Research Article

Conjugacy of a Discrete Semidynamical System in a Neighbourhood of the Nontrivial Invariant Manifold

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The conjugacy of a discrete semidynamical system and its partially decoupled discrete semidynamical system in a Banach space is proved in a neighbourhood of the nontrivial invariant manifold.

1. Introduction

The conjugacy for noninvertible mappings in a Banach space was considered by Aulbach and Garay [1–3]. For noninvertible mappings in a complete metric space it was extended and generalized by Reinfelds [4–9]. In the present paper we consider the case when the linear part of the noninvertible mapping depends on the behaviour of variables in a neighbourhood of the nontrivial invariant manifold.

2. Invariant Manifold

Let E and F be Banach spaces, $B(a) = \{r \in F \mid |r| \leq a\}$, and $a > 0$. Consider the following mapping $S : E \times B(a) \rightarrow E \times B(a)$ defined by

$$\begin{aligned}x_1 &= g(x) + \Psi(x, r) = X(x, r), \\r_1 &= A(x)r + \Phi(x, r) = R(x, r),\end{aligned}\tag{1}$$

where the derivative of the diffeomorphism $g : E \rightarrow E$ is uniformly continuous $\|Dg(x) - Dg(x')\| \leq \omega(|x - x'|)$, mappings A , Ψ , and Φ are Lipschitzian,

$$\begin{aligned}\|A(x) - A(x')\| &\leq \gamma |x - x'|, \\|\Psi(x, r) - \Psi(x', r')| &\leq \varepsilon (|x - x'| + |r - r'|), \\|\Phi(x, r) - \Phi(x', r')| &\leq \varepsilon (|x - x'| + |r - r'|), \\ \sup_x \|A(x)\| + 2\varepsilon &< 1, \\ \sup_x |\Phi(x, 0)| &\leq a \left(1 - \sup_x \|A(x)\| - \varepsilon\right).\end{aligned}\tag{2}$$

At the beginning we will modify the previous results on the existence of invariant manifolds of Neïmark and Sacker [10, 11] for (1).

Lemma 1. *If*

$$\begin{aligned}\left(\sup_x \|A(x)\| + 4\varepsilon + \gamma \frac{\sup_x |\Phi(x, 0)|}{1 - \sup_x \|A(x)\| - \varepsilon}\right) \\ \times \sup_x \|Dg(x)^{-1}\| \leq 1\end{aligned}\tag{3}$$

then there exists a continuous mapping $u : E \rightarrow F$ satisfying the following properties:

- (i) $u(g(x) + \Psi(x, u(x))) = A(x)u(x) + \Phi(x, u(x))$;
- (ii) $|u(x) - u(x')| \leq |x - x'|$;
- (iii) $\|u\| \leq \sup_x |\Phi(x, 0)| / (1 - \sup_x \|A(x)\| - \varepsilon)$.

Proof. The set of continuous mappings $u : \mathbf{E} \rightarrow \mathbf{F}$,

$$\mathcal{K} = \left\{ u \in C(\mathbf{E}, \mathbf{F}) \mid \sup_x |u(x)| < +\infty \right\} \quad (4)$$

equipped with the norm

$$\|u\| = \sup_x |u(x)| \quad (5)$$

is a Banach space. The set

$$\mathcal{K}_1 = \left\{ u \in \mathcal{K} \mid \|u\| \leq a, |u(x) - u(x')| \leq |x - x'| \right\} \quad (6)$$

is a closed subset of the Banach space $\mathcal{K}$.

