Global and Blow-Up Solutions for Nonlinear Hyperbolic Equations with Initial-Boundary Conditions

Ülkü Dinlemez¹ and Esra Aktaş²

¹ Department of Mathematics, Faculty of Science, Gazi University, Teknikokullar, Ankara, Turkey
² Incirli Mahallesi, Katedras Sokak, Yumusempre Caddesi 5/18, Incirli, Ankara, Turkey

Correspondence should be addressed to Ülkü Dinlemez; ulku@gazi.edu.tr

Received 24 December 2013; Revised 7 March 2014; Accepted 20 March 2014; Published 13 April 2014

Copyright © 2014 Ü. Dinlemez and E. Aktaş. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider an initial-boundary value problem to a nonlinear string equation with linear damping term. It is proved that under suitable conditions the solution is global in time and the solution with an negative initial energy blows up in finite time.

1. Introduction

We study the damped nonlinear string equation with source term $|u|^{\alpha}u$:

$$u_{tt} + u_t = \sigma \left( |u_x|^2 \right) u_x + |u|^{\alpha}u, \quad (x, t) \in (0, 1) \times [0, T],$$

(1)

where $1 < \alpha$, $\sigma(s)$ is a smooth function for $0 \leq s$ with the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in [0, 1],$$

(2)

and boundary conditions

$$u(1, t) = 0, \quad t \in (0, T),$$

$$\sigma \left( |u_x(0, t)|^2 u_x(0, t) \right) - u_t(0, t) = 2\phi(t), \quad t \in [0, T],$$

(3)

$$\sigma \left( |u_x(0, t)|^2 u_x(0, t) \right) + u_t(0, t) = 2\psi(t), \quad t \in [0, T].$$

The problem (1)–(3) can be regarded as a nonlinear string with vertical displacement function $u(x, t)$ in $\mathbb{R}$. And this problem has nonlinear mechanical damping of the form $|u|^{\alpha}u$. The right end of the string makes it steady. The input $\phi(t)$ function and the output $\psi(t)$ function are applied on the left.

Wu and Li [1] studied the motion for a nonlinear beam model with nonlinear damping $a|\phi|^{p-1}\phi$ and external forcing $b|\phi|^{p-1}\phi$ terms. They showed that this model has a unique global solution and blow-up solution under the same conditions. Levine et al. [2] and Levine and Serrin [3] studied abstract version. Georgiev and Todorova [4] studied nonlinear wave equations involving the nonlinear damping term $|u_t|^{m-1}u_t$ and source term of type $|u_t|^{p-1}u_t$. They proved global existence theorem with large initial data for $1 < p \leq m$.

In this paper we first find energy equation for the problem (1)–(3). Then we prove the solutions of the problem (1)–(3) are global in time under some conditions on the function $\sigma(s)$, input $\phi(t)$, and the output $\psi(t)$. Finally we establish a blow-up result for solutions with a negative initial energy. Our approach is similar to the one in [5].

2. Main Results

Now we give the following lemma for energy equation for the problem (1)–(3).
Lemma 1. Let $1 < \alpha$ and $u(x, t)$ be a solution of the problem (1)–(3). Then the energy equation of the problem (1)–(3) is

$$E(t) = \frac{1}{2} \|u_t\|^2 - \frac{1}{\alpha + 2} \|u\|_{\alpha+2}^2 + \frac{1}{\alpha + 2} \int_0^1 |u_x|^2 \sigma(\xi) \, d\xi \, dx,$$

$$\frac{d}{dt} E(t) = \phi^2(t) - \psi^2(t) - \|u_t\|^2.$$  

Proof. Multiplying (1) with $u_t$ and integrating over $(0, 1)$, then we get

$$\frac{d}{dt} \left\{ \frac{1}{2} \|u_t\|^2 - \frac{1}{\alpha + 2} \|u\|_{\alpha+2}^2 \right\} = \int_0^1 \left( \sigma \left( |u_x|^2 \right) u_t \right) u_t \, dx - \frac{1}{\alpha + 2} \int_0^1 |u_x|^2 \sigma(\xi) \, d\xi \, dx.$$  

