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Research Article

The Partial Averaging of Fuzzy Differential Inclusions on Finite Interval

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The substantiation of a possibility of application of partial averaging method on finite interval for differential inclusions with the fuzzy right-hand side with a small parameter is considered.

1. Introduction

In 1990, Aubin [1] and Baidosov [2, 3] introduced differential inclusions with the fuzzy right-hand side. Their approach is based on usual differential inclusions. In 1995, Hüllermeier [4–6] introduced the concept of R-solution similar to how it has been done in [7]. Further in [8–20], various properties of solutions of fuzzy differential inclusions and their applications at modeling of various natural-science processes were considered.

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. After the systematic researches done by N. M. Krylov, N. N. Bogoliubov, Yu. A. Mitropolsky, and so forth, in 1930s, the averaging method gradually became one of the classical methods in analyzing nonlinear oscillations.

In works [21, 22], the possibility of application of schemes of full and partial averaging for differential inclusions with the fuzzy right-hand side, containing a small parameter, was proved. By proving these theorems, the scheme offered by Plotnikov et al. for a substantiation of schemes of an average of usual differential inclusions [23–28] was used. In this work, the possibility of application of partial averaging method for fuzzy differential inclusions without passage to reviewing of separate solutions is proved; that is, all estimations are spent for R-solution corresponding fuzzy systems.

2. Preliminaries

Let $\mathrm{comp}(R^n)(\mathrm{conv}(R^n))$ be a family of all nonempty (convex) compact subsets from the space R^n with the Hausdorff metric:

$$h(A,B) = \min_{r\geq 0}\{S_r(A)\supset B,\ S_r(B)\supset A\}, \tag{1}$$

where $A, B \in \mathrm{comp}(R^n)$ and $S_r(A)$ is r-neighborhood of set A.

Let E^n be a family of all $u : R^n \to [0,1]$ such that u satisfies the following conditions:

(1) u is normal; that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is fuzzy convex; that is,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}, \tag{2}$$

for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$;

(3) u is upper semicontinuous; that is, for any $x_0 \in R^n$ and $\varepsilon > 0$ exists $\delta(x_0, \varepsilon) > 0$ such that $u(x) < u(x_0) + \varepsilon$ whenever $\|x - x_0\| < \delta(x_0, \varepsilon)$, $x \in R^n$;

(4) the closure of the set $\{x \in R^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number and E^n is said to be a fuzzy number space.

Definition 1. The set $\{x \in R^n : u(x) \geq \alpha\}$ is called the *α-level* $[u]^\alpha$ of a fuzzy number $u \in E^n$, for $0 < \alpha \leq 1$. The closure of the set $\{x \in R^n : u(x) > 0\}$ is called the 0-level $[u]^0$ of a fuzzy number $u \in E^n$.

It is clear that the set $[u]^\alpha \in \operatorname{conv}(R^n)$, for all $0 \leq \alpha \leq 1$.

Theorem 2 (see [29] (stacking theorem)). *If $u \in E^n$, then*

(1) $[u]^\alpha \in conv(R^n)$, *for all* $\alpha \in [0, 1]$;

(2) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$, *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

(3) *if* $\{\alpha_k\}$ *is a nondecreasing sequence converging to* $\alpha > 0$, *then* $[u]^\alpha = \bigcap_{k\geq 1} [u]^{\alpha_k}$.

Conversely, if $\{A_\alpha : \alpha \in [0, 1]\}$ is the family of subsets of R^n satisfying conditions (1)–(3), then there exists $u \in E^n$ such that $[u]^\alpha = A_\alpha$ for $0 < \alpha \leq 1$ and $[u]^0 = \overline{\bigcup_{0<\alpha\leq 1} A_\alpha} \subset A_0$.

Let θ be the fuzzy number defined by $\theta(x) = 0$, if $x \neq 0$ and $\theta(0) = 1$.

Define $D : E^n \times E^n \to [0, \infty)$ by the relation

$$D(u, v) = \sup_{0\leq\alpha\leq 1} h\left([u]^\alpha, [v]^\alpha\right). \tag{3}$$

Then, D is a metric in E^n. Further, we know that [30]

(i) (E^n, D) is a complete metric space;

(ii) $D(u + w, v + w) = D(u, v)$, for all $u, v, w \in E^n$;

(iii) $D(\lambda u, \lambda v) = |\lambda| D(u, v)$, for all $u, v \in E^n$ and $\lambda \in R$.