Let us consider the mapping $u \mapsto \mathcal{L}u$, $u \in \mathcal{K}_1$ defined by the equality

$$\mathcal{L}u(x_1) = A(x)u(x) + \Phi(x, u(x)), \quad (7)$$

where

$$x_1 = g(x) + \Psi(x, u(x)). \quad (8)$$

If $u \in \mathcal{K}_1$ then

$$\|\mathcal{L}u\| \leq \left(\sup_x \|A(x)\| + \varepsilon \right) \|u\| + \sup_x |\Phi(x, 0)| \leq a. \quad (9)$$

We have

$$\begin{aligned} & |\mathcal{L}u(x_1) - \mathcal{L}u'(x'_1)| \\ & \leq \left(\sup_x \|A(x)\| + \varepsilon \right) \|u - u'\| \\ & \quad + \left(\sup_x \|A(x)\| + 2\varepsilon + \gamma \|u\| \right) |x - x'|, \\ & |x - x'| \\ & \leq \sup_x \|Dg(x)^{-1}\| |g(x) - g(x')| \\ & = \sup_x \|Dg(x)^{-1}\| |x_1 - x'_1 - \Psi(x, u(x)) + \Psi(x', u'(x'))| \\ & \leq \sup_x \|Dg(x)^{-1}\| (|x_1 - x'_1| + 2\varepsilon |x - x'| + \varepsilon \|u - u'\|). \end{aligned} \quad (10)$$

It follows

$$\begin{aligned} |x - x'| & \leq \frac{\sup_x \|Dg(x)^{-1}\|}{1 - 2\varepsilon \sup_x \|Dg(x)^{-1}\|} |x_1 - x'_1| \\ & \quad + \frac{\varepsilon \sup_x \|Dg(x)^{-1}\|}{1 - 2\varepsilon \sup_x \|Dg(x)^{-1}\|} \|u - u'\|. \end{aligned} \quad (11)$$

Then

$$\begin{aligned} & |\mathcal{L}u(x_1) - \mathcal{L}u'(x'_1)| \\ & \leq \left(\sup_x \|A(x)\| + \varepsilon \right. \\ & \quad \left. + \frac{(\sup_x \|A(x)\| + 2\varepsilon + \gamma \|u\|) \sup_x \|Dg(x)^{-1}\| \varepsilon}{1 - 2\varepsilon \sup_x \|Dg(x)^{-1}\|} \right) \\ & \quad \times \|u - u'\| \\ & \quad + \frac{(\sup_x \|A(x)\| + 2\varepsilon + \gamma \|u\|) \sup_x \|Dg(x)^{-1}\|}{1 - 2\varepsilon \sup_x \|Dg(x)^{-1}\|} \\ & \quad \times |x_1 - x'_1|. \end{aligned} \quad (12)$$

Let us note that

$$\frac{(\sup_x \|A(x)\| + 2\varepsilon + \gamma \|u\|) \sup_x \|Dg(x)^{-1}\|}{1 - 2\varepsilon \sup_x \|Dg(x)^{-1}\|} \leq 1. \quad (13)$$

We obtain

$$\begin{aligned} & |\mathcal{L}u(x_1) - \mathcal{L}u'(x'_1)| \leq \left(\sup_x \|A(x)\| + 2\varepsilon \right) \|u - u'\| \\ & \quad + |x_1 - x'_1|. \end{aligned} \quad (14)$$

We get that $\mathcal{L} \in \mathcal{K}_1$ is contraction and consequently we have the invariant manifold $r = u(x)$. $\square$

3. Conjugacy of Noninvertible Mappings

Definition 2. Two mappings $S, T : \mathbf{X} \rightarrow \mathbf{X}$ are *conjugate*, if there exists a homeomorphism $H : \mathbf{X} \rightarrow \mathbf{X}$ such that

$$S \circ H(x) = H \circ T(x). \quad (15)$$

Definition 3. Two discrete semidynamical systems $S^n, T^n : \mathbf{X} \rightarrow \mathbf{X}$ ($n \in \mathbb{N}$) are *conjugate*, if there exists a homeomorphism $H : \mathbf{X} \rightarrow \mathbf{X}$ such that

$$S^n \circ H(x) = H \circ T^n(x). \quad (16)$$

It is easily verified that two discrete semidynamical systems S^n and T^n , generated by mappings S and T , are conjugate if and only if the mappings S and T are conjugate.

