Research Article

Multiple Positive Solutions for a Coupled System of p -Laplacian Fractional Order Two-Point Boundary Value Problems

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This paper establishes the existence of at least three positive solutions for a coupled system of p -Laplacian fractional order two-point boundary value problems, $D_{0+}^{\beta_1}(\phi_p(D_{0+}^{\alpha_1}u(t))) = f_1(t, u(t), v(t))$, $t \in (0, 1)$, $D_{0+}^{\beta_2}(\phi_p(D_{0+}^{\alpha_2}v(t))) = f_2(t, u(t), v(t))$, $t \in (0, 1)$, $u(0) = D_{0+}^{\alpha_1}u(0) = 0$, $\gamma u(1) + \delta D_{0+}^{\alpha_2}u(1) = 0$, $D_{0+}^{\alpha_1}u(0) = D_{0+}^{\alpha_1}u(1) = 0$, $v(0) = D_{0+}^{\alpha_2}v(0) = 0$, $\gamma v(1) + \delta D_{0+}^{\alpha_1}v(1) = 0$, $D_{0+}^{\alpha_2}v(0) = D_{0+}^{\alpha_2}v(1) = 0$, by applying five functionals fixed point theorem.

1. Introduction

The theory of differential equations offers a broad mathematical basis to understand the problems of modern society which are complex and interdisciplinary by nature. Fractional order differential equations have gained importance due to their applications to almost all areas of science, engineering, and technology. Among all the theories, the most applicable operator is the classical p -Laplacian, given by $\phi_p(s) = |s|^{p-2}s$, $p > 1$. These types of problems have a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design.

The positive solutions of boundary value problems associated with ordinary differential equations were studied by many authors [1–3] and extended to p -Laplacian boundary value problems [4–6]. Later, these results are further extended to fractional order boundary value problems [7–15] by applying various fixed point theorems on cones. Recently, researchers are concentrating on the theory of fractional order boundary value problems associated with p -Laplacian operator.

In 2012, Chai [16] investigated the existence and multiplicity of positive solutions for a class of boundary value

problem of fractional differential equation with p -Laplacian operator,

$$\begin{aligned} D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t))) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) + \sigma D_{0+}^{\gamma}u(1) &= 0, \quad D_{0+}^{\alpha}u(0) = 0, \end{aligned} \quad (1)$$

by means of the fixed point theorem on cones.

This paper is concerned with the existence of positive solutions for a coupled system of p -Laplacian fractional order boundary value problems:

$$D_{0+}^{\beta_1}(\phi_p(D_{0+}^{\alpha_1}u(t))) = f_1(t, u(t), v(t)), \quad t \in (0, 1), \quad (2)$$

$$D_{0+}^{\beta_2}(\phi_p(D_{0+}^{\alpha_2}v(t))) = f_2(t, u(t), v(t)), \quad t \in (0, 1), \quad (3)$$

$$u(0) = D_{0+}^{\alpha_1}u(0) = 0, \quad \gamma u(1) + \delta D_{0+}^{\alpha_2}u(1) = 0, \quad (4)$$

$$D_{0+}^{\alpha_1}u(0) = D_{0+}^{\alpha_1}u(1) = 0,$$

$$v(0) = D_{0+}^{\alpha_2}v(0) = 0, \quad \gamma v(1) + \delta D_{0+}^{\alpha_1}v(1) = 0, \quad (5)$$

$$D_{0+}^{\alpha_2}v(0) = D_{0+}^{\alpha_2}v(1) = 0,$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, γ, δ are positive real numbers, $2 < \alpha_i \leq 3$, $1 < \beta_i$, $q_i \leq 2$, $f_i : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}^+$ are continuous functions, and $D_{0+}^{\alpha_i}$, $D_{0+}^{\beta_i}$, $D_{0+}^{q_i}$, for $i = 1, 2$ are the standard Riemann-Liouville fractional order derivatives.

The rest of the paper is organized as follows. In Section 2, the Green functions for the homogeneous BVPs corresponding to (2), (4) are constructed and the bounds for the Green functions are estimated. In Section 3, sufficient conditions for the existence of at least three positive solutions for a coupled system of p -Laplacian fractional order BVP (2)–(5) are established, by using five functionals fixed point theorem. In Section 4, as an application, the results are demonstrated with an example.

2. Green Functions and Bounds

In this section, the Green functions for the homogeneous BVPs are constructed and the bounds for the Green functions are estimated, which are essential to establish the main results.

Let $G_1(t, s)$ be Green's function for the homogeneous BVP:

$$-D_{0+}^{\alpha_1} u(t) = 0, \quad t \in (0, 1), \quad (6)$$

$$u(0) = 0, \quad D_{0+}^{q_1} u(0) = 0, \quad \gamma u(1) + \delta D_{0+}^{q_2} u(1) = 0. \quad (7)$$