3. Fuzzy Differential Inclusion: R-Solution

Consider the fuzzy differential inclusion

$$\dot{x} \in F(t, x), \quad x(0) \in X_0, \tag{4}$$

where $x \in R^n$, $t \in [0, T] \subset R_+$, $F : [0, T] \times R^n \to E^n$, $X_0 \in E^n$.

We interpret (4) as a family of differential inclusions (see [7, 9, 10]):

$$\dot{x}_\alpha \in \left[F(t, x_\alpha)\right]^\alpha, \quad x_\alpha(0) \in \left[X_0\right]^\alpha, \quad \alpha \in [0, 1]. \tag{5}$$

An α-solution $x_\alpha(\cdot)$ of (4) is understood to be an absolutely continuous function $x_\alpha : [0, T] \to R^n$ which satisfies (5) almost everywhere. Let X_α denote the α-solution set of (5) and let $X_\alpha(t) = \{x_\alpha(t) : x_\alpha(\cdot) \in X_\alpha\}$. Clearly, a family of subsets $X_t = \{X_\alpha(t) : \alpha \in [0, 1]\}$ cannot satisfy the conditions of Theorem 2 (see [5, 6, 9]).

Therefore, we will consider an R-solution of fuzzy differential inclusion (4).

Definition 3. The upper semicontinuous fuzzy mapping $X : [0, T] \to E^n$ which satisfies the system

$$\sup_{\alpha\in[0,1]} h\left([X(t+\sigma)]^\alpha, \bigcup_{x\in[X(t)]^\alpha} \left\{x + \int_t^{t+\sigma} [F(s, x)]^\alpha ds\right\}\right) = o(\sigma), \quad X(0) = X_0 \tag{6}$$

is called the *R-solution $X(\cdot)$ of differential inclusion* (4), where $\lim_{\sigma\to 0_+}(o(\sigma)/\sigma) = 0$.

Theorem 4. *Suppose that the following conditions hold:*

(1) *fuzzy mapping $F(\cdot, x)$ is measurable, for all $x \in R^n$;*

(2) *there exists $\lambda > 0$ such that, for all $x', x'' \in R^n$,*

$$D\left(F\left(t, x'\right), F\left(t, x''\right)\right) \leq \lambda \left\|x' - x''\right\|, \tag{7}$$

for almost every $t \in [0, T]$;

(3) *there exists $\gamma > 0$ such that $D(F(t, x), \widehat{0}) \leq \gamma$, for almost every $t \in [0, T]$ and every $x \in R^n$;*

(4) *for all $\beta \in [0, 1]$, $x', x'' \in R^n$ and almost every $t \in [0, T]$,*

$$\beta F\left(t, x'\right) + (1 - \beta) F\left(t, x''\right) \subset F\left(t, \beta x' + (1 - \beta) x''\right). \tag{8}$$

Then, there exists a unique R-solution $X(\cdot)$ of fuzzy system (4) *defined on the interval* $[0, \tau] \subseteq [0, T]$.

Proof. Let $S_r(X_0) = \{X \in E^n : D(X, X_0) \leq r\}$ and $\tau = \min\{T, r/\gamma\}$.

By [5, 6], it follows that a family of subsets $X_t = \{X_\alpha(t) : \alpha \in [0, 1]\}$ satisfy the conditions of Theorem 2; that is, $X_t \in E^n$, for every $t \in [0, \tau]$.

Divide the interval $[0, \tau]$ into partial intervals by the points $t_k^P = k\tau 2^{-p}$, $k = 0, \ldots, P$, $P = 2^p$, $p \in N$. We use Euler algorithm; let the mapping $X^P(\cdot)$ be given by

$$\left[X^P(t)\right]^\alpha = \bigcup_{x\in[X(t_k^P)]^\alpha} \left\{x + \int_{t_k^P}^t [F(s, x)]^\alpha ds\right\}, \tag{9}$$

where $t \in [t_k^P, t_{k+1}^P]$, $k \in \{0, \ldots, P\}$, $X(0) = X_0$, $\alpha \in [0, 1]$.

