ &\quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s + \xi) d\xi \right. \\ &\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x_j'(s + \xi) d\xi \right] ds \\ &\quad + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)) \\ &\leq \frac{1}{1 - \delta_i} |x_i|_0 \left\{ \sum_{j=1}^n \int_0^\omega [a_{ij}(s) + b_{ij}(s) + c_{ij}(s)] |x_j|_1 ds + \gamma_i \right\} \\ &\leq \frac{\beta_{ij}^r + \gamma_i}{1 - \delta_i} |x_i|_0 \leq \delta_i |x_i|_0, \quad i = 1, 2, \dots, n. \end{aligned} \quad (43)$$

In view of (42) and (43), we obtain

$$\|x\|_0 \leq \|Tx\|_0 \leq \max_{1 \leq i \leq n} \{\delta_i\} \|x\|_0 < \|x\|_0, \quad (44)$$

which is a contradiction. Finally, we prove that $Tx \not\leq x$, for all $x \in E$, and $\|x\|_1 = R$ also hold. For this case, suppose, for the sake of contradiction, that there exist $x \in E$ and $\|x\|_1 = R$ such that $Tx \leq x$. Furthermore, for any $t \in [0, \omega]$, we have

$$\begin{aligned} x_i(t) - T_i x(t) &\geq \delta_i |x_i - T_i x|_1 \geq 0, \\ \text{for any } t \in [0, \omega], \quad i &= 1, 2, \dots, n. \end{aligned} \quad (45)$$

In addition, for any $t \in [0, \omega]$, we find

$$\begin{aligned} (T_i x)(t) &= \int_t^{t+\omega} G_i(t, s) x_i(s) \\ &\quad \times \left[\sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) \int_{-\tau_{ij}}^0 f_{ij}(\xi) x_j(s + \xi) d\xi \right. \\ &\quad \left. + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 g_{ij}(\xi) x_j'(s + \xi) d\xi \right] ds \\ &\quad + \sum_{t \leq t_k < t+\omega} G_i(t, t_k) I_{ik}(x_i(t_k)) \\ &> \frac{\delta_i^2}{1 - \delta_i} |x_i|_1 \sum_{j=1}^n \int_0^\omega [\delta_j a_{ij}(s) - b_{ij}(s) - c_{ij}(s)] ds |x_j|_1 \\ &= \frac{\delta_i^2}{1 - \delta_i} |x_i|_1 \sum_{j=1}^n (\alpha_{ij} |x_j|_1), \quad i = 1, 2, \dots, n. \end{aligned} \quad (46)$$

Thus, we have

$$\begin{aligned} \|Tx\|_0 &= \sum_{i=1}^n |T_i x|_0 > \sum_{i=1}^n \frac{\delta_i^2}{1 - \delta_i} |x_i|_1 \sum_{j=1}^n (\alpha_{ij} |x_j|_1) \\ &\geq \min_{1 \leq i \leq n} \left\{ \frac{\delta_i^2 \min_{1 \leq j \leq n} \{\alpha_{ij}\}}{1 - \delta_i} \right\} \sum_{i=1}^n |x_i|_1 \sum_{j=1}^n |x_j|_1 \quad (47) \\ &\geq \min_{1 \leq i \leq n} \left\{ \frac{\delta_i^2 \min_{1 \leq j \leq n} \{\alpha_{ij}\}}{1 - \delta_i} \right\} R^2 = R. \end{aligned}$$

From (45) and (47), we obtain

$$\|x\|_1 \geq \|Tx\|_1 \geq \|Tx\|_0 > R, \quad (48)$$

which is a contradiction. Therefore, condition (ii) of Lemma 7 holds. By Lemma 7, we see that T has at least one nonzero fixed point in E . Thus, the system (6) has at least one positive ω -periodic solution. Therefore, it follows from Lemma 6 that system (2) has a positive ω -periodic solution. The proof of Theorem 11 is complete. $\square$

Remark 12. If $I_{ik}(x(t_k)) = 0$, we can easily derive the corresponding results in [10]. So we extend the corresponding results in [10].

4. Example

In this section, we give an example to show the effectiveness of our result.