Applying integration by parts in the right hand side of (6), we find

$$\int_0^1 \left( \sigma \left( |u_x|^2 \right) u_x \right) u_t \, dx = -\sigma \left( |u_x(x, 0, t)|^2 \right) u_x(x, 0, t) u_t(x, 0, t) - \frac{1}{\alpha + 1} \int_0^1 |u_x|^2 \sigma(\xi) \, d\xi \, dx.$$  

And using boundary conditions in equality (7), we obtain

$$\frac{d}{dt} \left\{ \frac{1}{2} \|u_t\|^2 - \frac{1}{\alpha + 2} \|u\|_{\alpha+2}^2 \right\} = \phi^2(t) - \psi^2(t) - \|u_t\|^2.$$  

Hence the proof is completed. 

Next we give the following theorem for global solutions in time.

Theorem 2. Assume that $u(x, t)$ is a solution of the problem (1)–(3) with $1 < \alpha$ and

(i) $\sigma(s)$ satisfies the following condition:

$$|s|^\alpha \leq \sigma(s), \quad \text{for } s \in \mathbb{R}^+ \cup \{0\},$$  

(ii) the input and the output functions satisfy

$$\phi^2(t) \leq \psi^2(t).$$  

Then the solution $u(x, t)$ is global in time.

Proof. Let

$$G(t) := E(t) + \frac{2}{\alpha + 2} \|u\|_{\alpha+2}^2$$

$$= \frac{1}{2} \|u_t\|^2 + \frac{1}{\alpha + 2} \int_0^1 |u_x|^2 \sigma(\xi) \, d\xi \, dx + \frac{1}{\alpha + 2} \|u\|_{\alpha+2}^2.$$  

Differentiating $G(t)$ with respect to $t$ and using (5), we get

$$\frac{d}{dt} G(t) = \phi^2(t) - \psi^2(t) - \|u_t\|^2 + 2 \int_0^1 |u|^\alpha u u_t \, dx.$$  

Using the Cauchy-Schwarz inequality in the last term of (12), we obtain

$$2 \int_0^1 |u|^\alpha u u_t \, dx \leq 2 \int_0^1 |u|^\alpha+1 |u_t| \, dx$$

and it follows from (12), (13), and (10) that we have

$$\frac{d}{dt} G(t) \leq \|u\|_{\alpha+1}^2 + \|u_t\|^2.$$  

By assumption (9) and integrating over $(0, |u_t|^2)$ and $(0, 1)$, respectively, we yield

$$\frac{1}{\alpha + 1} \|u_x\|_{2(\alpha+1)}^2 \leq \frac{1}{2} \int_0^1 |u_x|^2 \sigma(\xi) \, d\xi.$$  

Furthermore, we have

$$|u(x, t)|^2(\alpha+1) = \left| \int_x^1 u_k(\xi, t) \, d\xi \right|^2$$

$$\leq \int_x^1 |u_k(\xi, t)|^2(\alpha+1) \, d\xi$$

and then

$$\|u(x, t)\|_{2(\alpha+1)}^2 \leq \|u_k(x, t)\|_{2(\alpha+1)}^2.$$  

Combining (11), (14), (15), and (17), we get

$$\frac{d}{dt} G(t) \leq \frac{1}{\xi_1} G(t),$$

where $\xi_1 = \min\{1/2, 1/(\alpha + 1)\}$. Using Gronwall’s inequality, we have

$$G(t) \leq G(0) e^{(\xi_1)t}.$$  

Therefore together with the continuation principle and the definition of $G(t)$ we complete the proof of Theorem 2.

Then we give the following theorem for the blow-up solutions of the problem (1)–(3).

Theorem 3. Let $u(x, t)$ be a solution of the problem (1)–(3) with $1 < \alpha$. Assume that

(i) there exists $1 < \varepsilon < (\alpha + 2)/2$ such that the function $\sigma(s)$ satisfies

$$\sigma(s) s \leq \frac{\varepsilon}{2} \int_0^1 \sigma(\xi) \, d\xi,$$  

for $s \in \mathbb{R}^+ \cup \{0\},$  

(ii) the input and the output functions satisfy

$$\phi^2(t) \leq \psi^2(t).$$  

Then there exists $1 < \varepsilon < \alpha + 2$, such that the function $\sigma(s)$ satisfies

$$\sigma(s) s \leq \frac{\varepsilon}{2} \int_0^1 \sigma(\xi) \, d\xi,$$  

for $s \in \mathbb{R}^+ \cup \{0\},$  

(iii) the solution $u(x, t)$ is global in time.
(ii) the initial values satisfy
\[ E(0) \leq 0, \quad 0 < \int_0^1 u_0(x) u_1(x) \, dx, \]  
(21)