By [7, 28], it follows that the sequence $\{[X^P(\cdot)]^\alpha\}_{p=1}^\infty$ is equicontinuous and fundamental and its limit is a unique R-solution $[X(\cdot)]^\alpha$ of differential inclusion (5) and $[X(t)]^\alpha = X_\alpha(t)$, for every $t \in [0, \tau]$ and $\alpha \in [0, 1]$. This concludes the proof. □

Also, we consider the differential inclusion

$$\dot{y} \in G(t, y), \quad y(0) \in Y_0, \tag{10}$$

where $y \in R^n$, $t \in [0, T] \subset R_+$, $F : [0, T] \times R^n \to E^n$, $Y_0 \in E^n$.

Lemma 5. *Let $F(t,x)$ and $G(t,y)$ satisfy conditions (1)–(4) of Theorem 4 and there exist $\eta > 0$ and $\mu > 0$ such that*

$$D\left(\int_{t_1}^{t_2} F(s,x)\,ds, \int_{t_1}^{t_2} G(s,x)\,ds\right) < \eta\ (t_2 - t_1), \qquad D(X_0, Y_0) < \mu, \tag{11}$$

for every $x \in R^n$ and $t_2 > t_1$, $t_1, t_2 \in [0,T]$.

Then $D(X(t),Y(t)) \le \mu e^{\lambda t} + (\eta/\lambda)(e^{\lambda t} - 1)$, for every $t \in [0,T]$.

Proof. Divide the interval $[0,T]$ into partial intervals by the points $t_k^m = k\Delta$, $\Delta = (T/m)$, $k = 0,\ldots,m$, $m \in N$. By Definition 3, we have

$$\begin{aligned}
&D(X(t),Y(t)) \\
&\le \sup_{\alpha\in[0,1]} h\left(\bigcup_{x\in[X(t_k)]^\alpha}\left\{x + \int_{t_k}^{t}[F(s,x)]^\alpha ds\right\},\right. \\
&\qquad \left.\bigcup_{y\in[Y(t_k)]^\alpha}\left\{y + \int_{t_k}^{t}[G(s,y)]^\alpha ds\right\}\right) \\
&\quad + o(\Delta) \\
&\le \sup_{\alpha\in[0,1]} h\left([X(t_k)]^\alpha + \int_{t_k}^{t}\left[F\left(s,[X(t_k)]^\alpha\right)\right]^\alpha ds,\right. \\
&\qquad \left.[Y(t_k)]^\alpha + \int_{t_k}^{t}\left[G\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds\right) \\
&\quad + o(\Delta) \\
&\le \sup_{\alpha\in[0,1]} h\left([X(t_k)]^\alpha + \int_{t_k}^{t}\left[F\left(s,[X(t_k)]^\alpha\right)\right]^\alpha ds,\right. \\
&\qquad \left.[X(t_k)]^\alpha + \int_{t_k}^{t}\left[F\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds\right) \\
&\quad + \sup_{\alpha\in[0,1]} h\left([X(t_k)]^\alpha + \int_{t_k}^{t}\left[F\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds,\right. \\
&\qquad \left.[Y(t_k)]^\alpha + \int_{t_k}^{t}\left[F\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds\right) \\
&\quad + \sup_{\alpha\in[0,1]} h\left([Y(t_k)]^\alpha + \int_{t_k}^{t}\left[F\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds,\right. \\
&\qquad \left.[Y(t_k)]^\alpha + \int_{t_k}^{t}\left[G\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds\right) \\
&\quad + o(\Delta) \\
&\le \sup_{\alpha\in[0,1]} h\left(\int_{t_k}^{t}\left[F\left(s,[X(t_k)]^\alpha\right)\right]^\alpha ds,\right. \\
&\qquad \left.\int_{t_k}^{t}\left[F\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds\right) \\
&\quad + \sup_{\alpha\in[0,1]} h\left([X(t_k)]^\alpha, [Y(t_k)]^\alpha\right) \\
&\le \sup_{\alpha\in[0,1]} h\left(\int_{t_k}^{t}\left[F\left(s,[X(t_k)]^\alpha\right)\right]^\alpha ds,\right. \\
&\qquad \left.\int_{t_k}^{t}\left[G\left(s,[Y(t_k)]^\alpha\right)\right]^\alpha ds\right) + o(\Delta) \\
&\le \int_{t_k}^{t} \lambda D(X(t_k),Y(t_k))\,ds \\
&\qquad + D(X(t_k),Y(t_k)) + \eta(t - t_k) + o(\Delta) \\
&\le ((t - t_k)\lambda + 1) D(X(t_k),Y(t_k)) \\
&\qquad + \eta(t - t_k) + o(\Delta) \le \mu e^{\lambda t} + \frac{\eta}{\lambda}\left(e^{\lambda t} - 1\right)
\end{aligned} \tag{12}$$