Suppose that mapping (1) has an invariant manifold given by Lipschitzian mapping $u : \mathbf{E} \rightarrow \mathbf{F}$ such that

$$\begin{aligned} & \sup_x \|u(x)\| \leq \delta, \\ & |u(x) - u(x')| \leq |x - x'|. \end{aligned} \quad (17)$$

Our aim is to find a simpler mapping conjugated with (1).

Theorem 4. If $\sup_x (\| (Dg(x))^{-1} \| \| A(x) \|) + 5\varepsilon \sup_x \| (Dg(x))^{-1} \| < 1$, then there exists a continuous mapping $v : \mathbf{E} \times \mathbf{B}(\delta) \rightarrow \mathbf{E}$ which is Lipschitzian with respect to the second variable such that mappings (1) and

$$\begin{aligned} x_1 &= X(x, u(x)), \\ r_1 &= R(x + v(x, r), r) \end{aligned} \quad (18)$$

are conjugated in a small neighbourhood of the invariant manifold $r = u(x)$.

We will seek the mapping establishing the conjugacy of (1) and (18) in the form

$$H(x, r) = (x + v(x, r), r). \quad (19)$$

We get the following functional equation:

$$\begin{aligned} X(x + v(x, r), r) &= X(x, u(x)) \\ &+ v(X(x, u(x)), R(x + v(x, r), r)) \end{aligned} \quad (20)$$

or equivalently

$$\begin{aligned} v(x, r) &= (Dg(x))^{-1} (Dg(x) v(x, r) - X(x + v(x, r), r) \\ &+ X(x, u(x)) \\ &+ v(X(x, u(x)), R(x + v(x, r), r))). \end{aligned} \quad (21)$$

The proof of the theorem consists of four lemmas.

Lemma 5. The functional equation (20) has a unique solution in $\mathcal{M}_1$.

Proof. The set of continuous mappings $v : \mathbf{E} \times \mathbf{B}(\delta) \rightarrow \mathbf{E}$,

$$\mathcal{M} = \left\{ v \in \mathbf{C}(\mathbf{E} \times \mathbf{B}(\delta), \mathbf{E}) \mid \sup_{x,r} \frac{|v(x, r)|}{|r - u(x)|} < +\infty \right\} \quad (22)$$

becomes a Banach space if we use the norm $\|v\| = \sup_{x,r} (|v(x, r)| / |r - u(x)|)$. The set

$$\mathcal{M}_1 = \{ v \in \mathcal{M} \mid \|v\| \leq 1, |v(x, r) - v(x, r')| \leq |r - r'| \} \quad (23)$$

is a closed subset of the Banach space $\mathcal{M}$.

Let us consider the mapping $v \mapsto \mathcal{L}v$, $v \in \mathcal{M}_1$ defined by the equality

$$\begin{aligned} \mathcal{L}v(x, r) &= (Dg(x))^{-1} v(X(x, u(x)), R(x + v(x, r), r)) \\ &+ (Dg(x))^{-1} (Dg(x) v(x, r) - g(x + v(x, r)) + g(x) \\ &- \Psi(x + v(x, r), r) + \Psi(x, u(x))). \end{aligned} \quad (24)$$

First we obtain

$$\begin{aligned} |Lv(x, r)| &\leq \| (Dg(x))^{-1} \| |R(x + v(x, r), r) - u(X(x, u(x)))| \\ &+ \| (Dg(x))^{-1} \| |Dg(x) v(x, r) - g(x + v(x, r)) + g(x)| \\ &+ \| (Dg(x))^{-1} \| |\Psi(x + v(x, r), r) - \Psi(x, u(x))| \\ &\leq \| (Dg(x))^{-1} \| \\ &\times (\|A(x)\| + \gamma|r| + 2\varepsilon + \omega(|r - u(x)|) + 2\varepsilon) \\ &\times |r - u(x)|. \end{aligned} \quad (25)$$