Lemma 1. Let $d = \delta\Gamma(\alpha_1) + \gamma\Gamma(\alpha_1 - q_2) \neq 0$. If $h \in C[0, 1]$, then the fractional order BVP

$$D_{0+}^{\alpha_1} u(t) + h(t) = 0, \quad t \in (0, 1), \quad (8)$$

with (7) has a unique solution

$$u(t) = \int_0^1 G_1(t, s) h(s) ds, \quad (9)$$

where

$$G_1(t, s) = \begin{cases} G_{11}(t, s), & 0 \leq t \leq s \leq 1, \\ G_{12}(t, s), & 0 \leq s \leq t \leq 1, \end{cases} \quad (10)$$

$$G_{11}(t, s) = \frac{1}{d} \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] [t(1-s)]^{\alpha_1-1},$$

$$G_{12}(t, s) = G_{11}(t, s) - \frac{1}{d} \left[\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right] (t-s)^{\alpha_1-1}. \quad (11)$$

Proof. Let $u \in C^{[\alpha_1+1]}[0, 1]$ be the solution of fractional order BVP (8), (7). Then

$$I_{0+}^{\alpha_1} D_{0+}^{\alpha_1} u(t) = -I_{0+}^{\alpha_1} h(t), \quad (12)$$

and hence

$$u(t) = \frac{-1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} h(s) ds + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + c_3 t^{\alpha_1-3}. \quad (13)$$

Using the boundary conditions (7), c_1 , c_2 , and c_3 are determined as

$$c_1 = \frac{1}{d} \int_0^1 \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] (1-s)^{\alpha_1-1} h(s) ds, \\ c_2 = 0, \quad c_3 = 0. \quad (14)$$

Hence, the unique solution of (8), (7) is

$$u(t) = \int_0^t \left\{ \frac{1}{d} \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] [t(1-s)]^{\alpha_1-1} - \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} \right\} h(s) ds \\ + \int_t^1 \frac{1}{d} \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] \\ \times [t(1-s)]^{\alpha_1-1} h(s) ds \\ = \int_0^1 G_1(t, s) h(s) ds. \quad (15)$$

Lemma 2. Let $y(t) \in C[0, 1]$ and $2 < \alpha_1 \leq 3$, $1 < \beta_1 \leq 2$. Then the fractional order BVP

$$D_{0+}^{\beta_1} (\phi_p(D_{0+}^{\alpha_1} u(t))) = y(t), \quad t \in (0, 1), \quad (16)$$

with (4) has a unique solution

$$u(t) = \int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) y(\tau) d\tau \right) ds, \quad (17)$$

where

$$H_1(t, s) = \begin{cases} \frac{[t(1-s)]^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leq t \leq s \leq 1, \\ \frac{[t(1-s)]^{\beta_1-1} - (t-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leq s \leq t \leq 1. \end{cases} \quad (18)$$

Proof. An equivalent integral equation for (16) is given by

$$\phi_p(D_{0+}^{\alpha_1} u(t)) = \frac{1}{\Gamma(\beta_1)} \int_0^t (t-\tau)^{\beta_1-1} y(\tau) d\tau + c_1 t^{\beta_1-1} + c_2 t^{\beta_1-2}. \quad (19)$$

Using the conditions $D_{0+}^{\alpha_1} u(0) = 0$, $D_{0+}^{\alpha_1} u(1) = 0$, c_1 , and c_2 are determined as $c_1 = (-1/\Gamma(\beta_1)) \int_0^1 (1-\tau)^{\beta_1-1} y(\tau) d\tau$ and $c_2 = 0$. Then,

$$\phi_p(D_{0+}^{\alpha_1} u(t)) = \frac{1}{\Gamma(\beta_1)} \int_0^t (t-\tau)^{\beta_1-1} y(\tau) d\tau - \frac{1}{\Gamma(\beta_1)} \int_0^1 [t(1-\tau)]^{\beta_1-1} y(\tau) d\tau \\ = - \int_0^1 H_1(t, \tau) y(\tau) d\tau. \quad (20)$$

Therefore,

$$\phi_p^{-1}(\phi_p(D_{0+}^{\alpha_1} u(t))) = -\phi_p^{-1}\left(\int_0^1 H_1(t, \tau) y(\tau) d\tau\right). \quad (21)$$

Consequently,

$$D_{0+}^{\alpha_1} u(t) + \phi_q\left(\int_0^1 H_1(t, \tau) y(\tau) d\tau\right) = 0. \quad (22)$$

Hence, $u(t) = \int_0^1 G_1(t, s) \phi_q\left(\int_0^1 H_1(s, \tau) y(\tau) d\tau\right) ds$ is the solution of fractional order BVP (16) and (4). $\square$

Lemma 3. Assume that $\delta(q_2 - 1) > \gamma\Gamma(\alpha_1 - q_2)/\Gamma(\alpha_1)$. Then Green's function $G_1(t, s)$ satisfies the following inequalities:

- (i) $G_1(t, s) \geq 0$, for all $(t, s) \in [0, 1] \times [0, 1]$,
- (ii) $G_1(t, s) \leq G_1(1, s)$, for all $(t, s) \in [0, 1] \times [0, 1]$,
- (iii) $G_1(t, s) \geq (1/4^{\alpha_1-1})G_1(1, s)$, for all $(t, s) \in I \times [0, 1]$,

where $I = [1/4, 3/4]$.