Example 1. Consider the following nonimpulsive system:

$$\begin{aligned} x_1'(t) &= x_1(t) \left[\frac{1 - \sin t}{24} - (10 + \sin t) x_1(t) \right. \\ &\quad - (9 - \cos t) x_2(t) - (3 - 2 \cos t) \\ &\quad \times \int_{-\tau_{11}}^0 f_{11}(\xi) x_1(t + \xi) d\xi - \frac{5 + \sin t}{15} \\ &\quad \times \int_{-\tau_{12}}^0 f_{12}(\xi) x_2(t + \xi) d\xi - (1 + \sin t) \\ &\quad \times \int_{-\sigma_{11}}^0 g_{11}(\xi) x_1'(t + \xi) d\xi - \frac{4 - 3 \cos t}{6} \\ &\quad \left. \times \int_{-\sigma_{12}}^0 g_{12}(\xi) x_2'(t + \xi) d\xi \right], \end{aligned}$$

$$\begin{aligned} x_2'(t) &= x_2(t) \left[\frac{1 + \cos t}{8\pi} - (12 + 3 \cos t) x_1(t) \right. \\ &\quad - (7 - \sin t) x_2(t) - (2 + 3 \sin t) \\ &\quad \times \int_{-\tau_{21}}^0 f_{21}(\xi) x_2(t + \xi) d\xi - (1 - \cos t) \\ &\quad \times \int_{-\tau_{22}}^0 f_{22}(\xi) x_2(t + \xi) d\xi - \frac{4 + \sin t}{12} \\ &\quad \times \int_{-\sigma_{21}}^0 g_{21}(\xi) x_2'(t + \xi) d\xi - \frac{5 - \cos t}{10} \\ &\quad \left. \times \int_{-\sigma_{22}}^0 g_{22}(\xi) x_2'(t + \xi) d\xi \right], \end{aligned} \quad (49)$$

where $\tau_{ij}, \sigma_{ij} \in (0, +\infty)$ ($i, j = 1, 2$), and $f_{ij}, g_{ij} \in C(R, R^+)$ satisfying $\int_{-\tau_{ij}}^0 f_{ij}(\xi) d\xi = 1$, $\int_{-\sigma_{ij}}^0 g_{ij}(\xi) d\xi = 1$, $i, j = 1, 2$. Obviously

$$\begin{aligned} r_1(t) &= \frac{1 - \sin t}{24}, & r_2 &= \frac{1 + \cos t}{8\pi}, \\ a_{11}(t) &= 10 + \sin t, \\ a_{12}(t) &= 9 - \cos t, & a_{21}(t) &= 12 + 3 \cos t, \\ a_{22}(t) &= 7 - \sin t, & b_{11}(t) &= 3 - 2 \cos t, \\ b_{12}(t) &= \frac{5 + \sin t}{15}, \\ b_{21}(t) &= 2 + 3 \sin t, & b_{22}(t) &= 1 - \cos t, \\ c_{11}(t) &= 1 + \sin t, & c_{12}(t) &= \frac{4 - 3 \cos t}{6}, \\ c_{21}(t) &= \frac{4 + \sin t}{12}, \\ c_{22}(t) &= \frac{5 - \cos t}{10}, & r_1^L &= r_2^L = 0. \end{aligned} \quad (50)$$

Furthermore, we obtain

$$\begin{aligned} \delta_1 &= e^{-(\pi/12)} < 1, & \delta_2 &= e^{-(1/4)} < 1, \\ \alpha_{11} &= 20\pi\delta_1 - 8\pi \approx 23.8007, \\ \alpha_{12} &= 18\pi\delta_2 - 2\pi \approx 37.2423, \\ \alpha_{21} &= 24\pi\delta_1 - \frac{14\pi}{3} \approx 44.0594, \\ \alpha_{22} &= 14\pi\delta_2 - 3\pi \approx 24.4284, \end{aligned}$$

$$\min_{1 \leq j \leq 2} \{\alpha_{1j}\} = \alpha_{11} = 20\pi\delta_1 - 8\pi \approx 23.8007,$$