Here we used Hadamard lemma:

$$g(x') - g(x) = \int_0^1 Dg(x + \theta(x' - x)) d\theta(x' - x). \quad (26)$$

Next we get

$$\begin{aligned} |Lv(x, r) - Lv(x, r')| &\leq \| (Dg(x))^{-1} \| |R(x + v(x, r), r) - R(x + v(x, r'), r')| \\ &+ \| (Dg(x))^{-1} \| |Dg(x) (v(x, r) - v(x, r')) \\ &- g(x + v(x, r)) + g(x + v(x, r'))| \\ &+ \| (Dg(x))^{-1} \| \\ &\times |\Psi(x + v(x, r), r) - \Psi(x + v(x, r'), r')| \\ &\leq \| (Dg(x))^{-1} \| (\|A(x)\| + 2\gamma \max\{|r - u(x)|, |r'|\}) + 2\varepsilon \\ &\times |r - r'| \\ &+ \| (Dg(x))^{-1} \| \\ &\times (\omega(\max\{|r - u(x)|, |r' - u(x)|\}) + 2\varepsilon) |r - r'|. \end{aligned} \quad (27)$$

In addition,

$$\begin{aligned} |Lv(x, r) - Lv'(x, r)| &\leq \| (Dg(x))^{-1} \| |R(x + v(x, r), r) - R(x + v'(x, r), r)| \\ &+ \| (Dg(x))^{-1} \| |v(X(x, u(x)), R(x + v(x, r), r)) \\ &- v'(X(x, u(x)), R(x + v(x, r), r))| \\ &+ \| (Dg(x))^{-1} \| |Dg(x) (v(x, r) - v'(x, r)) \\ &- g(x + v(x, r)) + g(x + v'(x, r))| \end{aligned}$$

$$\begin{aligned}
& + \|(Dg(x))^{-1}\| \\
& \times |\Psi(x + v(x, r), r) - \Psi(x + v'(x, r), r)| \\
& \leq \|(Dg(x))^{-1}\| (\|A(x)\| + 2\gamma|r| + 3\varepsilon) \|v - v'\| |r - u(x)| \\
& + \|(Dg(x))^{-1}\| (\omega(|r - u(x)|) + \varepsilon) \\
& \times \|v - v'\| |r - u(x)|.
\end{aligned} \tag{28}$$

We choose $\delta > 0$, where $\max\{|r|, |r'|\} = \delta \leq a$, such that

$$\begin{aligned}
& \sup_x (\|(Dg(x))^{-1}\| \|A(x)\|) \\
& + (5\varepsilon + \omega(8\delta) + 4\gamma\delta) \sup_x \|(Dg(x))^{-1}\| < 1.
\end{aligned} \tag{29}$$

Then $\|\mathcal{L}v\| \leq 1$, $|\mathcal{L}v(x, r) - \mathcal{L}v(x, r')| \leq |r - r'|$, the mapping $\mathcal{L}$ is a contraction, and consequently the functional equation (20) has unique solution in $\mathcal{M}_1$. $\square$

Next we will prove that the mapping H is a homeomorphism in the small neighbourhood of the invariant manifold $r = u(x)$. Let us consider the functional equation

$$\begin{aligned}
& X(x + v_1(x, r), u(x + v_1(x, r))) \\
& = X(x, r) + v_1(X(x, r), R(x, r))
\end{aligned} \tag{30}$$

or equivalently

$$\begin{aligned}
& v_1(x, r) \\
& = (Dg(x))^{-1} (Dg(x) v_1(x, r) \\
& \quad - X(x + v_1(x, r), u(x + v_1(x, r))) \\
& \quad + X(x, r) + v_1(X(x, r), R(x, r))).
\end{aligned} \tag{31}$$

Lemma 6. *The functional equation (30) has a unique solution in $\mathcal{M}_2$.*

Proof. The set

$$\mathcal{M}_2 = \{v \in \mathcal{M} \mid \|v\| \leq 1\} \tag{32}$$

is a closed subset of the Banach space $\mathcal{M}$.