Proof. Green's function $G_1(t, s)$ is given in (10). For $0 \leq t \leq s \leq 1$,

$$\begin{aligned} G_{11}(t, s) &= \frac{1}{d} \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] [t(1-s)]^{\alpha_1-1} \\ &\geq 0. \end{aligned} \quad (23)$$

For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} G_{12}(t, s) &= \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] \frac{[t-ts]^{\alpha_1-1}}{d} \\ &\quad - \left[\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right] \frac{(t-s)^{\alpha_1-1}}{d} \\ &\geq \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] \frac{[t-ts]^{\alpha_1-1}}{d} \\ &\quad - \left[\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right] \frac{(t-ts)^{\alpha_1-1}}{d} \\ &\geq \frac{\delta}{d} [(1-s)^{-q_2} - 1] [t-ts]^{\alpha_1-1} \geq 0. \end{aligned} \quad (24)$$

Hence, the inequality (i) is proved. For $0 \leq t \leq s \leq 1$,

$$\begin{aligned} \frac{\partial G_{11}(t, s)}{\partial t} &= \frac{1}{d} \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] (1-s)^{\alpha_1-1} \\ &\quad \times (\alpha_1 - 1) t^{\alpha_1-2} \geq 0. \end{aligned} \quad (25)$$

Therefore, $G_{11}(t, s)$ is increasing with respect to t , which implies that $G_{11}(t, s) \leq G_{11}(1, s)$. Now, for $0 \leq s \leq t \leq 1$,

$$\begin{aligned} \frac{\partial G_{12}(t, s)}{\partial t} &= \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] \\ &\quad \times \frac{(\alpha_1 - 1) [t-ts]^{\alpha_1-2} (1-s)}{d} \\ &\quad - \left[\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right] \frac{(\alpha_1 - 1) (t-s)^{\alpha_1-2}}{d} \\ &\geq \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] \\ &\quad \times \frac{(\alpha_1 - 1) [t-ts]^{\alpha_1-2} (1-s)}{d} \\ &\quad - \left[\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right] \frac{(\alpha_1 - 1) (t-ts)^{\alpha_1-2}}{d} \\ &= \frac{(\alpha_1 - 1) (t-ts)^{\alpha_1-2}}{d} \\ &\quad \times \left[\left(\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right) (1-s) \right. \\ &\quad \left. - \left(\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right) \right] \\ &= \frac{(\alpha_1 - 1) (t-ts)^{\alpha_1-2}}{d} \\ &\quad \times \left[\delta \left((1-s)^{-(q_2-1)} - 1 \right) - \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} s \right] \\ &= \frac{(\alpha_1 - 1) (t-ts)^{\alpha_1-2}}{d} \\ &\quad \times \left[\delta \left((q_2 - 1) s + \frac{(q_2 - 1) q_2}{2} s^2 + \dots \right) \right. \\ &\quad \left. - \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} s \right] \\ &= \frac{(\alpha_1 - 1) (t-ts)^{\alpha_1-2}}{d} \\ &\quad \times \left[\left(\delta (q_2 - 1) - \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right) s + O(s^2) \right] \\ &\geq 0. \end{aligned} \quad (26)$$

Therefore, $G_{12}(t, s)$ is increasing with respect to t , which implies that $G_{12}(t, s) \leq G_{12}(1, s)$. Hence, the inequality (ii) is proved. Now, the inequality (iii) can be established.

Let $0 \leq t \leq s \leq 1$ and $t \in I$. Then

$$\begin{aligned} G_{11}(t, s) &= \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] [t(1-s)]^{\alpha_1-1} \\ &= t^{\alpha_1-1} \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] (1-s)^{\alpha_1-1} \\ &= t^{\alpha_1-1} G_{11}(1, s) \\ &\geq \frac{1}{4^{\alpha_1-1}} G_{11}(1, s). \end{aligned} \quad (27)$$

Let $0 \leq s \leq t \leq 1$ and $t \in I$. Then

$$\begin{aligned} G_{12}(t, s) &= \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] [t(1-s)]^{\alpha_1-1} \\ &\quad - \left[\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right] (t-s)^{\alpha_1-1} \\ &\geq \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] [t(1-s)]^{\alpha_1-1} \\ &\quad - \left[\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right] (t-ts)^{\alpha_1-1} \\ &= t^{\alpha_1-1} \left[\frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \delta(1-s)^{-q_2} \right] \\ &\quad - \left(\delta + \frac{\gamma\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} \right) (1-s)^{\alpha_1-1} \\ &= t^{\alpha_1-1} G_{12}(1, s) \\ &\geq \frac{1}{4^{\alpha_1-1}} G_{12}(1, s). \end{aligned} \quad (28)$$

Hence the inequality (iii) is proved. $\square$

Lemma 4. For $t, s \in [0, 1]$, Green's function $H_1(t, s)$ satisfies the following inequalities:

$$(i) \ H_1(t, s) \geq 0,$$

$$(ii) \ H_1(t, s) \leq H_1(s, s).$$

Proof. Green's function $H_1(t, s)$ is given in (18). Clearly, it is observed that, for $0 \leq t \leq s \leq 1$, $H_1(t, s) \geq 0$.