(iii) the input and output functions satisfy
\[ \phi^2(t) \leq \psi^2(t), \]
\[ (\psi(t) + \phi(t)) \left( \int_0^t (\psi(s) - \phi(s)) \, ds + u_0(0) \right) \leq 0, \]  
(22)

(iv) \( u(x,t) \) satisfies
\[ u(0,t) \leq \|u\|, \]
\[ \frac{d}{dt} \|u\|^2 \leq (1 - \gamma) M, \]  
(23)

where \( \eta \) is some positive constant independent of the initial value \( \alpha \) and \( N(t) \) are given by (25).

Proof. We define
\[ M(t) := -E(t), \quad \gamma := \frac{\alpha}{2(\alpha + 2)}. \]  
(24)

\[ N(t) := M^{-\gamma}(t) + \int_0^1 u(x,t) u_t(x,t) \, dx. \]  
(25)

By virtue of (5), (21), (22), and (24), we get
\[ \frac{dM(t)}{dt} = \|u_t\|^2 + \psi^2(t) - \phi^2(t) \geq 0, \]  
(26)

\[ 0 \leq M(0) \leq M(t), \quad \text{for } 0 \leq t. \]  
(27)

Taking a derivative of (25) and using (26), we have
\[ \frac{dN(t)}{dt} = (1 - \gamma) M^{-\gamma}(t) M'(t) + \int_0^1 u_t^2 \, dx + \int_0^1 uu_t \, dx \]
\[ = (1 - \gamma) M^{-\gamma}(t) \left( \|u_t\|^2 + \psi^2(t) - \phi^2(t) \right) + \|u_t\|^2 \]
\[ + \int_0^1 uu_t \, dx. \]  
(28)

Using (22) in (28), we obtain
\[ \left( 1 + \frac{\epsilon}{2} \right) \|u_t\|^2 + \left( 1 - \frac{\epsilon}{\alpha + 2} \right) \|u_t^{\alpha + 2}\|^2 + \frac{1}{2} \|u_t^{\alpha + 2}\|^2 \]
\[ + \int_0^1 \left( \frac{\epsilon}{2} \|u_t\|^2 \sigma(\xi) d\xi - \sigma(\|u_t\|^2) u_x^2 \right) \, dx \]
\[ + \epsilon M(t) - \int_0^1 uu_t \, dx \leq \frac{dN(t)}{dt}. \]  
(32)

Thanks to Young's inequality,
\[ AB \leq \frac{\delta p}{p} A^p + \frac{\delta q}{q} B^q, \quad 0 \leq A, B \forall 0 < \delta, \quad \frac{1}{p} + \frac{1}{q} = 1, \]  
(33)

for \( \int_0^1 uu_t \, dx \) with \( p = q = 2 \) and \( \gamma = 2 \), and then we get
\[ \int_0^1 uu_t \, dx \leq \int_0^1 \|u_t\|^2 \, dx \leq \|u_t\|^2 + \frac{1}{4} \|u\|^2. \]  
(34)

From embedding for \( L^p(0,1) \) and using (iv), we have \( \|u\|_2^2 \leq \|u\|_{\alpha + 2}^2 \) and putting (34) in (32) we have
\[ \left( 1 + \frac{\epsilon}{2} \right) \|u_t\|^2 + \left( 1 - \frac{\epsilon}{\alpha + 2} \right) \|u_t^{\alpha + 2}\|^2 + \frac{1}{2} \|u_t^{\alpha + 2}\|^2 \]
\[ + \int_0^1 \left( \frac{\epsilon}{2} \|u_t\|^2 \sigma(\xi) d\xi - \sigma(\|u_t\|^2) u_x^2 \right) \, dx + \epsilon M(t) \]
\[ - \|u_t\|^2 - \frac{1}{4} \|u\|^2 \leq \frac{dN(t)}{dt}. \]  
(35)
From (20), we get
\[ \varepsilon M(t) + \frac{\varepsilon}{2} \| u_t \|_2^2 + \left( \frac{1}{2} - \frac{\varepsilon}{\alpha + 2} \right) \| u \|_{\alpha+2}^{\alpha+2} + \frac{1}{4} \| u \|_2^4 \leq \frac{dN(t)}{dt}. \]  
(36)