Let us consider the mapping $v_1 \mapsto \mathcal{L}v_1$, $v_1 \in \mathcal{M}_2$ defined by the equality

$$\begin{aligned}
& \mathcal{L}v_1(x, r) \\
& = (Dg(x))^{-1} v_1(X(x, r), R(x, r)) + (Dg(x))^{-1} \\
& \quad \times (Dg(x) v_1(x, r) - g(x + v_1(x, r)) + g(x) \\
& \quad - \Psi(x + v_1(x, r), u(x + v_1(x, r))) + \Psi(x, r)).
\end{aligned} \tag{33}$$

We have

$$\begin{aligned}
& |Lv_1(x, r)| \\
& \leq \|(Dg(x))^{-1}\| |R(x, r) - u(X(x, r))| + \|(Dg(x))^{-1}\| \\
& \quad \times |Dg(x) v_1(x, r) - g(x + v_1(x, r)) + g(x)| \\
& \quad + \|(Dg(x))^{-1}\| \\
& \quad \times |\Psi(x + v_1(x, r), u(x + v_1(x, r))) - \Psi(x, r)| \\
& \leq \|(Dg(x))^{-1}\| (\|A(x)\| + 2\varepsilon + \omega(|r - u(x)|) + 3\varepsilon) \\
& \quad \times |r - u(x)|.
\end{aligned} \tag{34}$$

We obtain

$$\begin{aligned}
& |Lv_1(x, r) - Lv'_1(x, r)| \\
& \leq \|(Dg(x))^{-1}\| \\
& \quad \times |v_1(X(x, r), R(x, r)) - v'_1(X(x, r), R(x, r))| \\
& \quad + \|(Dg(x))^{-1}\| \\
& \quad \times |Dg(x) (v_1(x, r) - v'_1(x, r)) \\
& \quad \quad - g(x + v_1(x, r)) + g(x + v'_1(x, r))| + \|(Dg(x))^{-1}\| \\
& \quad \times |\Psi(x + v_1(x, r), u(x + v_1(x, r))) \\
& \quad \quad - \Psi(x + v'_1(x, r), u(x + v'_1(x, r)))| \\
& \leq \|(Dg(x))^{-1}\| (\|A(x)\| + 2\varepsilon + \omega(|r - u(x)|) + 2\varepsilon) \\
& \quad \times \|v_1 - v'_1\| |r - u(x)|.
\end{aligned} \tag{35}$$

We get that $\mathcal{L}$ is a contraction and consequently the functional equation (30) has a unique solution in $\mathcal{M}_2$. $\square$

Consider the mapping G defined by equality $G(x, r) = (x + v_1(x, r), r)$.

Lemma 7. *One has $G \circ H = id$.*

Proof. Let us consider the functional equation

$$\begin{aligned}
& X(x + v_2(x, r), u(x + v_2(x, r))) \\
& = X(x, u(x)) + v_2(X(x, u(x)), R(x + v(x, r), r))
\end{aligned} \tag{36}$$

or equivalently

$$\begin{aligned}
& v_2(x, r) \\
& = (Dg(x))^{-1} \\
& \quad \times (Dg(x) v_2(x, r) - X(x + v_2(x, r), u(x + v_2(x, r))) \\
& \quad + X(x, u(x)) + v_2(X(x, u(x)), R(x + v(x, r), r))).
\end{aligned} \tag{37}$$

It is easily verified that the functional equation (36) has the trivial solution. Let us prove the uniqueness of the solution in $\mathcal{M}_3$, where

$$\mathcal{M}_3 = \{v_2 \in \mathcal{M} \mid \|v_2\| \leq 3\} \quad (38)$$

is a closed subset of the Banach space $\mathcal{M}$. We get

$$\begin{aligned} & |v_2(x, r)| \\ & \leq \|(Dg(x))^{-1}\| \|v_2\| |R(x + v(x, r), r) - u(X(x, u(x)))| \\ & \quad \times \|(Dg(x))^{-1}\| \\ & \quad \times |Dg(x) v_2(x, r) - g(x + v_2(x, r)) + g(x)| \\ & \quad + \|(Dg(x))^{-1}\| \\ & \quad \times |\Psi(x + v_2(x, r), u(x + v_2(x, r))) - \Psi(x, u(x))| \\ & \leq \|(Dg(x))^{-1}\| \\ & \quad \times (\|A(x)\| + \gamma|r| + 2\varepsilon + \omega(3|r - u(x)|) + 2\varepsilon) \\ & \quad \times \|v_2\| |r - u(x)|. \end{aligned} \quad (39)$$