For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} H_1(t, s) &= \frac{1}{\Gamma(\beta_1)} [t^{\beta_1-1}(1-s)^{\beta_1-1} - (t-s)^{\beta_1-1}] \\ &\geq \frac{1}{\Gamma(\beta_1)} [t^{\beta_1-1}(1-s)^{\beta_1-1} - (t-ts)^{\beta_1-1}] \\ &= \frac{1}{\Gamma(\beta_1)} [t^{\beta_1-1}(1-s)^{\beta_1-1} - t^{\beta_1-1}(1-s)^{\beta_1-1}] \\ &= 0. \end{aligned} \quad (29)$$

Hence, the inequality (i) is proved. Now we establish the inequality (ii), for $0 \leq t \leq s \leq 1$,

$$\frac{\partial H_1(t, s)}{\partial t} = \frac{1}{\Gamma(\beta_1)} [(\beta_1 - 1)t^{\beta_1-2}(1-s)^{\beta_1-1}] > 0. \quad (30)$$

Therefore, $H_1(t, s)$ is increasing with respect to t , for $s \in [0, 1]$, which implies that $H_1(t, s) \leq H_1(s, s)$. Similarly, it can be proved that $H_1(t, s) \leq H_1(s, s)$ for $0 \leq s \leq t \leq 1$. Hence the inequality (ii) is proved. $\square$

Lemma 5. Green's function $H_1(t, s)$ satisfies the following inequality: (A) there exists a positive function $\gamma_1^*(s) \in C(0, 1)$ such that

$$\min_{t \in I} H_1(t, s) \geq \gamma_1^*(s) H_1(s, s), \quad \text{for } 0 < s < 1. \quad (31)$$

Proof. Since $H_1(t, s)$ is monotonic function, for all $t, s \in [0, 1]$, we have

$$\max_{0 \leq t \leq 1} H_1(t, s) = H_1(s, s) = \frac{1}{\Gamma(\beta_1)} [s(1-s)]^{\beta_1-1}. \quad (32)$$

From (i) of Lemma 4, $H_1(t, s) \geq 0$, for $t, s \in [0, 1]$. For $s \in (0, 1/4)$, $H_1(t, s)$ is increasing with respect to t for $t \in (0, s/(1 - (1-s)^{(\beta_1-1)/(\beta_1-2)}))$ and decreasing with respect to t for $t \in (s/(1 - (1-s)^{(\beta_1-1)/(\beta_1-2)}), 1/4)$. For $s \in (1/4, 1)$, $H_1(t, s)$ is decreasing with respect to t for $s \leq t$ and increasing with respect to t for $s \geq t$. If we define

$$h_1(t, s) = \frac{[t(1-s)]^{\beta_1-1} - (t-s)^{\beta_1-1}}{\Gamma(\beta_1)}, \quad (33)$$

$$h_2(t, s) = \frac{[t(1-s)]^{\beta_1-1}}{\Gamma(\beta_1)}.$$

Then,

$$\begin{aligned} \min_{t \in I} H_1(t, s) &= \begin{cases} h_1\left(\frac{3}{4}, s\right), & s \in \left(0, \frac{1}{4}\right], \\ \min\left\{h_1\left(\frac{3}{4}, s\right), h_2\left(\frac{1}{4}, s\right)\right\}, & s \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ h_2\left(\frac{1}{4}, s\right), & s \in \left[\frac{3}{4}, 1\right), \end{cases} \\ &= \begin{cases} h_1\left(\frac{3}{4}, s\right), & s \in (0, \xi], \\ h_2\left(\frac{1}{4}, s\right), & s \in [\xi, 1), \end{cases} \\ &= \begin{cases} \frac{[(3/4)(1-s)]^{\beta_1-1} - ((3/4)-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & s \in (0, \xi], \\ \frac{1}{\Gamma(\beta_1)} \frac{(1-s)^{\beta_1-1}}{4^{\beta_1-1}}, & s \in [\xi, 1), \end{cases} \\ &\geq \begin{cases} \frac{[(3/4)(1-s)]^{\beta_1-1} - ((3/4)-s)^{\beta_1-1}}{[s(1-s)]^{\beta_1-1}} \\ \quad \times H_1(s, s), & s \in (0, \xi], \\ \frac{1}{(4s)^{\beta_1-1}} H_1(s, s) & s \in [\xi, 1), \end{cases} \\ &= \gamma_1^*(s) H_1(s, s), \end{aligned} \quad (34)$$

where

$$\gamma_1^*(s) = \begin{cases} \frac{[(3/4)(1-s)]^{\beta_1-1} - ((3/4)-s)^{\beta_1-1}}{[s(1-s)]^{\beta_1-1}}, & s \in (0, \xi], \\ \frac{1}{(4s)^{\beta_1-1}}, & s \in [\xi, 1), \end{cases} \quad (35)$$

and $\xi \in (1/4, 3/4)$ satisfy the equation $[(3/4)(1-\xi)]^{\beta_1-1} - ((3/4)-\xi)^{\beta_1-1} = ((1-\xi)/4)^{\beta_1-1}$. In particular, $\xi = 0.5$ if $\beta_1 = 2$; $\xi \rightarrow 0.5$ as $\beta_1 \rightarrow 2$; and $\xi \rightarrow 0.76$ as $\beta_1 \rightarrow 1$. Hence the inequality in (31) holds. $\square$

In a similar manner, the results of the Green functions $G_2(t, s)$ and $H_2(t, s)$ for the homogeneous BVPs corresponding to the fractional order BVP (3) and (5) are obtained.

Remark 6. Consider the following.

$G_1(t, s) \geq \eta G_1(1, s)$ and $G_2(t, s) \geq \eta G_2(1, s)$, for all $(t, s) \in I \times [0, 1]$, where $\eta = \min\{1/4^{\alpha_1-1}, 1/4^{\alpha_2-1}\}$.