for every $t \in [0,T]$. This concludes the proof. □

Remark 6. If $X_0 = Y_0$, then $D(X(t),Y(t)) \le (\eta/\lambda)(e^{\lambda t} - 1)$, for every $t \in [0,T]$.

4. The Method of Partial Averaging

Now, consider the fuzzy differential inclusion with a small parameter

$$\dot{x} \in \varepsilon F(t,x), \quad x(0) \in X_0, \tag{13}$$

where $X \in R^n$, $t \in R_+$, $F : R_+ \times R^n \to E^n$, $X_0 \in E^n$, and $\varepsilon > 0$ is a small parameter.

In this work, we associate the following partial averaged fuzzy differential inclusion with the inclusion (10):

$$\dot{y} \in \varepsilon G(t,y), \quad y(0) \in X_0, \tag{14}$$

where $G : R_+ \times R^n \to E^n$ such that

$$\lim_{T\to\infty} D\left(\frac{1}{T}\int_0^T F(t,x)\,dt, \frac{1}{T}\int_0^T G(t,x)\,dt\right) = 0. \tag{15}$$

Theorem 7. *Let in domain $Q = \{(t,x) : t \ge 0,\ x \in D \in conv(R^n)\}$ the following conditions hold:*

(1) *mappings $F(\cdot,x)$, $G(\cdot,x)$ are measurable on R_+;*

(2) *mappings $F(t,\cdot)$, $G(t,\cdot)$ satisfy a Lipschitz condition*

$$\begin{aligned}
D\left(F(t,x'), F(t,x'')\right) &\le \lambda\left\|x' - x''\right\|, \\
D\left(G(t,x'), G(t,x'')\right) &\le \lambda\left\|x' - x''\right\|,
\end{aligned} \tag{16}$$

with a Lipschitz constant $\lambda > 0$;

(3) *there exists* $\gamma > 0$ *such that*

$$D\left(F(t,x),\hat{0}\right) \le \gamma, \qquad D\left(G(t,x),\hat{0}\right) \le \gamma; \tag{17}$$

for almost every $t \in [0,T]$ *and every* $x \in R^n$;

(4) *for all* $\beta \in [0,1]$, $x', x'' \in R^n$ *and almost every* $t \in [0,T]$,

$$\begin{aligned} &\beta F\left(t,x'\right) + (1-\beta) F\left(t,x''\right) \\ &\subset F\left(t,\beta x' + (1-\beta) x''\right), \\ &\beta\, G\left(t,x'\right) + (1-\beta) G\left(t,x''\right) \\ &\subset G\left(t,\beta x' + (1-\beta)\ x''\right). \end{aligned} \tag{18}$$

(5) *limit* (15) *exists uniformly with respect to* x *in the domain* G;

(6) *for any* X_0 $([X_0]^0 \subset D' \subset D)$, $\varepsilon \in (0,\nu]$, *and* $t > 0$, *the R-solution of the inclusion* (10) *together with a* ρ*-neighborhood belongs to the domain* G; *that is,* $[X(t)]^0 + S_\rho(0) \subset D$, *for every* $t > 0$.

Then, for any $\eta \in (0,\rho]$ *and* $L > 0$, *there exists* $\varepsilon_0(\eta, L) > 0$ *such that, for all* $\varepsilon \in (0,\varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$, *the following inequality holds:*

$$D(X(t), Y(t)) < \eta, \tag{19}$$

where $X(\cdot)$, $Y(\cdot)$ *are the R-solutions of initial and partial averaged inclusions.*