It follows that $v_2(x, r) \equiv 0$. The mapping w_1 , where

$$w_1(x, r) = v(x, r) + v_1(x + v(x, r), r), \quad (40)$$

also satisfies the functional equation (36). Using the change of variables $x \mapsto x + v(x, r)$ in (30) we get

$$\begin{aligned} & X(x + w_1(x, r), u(x + w_1(x, r))) \\ & = X(x + v(x, r), r) \\ & \quad + v_1(X(x + v(x, r), r), R(x + v(x, r), r)). \end{aligned} \quad (41)$$

Using (20), we obtain

$$\begin{aligned} & X(x + w_1(x, r), u(x + w_1(x, r))) \\ & = X(x, u(x)) + v(X(x, u(x)), R(x + v(x, r), r)) \\ & \quad + v_1(X(x, u(x)) + v(X(x, u(x)), R(x + v(x, r), r)), \\ & \quad \quad R(x + v(x, r), r)) \\ & = X(x, u(x)) + w_1(X(x, u(x)), R(x + v(x, r), r)). \end{aligned} \quad (42)$$

Let us note that

$$|w_1(x, r)| \leq |r - u(x)| + |r - u(x + v(x, r))| \leq 3|r - u(x)|. \quad (43)$$

Therefore $\|w_1\| \leq 3$ and we have

$$v(x, r) + v_1(x + v(x, r), r) = 0. \quad (44)$$

We obtain that $G \circ H = \text{id}$. $\square$

Lemma 8. One has $H \circ G = \text{id}$.

Proof. The set of continuous mappings $v_3 : \mathbf{E} \times \mathbf{B}(\delta) \times \mathbf{B}(\delta) \rightarrow \mathbf{E}$,

$$\mathcal{N} = \left\{ v_3 \in \mathbf{C}(\mathbf{E} \times \mathbf{B}(\delta) \times \mathbf{B}(\delta), \mathbf{E}) \mid \sup_{x, r, z} \frac{|v_3(x, r, z)|}{\max(|r - u(x)|, |z - r|)} < \infty \right\} \quad (45)$$

becomes a Banach space if we use the norm $\|v_3\| = \sup_{x, r, z} (|v_3(x, r, z)| / \max(|r - u(x)|, |z - r|))$. The set

$$\mathcal{N}_1 = \{v_3 \in \mathcal{N} \mid \|v_3\| \leq 1, |v_3(x, r, z) - v_3(x, r, z')| \leq |z - z'|\} \quad (46)$$

is a closed subset of the Banach space $\mathcal{N}$.

Let us consider the functional equation

$$\begin{aligned} & X(x, r) + v_3(X(x, r), R(x, r), R(x + v_3(x, r, z), z)) \\ & = X(x + v_3(x, r, z), z) \end{aligned} \quad (47)$$

or equivalently

$$\begin{aligned} & v_3(x, r, z) \\ & = (Dg(x))^{-1} \\ & \quad \times (Dg(x) v_3(x, r, z) - g(x + v_3(x, r, z)) + g(x) \\ & \quad + \Psi(x, r) - \Psi(x + v_3(x, r, z), z) \\ & \quad + v_3(X(x, r), R(x, r), R(x + v_3(x, r, z), z))). \end{aligned} \quad (48)$$