Remark 7. Consider the following.

$H_1(t, s) \geq \gamma^*(s)H_1(s, s)$ and $H_2(t, s) \geq \gamma^*(s)H_2(s, s)$, for all $(t, s) \in I \times [0, 1]$, where $\gamma^*(s) = \min\{\gamma_1^*(s), \gamma_2^*(s)\}$.

3. Existence of Multiple Positive Solutions

In this section, the existence of at least three positive solutions for a coupled system of p -Laplacian fractional order BVP (2)–(5) is established by using five functionals fixed point theorem.

Let γ, β, θ be nonnegative continuous convex functionals on P and let α, ψ be nonnegative continuous concave functionals on P ; then for nonnegative numbers h', a', b', d' , and c' , convex sets are defined:

$$\begin{aligned} P(\gamma, c') &= \{y \in P : \gamma(y) < c'\}, \\ P(\gamma, \alpha, a', c') &= \{y \in P : a' \leq \alpha(y); \gamma(y) \leq c'\}, \\ Q(\gamma, \beta, d', c') &= \{y \in P : \beta(y) \leq d'; \gamma(y) \leq c'\}, \\ P(\gamma, \theta, \alpha, a', b', c') &= \{y \in P : a' \leq \alpha(y); \theta(y) \leq b'; \gamma(y) \leq c'\}, \\ Q(\gamma, \beta, \psi, h', d', c') &= \{y \in P : h' \leq \psi(y); \beta(y) \leq d'; \gamma(y) \leq c'\}. \end{aligned} \quad (36)$$

In obtaining multiple positive solutions of the p -Laplacian fractional order BVP (2)–(5), the following so-called five functionals fixed point theorem is fundamental.

Theorem 8 (see [17]). *Let P be a cone in the real Banach space B . Suppose that α and ψ are nonnegative continuous concave functionals on P and γ, β, θ are nonnegative continuous convex functionals on P , such that, for some positive numbers c' and e' , $\alpha(y) \leq \beta(y)$ and $\|y\| \leq e' \gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose further that $T : \overline{P(\gamma, c')} \rightarrow \overline{P(\gamma, c')}$ is completely continuous and there exist constants h', d', a' , and $b' \geq 0$ with $0 < d' < a'$ such that each of the following is satisfied:*

- (B1) $\{y \in P(\gamma, \theta, \alpha, a', b', c') : \alpha(y) > a'\} \neq \emptyset$ and $\alpha(Ty) > a'$ for $y \in P(\gamma, \theta, \alpha, a', b', c')$,
- (B2) $\{y \in Q(\gamma, \beta, \psi, h', d', c') : \beta(y) > d'\} \neq \emptyset$ and $\beta(Ty) > d'$ for $y \in Q(\gamma, \beta, \psi, h', d', c')$,
- (B3) $\alpha(Ty) > a'$ provided that $y \in P(\gamma, \alpha, a', c')$ with $\theta(Ty) > b'$,
- (B4) $\beta(Ty) < d'$ provided that $y \in Q(\gamma, \beta, \psi, h', d', c')$ with $\psi(Ty) < h'$.

Then, T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that $\beta(y_1) < d'$, $a' < \alpha(y_2)$ and $d' < \beta(y_3)$ with $\alpha(y_3) < a'$.

Consider the Banach space $B = E \times E$, where $E = \{u : u \in C[0, 1]\}$ equipped with the norm $\|(u, v)\| = \|u\|_0 + \|v\|_0$, for $(u, v) \in B$ and the norm, is defined as

$$\|(u, v)\|_0 = \max_{0 \leq t \leq 1} |u(t)|. \quad (37)$$

Define a cone $P \subset B$ by

$$P = \left\{ (u, v) \in B \mid u(t) \geq 0, v(t) \geq 0, t \in [0, 1], \right. \\ \left. \min_{t \in I} [u(t) + v(t)] \geq \eta \|(u, v)\| \right\}. \quad (38)$$

Define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P by

$$\begin{aligned} \alpha(u, v) &= \min_{t \in I} \{|u| + |v|\}, \\ \psi(u, v) &= \min_{t \in I_1} \{|u| + |v|\}, \\ \gamma(u, v) &= \max_{t \in [0, 1]} \{|u| + |v|\}, \\ \beta(u, v) &= \max_{t \in I_1} \{|u| + |v|\}, \\ \theta(u, v) &= \max_{t \in I} \{|u| + |v|\}, \end{aligned} \quad (39)$$

where $I_1 = [1/3, 2/3]$. For any $(u, v) \in P$,

$$\begin{aligned} \alpha(u, v) &= \min_{t \in I} \{|u| + |v|\} \leq \max_{t \in I_1} \{|u| + |v|\} = \beta(u, v), \\ \|(u, v)\| &\leq \frac{1}{\eta} \min_{t \in I} \{|u| + |v|\} \leq \frac{1}{\eta} \max_{t \in [0, 1]} \{|u| + |v|\} = \frac{1}{\eta} \gamma(u, v). \end{aligned} \quad (40)$$