Proof. Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals by the points $t_k = (kL/\varepsilon m)$, $k \in \{0,1,\ldots,m-1\}$. We denote fuzzy mappings $X^m(\cdot)$ and $Y^m(\cdot)$ such that

$$\begin{gathered} \left[X^m(t)\right]^\alpha = \bigcup_{x\in[X(t_k)]^\alpha} \left\{x + \varepsilon \int_{t_k}^{t} [F(s,x)]^\alpha \mathrm{d}s\right\}, \\ \left[X^m(0)\right]^\alpha = \left[X_0\right]^\alpha, \end{gathered} \tag{20}$$

$$\begin{gathered} \left[Y^m(t)\right]^\alpha = \bigcup_{y\in[Y(t_k)]^\alpha} \left\{y + \varepsilon \int_{t_k}^{t} [G(s,y)]^\alpha \mathrm{d}s\right\}, \\ \left[Y^m(0)\right]^\alpha = \left[X_0\right]^\alpha, \end{gathered} \tag{21}$$

for every $\alpha \in [0,1]$, $t \in [t_k, t_{k+1}]$, $k \in \{0,1,\ldots,m-1\}$.

Then

$$\begin{aligned} &D\left(X^m(t_k), X(t_k)\right) \\ &\le \sup_{\alpha\in[0,1]} h\left(\bigcup_{x\in[X^m(t_{k-1})]^\alpha} \left\{x + \varepsilon \int_{t_{k-1}}^{t_k} [F(s,x)]^\alpha \mathrm{d}s\right\},\right. \\ &\qquad \left.\bigcup_{x\in[X(t_{k-1})]^\alpha} \left\{x + \varepsilon \int_{t_{k-1}}^{t_k} [F(s,x)]^\alpha \mathrm{d}s\right\}\right) \\ &\quad + o\left(t_k - t_{k-1}\right) \\ &\le \left(1 + \varepsilon\left(t_k - t_{k-1}\right)\lambda\right) D\left(X^m(t_{k-1}), X(t_{k-1})\right) \\ &\quad + o\left(t_k - t_{k-1}\right) \le \frac{o\left(t_k - t_{k-1}\right)}{t_k - t_{k-1}}\left(e^{\lambda L} - 1\right). \end{aligned} \tag{22}$$

Also, we take

$$D\left(Y^m(t_k), Y(t_k)\right) \le \frac{o\left(t_k - t_{k-1}\right)}{t_k - t_{k-1}}\left(e^{\lambda L} - 1\right). \tag{23}$$

As for $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} &D\left(X^m(t), X^m(t_k)\right) \\ &\le \sup_{\alpha\in[0,1]} h \\ &\quad \times \left(\bigcup_{x\in[X^m(t_k)]^\alpha} \left\{x + \varepsilon \int_{t_k}^{t} [F(s,x)]^\alpha \mathrm{d}s\right\}, \left[X^m(t_k)\right]^\alpha\right) \\ &\le \varepsilon\gamma\left(t - t_k\right) \le \frac{\gamma L}{m}, \end{aligned} \tag{24}$$

$$D\left(Y^m(t), Y^m(t_k)\right) \le \varepsilon\gamma\left(t - t_k\right) \le \frac{\gamma L}{m}. \tag{25}$$

Using estimates (22)–(25), for any $\eta > 0$, there exists m_0 such that, for $m > m_0$, we have

$$\begin{gathered} D\left(X^m(t), X(t)\right) \le \frac{\eta}{4}, \\ D\left(Y^m(t), Y(t)\right) \le \frac{\eta}{4}. \end{gathered} \tag{26}$$

Taking into account Lemma 5, for any $\nu > 0$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0]$, the following inequality holds:

$$D\left(X^m(t_{k+1}), Y(t_{k+1})\right) \le \frac{\nu}{\lambda}\left(e^{\lambda L} - 1\right). \tag{27}$$

By combining (26) and (27) and choosing $m \ge \max\{m_0, 8\gamma L/\eta\}$ and $\nu < (\eta\lambda/4(e^{\lambda L} - 1))$, we obtain (19). The theorem is proved. □

5. Conclusion

If $F(\cdot, x)$ is continuous on $[0, T]$, then, instead of (5), it is possible to consider the following more simple equation:

$$\sup_{\alpha\in[0,1]} h\left([X(t+\sigma)]^{\alpha}, \bigcup_{x\in[X(t)]^{\alpha}} \{x+\sigma[F(t,x)]^{\alpha}\}\right) = o(\sigma), \quad X(0) = X_0, \tag{28}$$

and, similarly, we can prove all the results received earlier.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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