Choosing \( \varepsilon \) and \( \kappa = \min \{ \varepsilon/2, (1/2 - \varepsilon/(\alpha + 2)), 1/4 \} \), we obtain
\[ \kappa \left\{ M(t) + \| u_t \|_2^2 + \| u \|_{\alpha+2}^{\alpha+2} + \| u \|_2^2 \right\} \leq \frac{dN(t)}{dt}. \]  
(37)

Thanks to (21) and (27), we yield
\[ 0 < N(0) \leq N(t), \quad \forall 0 < t. \]  
(38)

Now we estimate \( [N(t)]^{1/(1-\gamma)} \). From Holder’s inequality,
\[ \left| \int_0^1 u u_t^\delta \, dx \right| \leq \| u \|_2 \| u_t \|_2 \leq \| u \|_{\alpha+2} \| u_t \|_2; \]  
(39)

then using Young’s inequality again we get
\[ \left| \int_0^1 u u_t^\delta \, dx \right|^{1/(1-\gamma)} \leq 2^{1/(1-\gamma)} \left( \frac{\delta^2}{2 (1 - \gamma)} \| u_t \|_2^2 \right)^{1/(1-\gamma)} + \left( \frac{1 - 2\gamma}{2 (1 - \gamma)} \right)^{1/(1-\gamma)} \delta^{-2/(1-2\gamma)} \| u \|_{\alpha+2}^{2(1-2\gamma)/(1-2\gamma)}, \]  
(40)

where \( 0 < \delta \) and \( 1/p + 1/q = 1 \) with \( p = 2(1 - \gamma) \). And so we have
\[ \left| \int_0^1 u u_t^\delta \, dx \right|^{1/(1-\gamma)} \leq \beta \left( \| u_t \|_2^2 + \| u \|_{\alpha+2}^{\alpha+2} \right), \]  
(41)

Choosing \( \beta = \max \{ \delta^2/(1 - \gamma)^{1/(1-\gamma)}, ((1 - 2\gamma)/(1 - \gamma))^{1/(1-\gamma)} \} \delta^{-2/(1-2\gamma)} \), we obtain
\[ \left| \int_0^1 u u_t^\delta \, dx \right|^{1/(1-\gamma)} \leq \beta \left( \| u_t \|_2^2 + \| u \|_{\alpha+2}^{\alpha+2} \right). \]  
(42)

Therefore we yield
\[ (N(t))^{1/(1-\gamma)} = \left( M^{-\gamma}(t) + \int_0^t u(x,t) u_t(x,t) \, dx \right)^{1/(1-\gamma)} \leq 2^{1/(1-\gamma)} \left( M(t) + \int_0^t u(x,t) u_t(x,t) \, dx \right)^{1/(1-\gamma)}, \]  
(43)

where \( C \) depends on \( \delta \) and \( \alpha \). From (37) and (43), we have
\[ \eta(N(t))^{1/(1-\gamma)} \leq \frac{dN(t)}{dt}, \]  
(44)

where \( \eta = \kappa/C \). Integrating (44) over \((0, t)\), then we get
\[ \frac{1}{(N(0))^{-\alpha/(\alpha+4)} - (\alpha/(\alpha+4)) \eta t} \leq (N(t))^{\alpha/(\alpha+4)}. \]  
(45)

Hence \( N(t) \) blows up in finite time \( T_{\text{max}} \). \( T_{\text{max}} \) is given by the inequality as below:
\[ T_{\text{max}} \leq \frac{\alpha + 4}{\alpha \eta} (N(0))^{-\alpha/(\alpha+4)}. \]  
(46)

Consequently the solution blows up in finite time. And the proof of Theorem 3 is now finished.

\[ \square \]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgment**

The authors would like to thank the referees for the careful reading of this paper and for the valuable suggestions to improve the presentation and style of the paper.

**References**