Let

$$\begin{aligned} L &= \min \left\{ \left(\int_0^1 G_1(1, s) \phi_q \left(\int_0^1 H_1(\tau, \tau) d\tau \right) ds \right)^{-1}, \right. \\ &\quad \left. \left(\int_0^1 G_2(1, s) \phi_q \left(\int_0^1 H_2(\tau, \tau) d\tau \right) ds \right)^{-1} \right\}, \\ M &= \max \left\{ \left(\int_{s \in I} \eta G_1(1, s) \phi_q \left(\int_{\tau \in I} \gamma^*(\tau) H_1(\tau, \tau) d\tau \right) ds \right)^{-1}, \right. \\ &\quad \left(\int_{s \in I} \eta G_2(1, s) \phi_q \right. \\ &\quad \left. \times \left(\int_{\tau \in I} \gamma^*(\tau) H_2(\tau, \tau) d\tau \right) ds \right)^{-1} \Big\}. \end{aligned} \quad (41)$$

Theorem 9. Suppose that there exist $0 < a' < b' < b'/\eta < c'$ such that f_i , for $i = 1, 2$ satisfies the following conditions:

- (A1) $f_i(t, u(t), v(t)) < \phi_p(a'L/2)$, $t \in [0, 1]$ and $u, v \in [\eta a', a']$,
- (A2) $f_i(t, u(t), v(t)) > \phi_p(b'M/2)$, $t \in I$ and $u, v \in [b', b'/\eta]$,
- (A3) $f_i(t, u(t), v(t)) < \phi_p(c'L/2)$, $t \in [0, 1]$ and $u, v \in [0, c']$.

Then, the fractional order BVP (2)–(5) has at least three positive solutions, (x_1, x_2) , (y_1, y_2) , and (z_1, z_2) such that $\beta(x_1, x_2) < a'$, $b' < \alpha(y_1, y_2)$ and $a' < \beta(z_1, z_2)$ with $\alpha(z_1, z_2) < b'$.

Proof. Let $T_1, T_2 : P \rightarrow E$ and $T : P \rightarrow B$ be the operators defined by

$$T_1(u, v)(t) = \int_0^1 G_1(t, s) \phi_q \times \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds,$$

$$T_2(u, v)(t) = \int_0^1 G_2(t, s) \phi_q \times \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds,$$

$$T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t)), \quad \text{for } (u, v) \in B. \quad (42)$$

It is obvious that a fixed point of T is the solution of the fractional order BVP (2)–(5). Three fixed points of T are sought. First, it is shown that $T : P \rightarrow P$. Let $(u, v) \in P$. Clearly, $T_1(u, v)(t) \geq 0$ and $T_2(u, v)(t) \geq 0$, for $t \in [0, 1]$. Also, for $(u, v) \in P$,

$$\begin{aligned} \|T_1(u, v)\|_0 &\leq \int_0^1 G_1(1, s) \phi_q \times \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds, \\ \|T_2(u, v)\|_0 &\leq \int_0^1 G_2(1, s) \phi_q \times \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds, \\ \min_{t \in I} T_1(u, v)(t) &= \min_{t \in I} \int_0^1 G_1(t, s) \phi_q \times \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\geq \eta \int_0^1 G_1(1, s) \phi_q \times \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\geq \eta \|T_1(u, v)\|_0. \end{aligned} \quad (43)$$

Similarly, $\min_{t \in I} T_2(u, v)(t) \geq \eta \|T_2(u, v)\|_0$. Therefore,

$$\begin{aligned} & \min_{t \in I} \{T_1(u, v)(t) + T_2(u, v)(t)\} \\ & \geq \eta \|T_1(u, v)\|_0 + \eta \|T_2(u, v)\|_0 \\ & = \eta (\|T_1(u, v)\|_0 + \|T_2(u, v)\|_0) \\ & = \eta \|(T_1(u, v), T_2(u, v))\| \\ & = \eta \|T(u, v)\|. \end{aligned} \quad (44)$$

Hence, $T(u, v) \in P$ and so $T : P \rightarrow P$. Moreover, T is completely continuous operator. From (40), for each $(u, v) \in P$, $\alpha(u, v) \leq \beta(u, v)$, and $\|(u, v)\| \leq (1/\eta)\gamma(u, v)$. It is shown that $T : \overline{P(\gamma, c')} \rightarrow \overline{P(\gamma, c')}$. Let $(u, v) \in \overline{P(\gamma, c')}$. Then $0 \leq |u| + |v| \leq c'$. Condition (A3) is used to obtain

$$\begin{aligned} & \gamma(T(u, v)(t)) \\ & = \max_{t \in [0, 1]} \left[\int_0^1 G_1(t, s) \phi_q \right. \\ & \quad \times \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ & \quad + \int_0^1 G_2(t, s) \phi_q \\ & \quad \times \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \Big] \\ & \leq \int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) \phi_p \left(\frac{c'L}{2} \right) d\tau \right) ds \\ & \quad + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) \phi_p \left(\frac{c'L}{2} \right) d\tau \right) ds \\ & < \frac{c'L}{2} \int_0^1 G_1(1, s) \phi_q \left(\int_0^1 H_1(\tau, \tau) d\tau \right) ds \\ & \quad + \frac{c'L}{2} \int_0^1 G_2(1, s) \phi_q \left(\int_0^1 H_2(\tau, \tau) d\tau \right) ds \\ & < \frac{c'}{2} + \frac{c'}{2} \\ & = c'. \end{aligned} \quad (45)$$