Let us consider the mapping $v_3 \mapsto \mathcal{L}v_3$, $v_3 \in \mathcal{N}_1$ defined by the equality

$$\begin{aligned} & \mathcal{L}v_3(x, r, z) \\ & = (Dg(x))^{-1} \\ & \quad \times (Dg(x) v_3(x, r, z) - g(x + v_3(x, r, z)) + g(x) \\ & \quad - \Psi(x + v_3(x, r, z), z) + \Psi(x, r) \\ & \quad + v_3(X(x, r), R(x, r), R(x + v_3(x, r, z), z))). \end{aligned} \quad (49)$$

We obtain

$$\begin{aligned} & |Lv_3(x, r, z)| \\ & \leq \|(Dg(x))^{-1}\| \max\{|R(x, r) - u(X(x, r))|, \\ & \quad |R(x + v_3(x, r, z), z) - R(x, r)|\} \end{aligned}$$

$$\begin{aligned}
& + \|(Dg(x))^{-1}\| \\
& \times |Dg(x) v_3(x, r, z) - g(x + v_3(x, r, z)) + g(x)| \\
& + \|(Dg(x))^{-1}\| |\Psi(x, r) - \Psi(x + v_3(x, r, z), z)| \\
& \leq \|(Dg(x))^{-1}\| (\|A(x)\| + \gamma |z| + 2\varepsilon) \\
& \times \max\{|r - u(x)|, |z - r|\} + \|(Dg(x))^{-1}\| \\
& \times (\omega(\max\{|r - u(x)|, |z - r|\}) + 2\varepsilon) \\
& \times \max\{|r - u(x)|, |z - r|\}.
\end{aligned} \tag{50}$$

In addition,

$$\begin{aligned}
& |Lv_3(x, r, z) - Lv_3(x, r, z')| \\
& \leq \|(Dg(x))^{-1}\| \\
& \times |Dg(x)(v_3(x, r, z) - v_3(x, r, z')) \\
& \quad - g(x + v_3(x, r, z)) + g(x + v_3(x, r, z'))| \\
& + \|(Dg(x))^{-1}\| \\
& \times |\Psi(x + v_3(x, r, z), z) - \Psi(x + v_3(x, r, z'), z')| \\
& + \|(Dg(x))^{-1}\| \\
& \times |R(x + v_3(x, r, z), z) - R(x + v_3(x, r, z'), z')| \\
& \leq \|(Dg(x))^{-1}\| \\
& \times (\omega(\max\{|r - u(x)|, |z - r|, |z' - r|\}) + 2\varepsilon) |z - z'| \\
& + \|(Dg(x))^{-1}\| \\
& \times (\|A\| + 2\varepsilon + 2\gamma \max\{|r - u(x)|, |z|, |z' - r|\}) \\
& \times |z - z'|.
\end{aligned} \tag{51}$$

Let $v_3 \in \mathcal{N}_1$ and $v'_3 \in \mathcal{N}_1 \cup \mathcal{N}_2$ where

$$\begin{aligned}
\mathcal{N}_2 = \left\{ v'_3 \in \mathcal{N} \mid \sup_{x, |r| \leq \delta, |z| \leq \delta} |v'_3(x, r, z)| \leq 8\delta, \right. \\
\left. |v'_3(x, r, z) - v'_3(x, r, z')| \leq |z - z'| \right\}.
\end{aligned} \tag{52}$$

We have

$$\begin{aligned}
& |Lv_3(x, r, z) - Lv'_3(x, r, z)| \\
& \leq \|(Dg(x))^{-1}\| \\
& \times |Dg(x)(v_3(x, r, z) - v'_3(x, r, z)) \\
& \quad - g(x + v_3(x, r, z)) + g(x + v'_3(x, r, z))|
\end{aligned}$$