$$\min_{1 \leq j \leq 2} \{\alpha_{2j}\} = \alpha_{22} = 14\pi\delta_2 - 3\pi \approx 24.4284,$$

$$\delta_1 a_{11}(t) - b_{11}(t) - c_{11}(t) > 0.009 > 0,$$

$$\delta_2 a_{12}(t) - b_{12}(t) - c_{12}(t) > 1 > 0,$$

$$\delta_1 a_{21}(t) - b_{21}(t) - c_{21}(t) > 4 > 0,$$

$$\delta_2 a_{22}(t) - b_{22}(t) - c_{22}(t) > 2 > 0,$$

$$(1 + r_1^L) \frac{\delta_1^2 \alpha_{11}}{1 - \delta_1} = \frac{e^{-(\pi/6)} (20\pi\delta_1 - 8\pi)}{1 - e^{-(\pi/12)}} \approx 65.2614 > 18$$

$$\geq \max_{0 \leq t \leq 2\pi} \{a_{11}(t) + b_{11}(t) + c_{11}(t)\},$$

$$(1 + r_1^L) \frac{\delta_1^2 \alpha_{12}}{1 - \delta_1} = \frac{e^{-(\pi/6)} (18\pi\delta_2 - 2\pi)}{1 - e^{-(\pi/12)}} \approx 102.1184 > \frac{347}{30}$$

$$\geq \max_{0 \leq t \leq 2\pi} \{a_{12}(t) + b_{12}(t) + c_{12}(t)\},$$

$$(1 + r_2^L) \frac{\delta_2^2 \alpha_{21}}{1 - \delta_2} = \frac{e^{-(1/2)} (24\pi\delta_1 - (14\pi/3))}{1 - e^{-(1/4)}} \approx 113.3411 > \frac{245}{12}$$

$$> \max_{0 \leq t \leq 2\pi} \{a_{21}(t) + b_{21}(t) + c_{21}(t)\},$$

$$(1 + r_2^L) \frac{\delta_2^2 \alpha_{22}}{1 - \delta_2} = \frac{e^{-(1/2)} (14\pi\delta_2 - 3\pi)}{1 - e^{-(1/4)}} \approx 62.8411 > \frac{53}{5}$$

$$> \max_{0 \leq t \leq 2\pi} \{a_{22}(t) + b_{22}(t) + c_{22}(t)\},$$

$$\frac{\delta_1^2 \min_{1 \leq j \leq 2} \{\alpha_{1j}\}}{1 - \delta_1} = \frac{e^{-(\pi/6)} (20\pi\delta_1 - 6\pi)}{1 - e^{-(\pi/12)}} \approx 65.2614,$$

$$\frac{\delta_2^2 \min_{1 \leq j \leq 2} \{\alpha_{2j}\}}{1 - \delta_2} = \frac{e^{-(1/2)} (14\pi\delta_2 - 3\pi)}{1 - e^{-(1/4)}} \approx 62.8411,$$

$$\min_{1 \leq i \leq 2} \left\{ \frac{\delta_i^2 \min_{1 \leq j \leq 2} \{\alpha_{ij}\}}{1 - \delta_i} \right\} = \frac{\delta_2^2 \min_{1 \leq j \leq 2} \{\alpha_{2j}\}}{1 - \delta_1} \approx 62.8411,$$

$$c_{11}^M + c_{12}^M = \frac{19}{6}, \quad c_{21}^M + c_{22}^M = \frac{61}{60},$$

$$\max_{1 \leq i \leq 2} \left\{ \sum_{j=1}^2 d_{ij}^M \right\} = \frac{19}{6}.$$

(51)

Therefore,

$$\frac{19}{6} = \max_{1 \leq i \leq 2} \left\{ \sum_{j=1}^2 d_{ij}^M \right\} < \min_{1 \leq i \leq 2} \left\{ \frac{\delta_i^2 \min_{1 \leq j \leq 2} \{\alpha_{ij}\}}{1 - \delta_i} \right\} \approx 62.8411. \quad (52)$$

Hence, $(A_1)-(A_3), (A_5)$ hold and $a_i^M \leq 1, i = 1, 2$. According to Theorem 11, system (49) has at least one positive 2π -periodic solution.

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