Therefore $T : \overline{P(\gamma, c')} \rightarrow \overline{P(\gamma, c')}$. Now conditions (B1) and (B2) of Theorem 8 are to be verified. It is obvious that

$$\begin{aligned} & \frac{b' + (b'/\eta)}{2} \in \left\{ (u, v) \in P \left(\gamma, \theta, \alpha, b', \frac{b'}{\eta}, c' \right) : \alpha(u, v) > b' \right\} \neq \emptyset, \\ & \frac{\eta a' + a'}{2} \in \left\{ (u, v) \in Q \left(\gamma, \beta, \psi, \eta a', a', c' \right) : \beta(u, v) < a' \right\} \neq \emptyset. \end{aligned} \quad (46)$$

Next, let $(u, v) \in P(\gamma, \theta, \alpha, b', b'/\eta, c')$ or $(u, v) \in Q(\gamma, \beta, \psi, \eta a', a', c')$. Then, $b' \leq |u(t)| + |v(t)| \leq b'/\eta$ and

$\eta a' \leq |u(t)| + |v(t)| \leq a'$. Now, condition (A2) is applied to get

$$\begin{aligned} & \alpha(T(u, v)(t)) \\ & = \min_{t \in I} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ & \quad + \int_0^1 G_2(t, s) \phi_q \\ & \quad \times \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \Big] \\ & \geq \eta \left[\int_0^1 G_1(1, s) \phi_q \left(\int_0^1 \gamma^*(\tau) H_1(\tau, \tau) \phi_p \left(\frac{b'M}{2} \right) d\tau \right) ds \right. \\ & \quad + \int_0^1 G_2(1, s) \phi_q \\ & \quad \times \left(\int_0^1 \gamma^*(\tau) H_2(\tau, \tau) \phi_p \left(\frac{b'M}{2} \right) d\tau \right) ds \Big] \\ & > \frac{b'M}{2} \int_{s \in I} \eta G_1(1, s) \phi_q \left(\int_{\tau \in I} \gamma^*(\tau) H_1(\tau, \tau) d\tau \right) ds \\ & \quad + \frac{b'M}{2} \int_{s \in I} \eta G_2(1, s) \phi_q \left(\int_{\tau \in I} \gamma^*(\tau) H_2(\tau, \tau) d\tau \right) ds \\ & \geq \frac{b'}{2} + \frac{b'}{2} = b'. \end{aligned} \quad (47)$$

Clearly, condition (A1) leads to

$$\begin{aligned} & \beta(T(u, v)(t)) \\ & = \max_{t \in I_1} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\ & \quad + \int_0^1 G_2(t, s) \phi_q \\ & \quad \times \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \Big] \\ & \leq \int_0^1 G_1(1, s) \phi_q \left(\int_0^1 H_1(s, \tau) \phi_p \left(\frac{a'L}{2} \right) d\tau \right) ds \\ & \quad + \int_0^1 G_2(1, s) \phi_q \left(\int_0^1 H_2(s, \tau) \phi_p \left(\frac{a'L}{2} \right) d\tau \right) ds \\ & < \frac{a'L}{2} \int_0^1 G_1(1, s) \phi_q \left(\int_0^1 H_1(\tau, \tau) d\tau \right) ds \\ & \quad + \frac{a'L}{2} \int_0^1 G_2(1, s) \phi_q \left(\int_0^1 H_2(\tau, \tau) d\tau \right) ds \\ & \leq \frac{a'}{2} + \frac{a'}{2} = a'. \end{aligned} \quad (48)$$

To see that (B3) is satisfied, let $(u, v) \in P(\gamma, \alpha, b', c')$ with $\theta(T(u, v)(t)) > b'/\eta$. Then

$$\begin{aligned}
 & \alpha(T(u, v)(t)) \\
 &= \min_{t \in I} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &\geq \eta \left[\int_0^1 G_1(1, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(1, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &\geq \eta \max_{t \in [0, 1]} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &\geq \eta \max_{t \in I} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &= \eta \theta(T(u, v)(t)) > b'.
 \end{aligned} \tag{49}$$

Finally, it is shown that (B4) holds. Let $(u, v) \in Q(\gamma, \beta, a', c')$ with $\psi(T(u, v)) < \eta a'$. Then, we have

$$\begin{aligned}
 & \beta(T(u, v)(t)) \\
 &= \max_{t \in I_1} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right]
 \end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \Big] \\
 &\leq \max_{t \in [0, 1]} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &\leq \int_0^1 G_1(1, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
 & \quad + \int_0^1 G_2(1, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
 &= \frac{1}{\eta} \left[\eta \int_0^1 G_1(1, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \eta \int_0^1 G_2(1, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &\leq \frac{1}{\eta} \min_{t \in I} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &\leq \frac{1}{\eta} \min_{t \in I_1} \left[\int_0^1 G_1(t, s) \phi_q \left(\int_0^1 H_1(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \right. \\
 & \quad \left. + \int_0^1 G_2(t, s) \phi_q \left(\int_0^1 H_2(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right] \\
 &= \frac{1}{\eta} \psi(T(u, v)(t)) < a'.
 \end{aligned} \tag{50}$$

It has been proved that all the conditions of Theorem 8 are satisfied. Therefore, the fractional order BVP (2)–(5) has at least three positive solutions, (x_1, x_2) , (y_1, y_2) , and (z_1, z_2) such that $\beta(x_1, x_2) < a'$, $b' < \alpha(y_1, y_2)$, and $a' < \beta(z_1, z_2)$ with $\alpha(z_1, z_2) < b'$. This completes the proof of the theorem. $\square$

4. Example

In this section, as an application, the result is demonstrated with an example.