$$\begin{aligned}
& + \|(Dg(x))^{-1}\| \\
& \times |\Psi(x + v_3(x, r, z), z) - \Psi(x + v'_3(x, r, z), z)| \\
& + \|(Dg(x))^{-1}\| \\
& \times |v_3(X(x, r), R(x, r), R(x + v_3(x, r, z), z)) \\
& \quad - v'_3(X(x, r), R(x, r), R(x + v_3(x, r, z), z))| \\
& + \|(Dg(x))^{-1}\| \\
& \times |v'_3(X(x, r), R(x, r), R(x + v_3(x, r, z), z)) \\
& \quad - v'_3(X(x, r), R(x, r), R(x + v'_3(x, r, z), z))| \\
& \leq \|(Dg(x))^{-1}\| (\omega(\max\{|r - u(x)|, |z - r|, 8\delta\}) + \varepsilon) \\
& \times \|v_3 - v'_3\| \max\{|r - u(x)|, |z - r|\} + \|(Dg(x))^{-1}\| \\
& \times \max\{|R(x, r) - u(X(x, r))|, \\
& \quad |R(x + v'_3(x, r, z), z) - R(x, r)|\} \|v_3 - v'_3\| \\
& + \|(Dg(x))^{-1}\| \\
& \times |R(x + v_3(x, r, z), z) - R(x + v'_3(x, r, z), z)| \\
& \leq \|(Dg(x))^{-1}\| (\omega(8\delta) + \varepsilon) \|v_3 - v'_3\| \\
& \times \max\{|r - u(x)|, |z - r|\} \\
& + \|(Dg(x))^{-1}\| (\|A\| + 3\varepsilon + \gamma |z|) \\
& \times \max\{|r - u(x)|, |z - r|\} \|v_3 - v'_3\| \\
& = \|(Dg(x))^{-1}\| (\|A(x)\| + 4\varepsilon + \omega(8\delta) + \gamma |z|) \\
& \times \|v_3 - v'_3\| \max\{|r - u(x)|, |z - r|\}.
\end{aligned} \tag{53}$$

Then $\|\mathcal{L}v_3\| \leq 1$, $|\mathcal{L}v_3(x, r, z) - \mathcal{L}v_3(x, r, z')| \leq |z - z'|$, the mapping $\mathcal{L}$ is a contraction, and consequently the functional equation (47) has a unique solution in $\mathcal{N}_1$. Moreover, this solution is also unique in the closed subset $\mathcal{N}_2$. Let us note that

$$v_3(x, r, r) = 0. \tag{54}$$

The mapping w_2 , where

$$w_2(x, r, z) = v_1(x, r) + v(x + v_1(x, r), z), \tag{55}$$

satisfies (47). Using the change of variables $(x, r) \mapsto (x + v_1(x, r), z)$ in (20) we get

$$\begin{aligned}
& X(x + w_2(x, r, z), z) \\
& = X(x + v_1(x, r), 0) \\
& \quad + v(X(x + v_1(x, r), 0), R(x + w_2(x, r, z), z)).
\end{aligned} \tag{56}$$

Using (30) we obtain

$$\begin{aligned}
 & X(x + w_2(x, r, z), z) \\
 &= X(x, r) + v_1(X(x, r), R(x, r)) \\
 &\quad + v(X(x, r) + v_1(X(x, r), R(x, r)), \\
 &\quad R(x + w_2(x, r, z), z)) \\
 &= X(x, r) \\
 &\quad + w_2(X(x, r), R(x, r), R(x + w_2(x, r, z), z)).
 \end{aligned} \tag{57}$$

Let us note that

$$|w_2(x, r, z) - w_2(x, r, z')| \leq |z - z'|, \tag{58}$$

$$\begin{aligned}
 |w_2(x, r, z)| &\leq |r - u(x)| + |z - r| \\
 &\quad + |r - u(x + v_1(x, r))| \\
 &\leq 4 \max\{|r - u(x)|, |z - r|\}.
 \end{aligned} \tag{59}$$

Therefore $w_2 \in \mathcal{N}_2$ and we have

$$v_1(x, r) + v(x + v_1(x, r), r) = 0. \tag{60}$$

It follows that $H \circ G = \text{id}$.

Finally we conclude that the mapping H is a homeomorphism establishing a conjugacy of the noninvertible mappings (1) and (18). $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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