Consider a coupled system of p -Laplacian fractional order BVP:

$$\begin{aligned} D_{0^+}^{1.7} (\phi_p(D_{0^+}^{2.6} u(t))) &= f_1(t, u(t), v(t)), \quad t \in (0, 1), \\ D_{0^+}^{1.9} (\phi_p(D_{0^+}^{2.8} v(t))) &= f_2(t, u(t), v(t)), \quad t \in (0, 1), \\ u(0) &= D_{0^+}^{0.5} u(0) = 0, \quad 5u(1) + 8D_{0^+}^{0.7} u(1) = 0, \\ D_{0^+}^{2.6} u(0) &= D_{0^+}^{2.6} u(1) = 0, \\ v(0) &= D_{0^+}^{0.5} v(0) = 0, \quad 5v(1) + 8D_{0^+}^{0.7} v(1) = 0, \\ D_{0^+}^{2.8} v(0) &= D_{0^+}^{2.8} v(1) = 0, \end{aligned} \quad (51)$$

where

$$\begin{aligned} f_1(t, u, v) &= \begin{cases} \frac{e^t}{99} + \frac{\sin(u+v)}{10} \\ \quad + \frac{87(u+v)^3}{10}, & 0 \leq u+v \leq 10, \\ \frac{e^t}{99} + \frac{87000}{10} \\ \quad + \frac{\sin(u+v)}{10}, & u+v > 10. \end{cases} \\ f_2(t, u, v) &= \begin{cases} \frac{3e^t}{159} + \frac{\cos(u+v)}{10} \\ \quad + \frac{93(u+v)^3}{10}, & 0 \leq u+v \leq 10, \\ \frac{3e^t}{159} + \frac{93000}{10} \\ \quad + \frac{\cos(u+v)}{10}, & u+v > 10, \end{cases} \end{aligned} \quad (52)$$

Then the Green functions $G_i(t, s)$ and $H_i(t, s)$, for $i = 1, 2$, are given by

$$\begin{aligned} G_1(t, s) &= \begin{cases} \frac{1}{16.24} \left[\frac{5\Gamma(1.9)}{\Gamma(2.6)} + 8(1-s)^{-0.7} \right] \\ \quad \times [t(1-s)]^{1.6}, & t \leq s, \\ \frac{1}{16.24} \left[\frac{5\Gamma(1.9)}{\Gamma(2.6)} + 8(1-s)^{-0.7} \right] \\ \quad \times [t(1-s)]^{1.6} - \frac{(t-s)^{1.6}}{\Gamma(2.6)}, & s \leq t, \end{cases} \\ G_2(t, s) &= \begin{cases} \frac{1}{18.64} \left[\frac{5\Gamma(2.1)}{\Gamma(2.6)} + 8(1-s)^{-0.7} \right] \\ \quad \times [t(1-s)]^{1.8}, & t \leq s, \\ \frac{1}{18.64} \left[\frac{5\Gamma(2.1)}{\Gamma(2.6)} + 8(1-s)^{-0.7} \right] \\ \quad \times [t(1-s)]^{1.8} - \frac{(t-s)^{1.8}}{\Gamma(2.8)}, & s \leq t, \end{cases} \end{aligned}$$

$$\begin{aligned} H_1(t, s) &= \begin{cases} \frac{[t(1-s)]^{0.7} - (t-s)^{0.7}}{\Gamma(1.7)}, & s \leq t, \\ \frac{[t(1-s)]^{0.7}}{\Gamma(1.7)}, & t \leq s, \end{cases} \\ H_2(t, s) &= \begin{cases} \frac{[t(1-s)]^{0.9} - (t-s)^{0.9}}{\Gamma(1.9)}, & s \leq t, \\ \frac{[t(1-s)]^{0.9}}{\Gamma(1.9)}, & t \leq s. \end{cases} \end{aligned} \quad (53)$$

Clearly, the Green functions $G_i(t, s)$ and $H_i(t, s)$, for $i = 1, 2$, are positive and f_1, f_2 are continuous and increasing on $[0, \infty)$. By direct calculations, $\eta = 0.08$, $p = 2$, $L = 33.16$, and $M = 1677.73$. Choosing $a' = 1$, $b' = 10$ and $c' = 900$ and then $0 < a' < b' < b'/\eta \leq c'$ and f_i , for $i = 1, 2$ satisfies

- (i) $f_i(t, u, v) < 16.5845 = \phi_p(a'L/2)$, $t \in [0, 1]$ and $u, v \in [0.08, 1]$,
- (ii) $f_i(t, u, v) > 8388.65 = \phi_p(b'M/2)$, $t \in [1/4, 3/4]$ and $u, v \in [10, 125]$,
- (iii) $f_i(t, u, v) < 14926.09 = \phi_p(c'L/2)$, $t \in [0, 1]$ and $u, v \in [0, 900]$.

Then, all the conditions of Theorem 9 are satisfied. Therefore, it follows from Theorem 8 that the p -Laplacian fractional order BVP (51) has at least three positive solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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