1. Introduction

Consider the second order half-linear difference equation:

\[ \Delta \left[ p_n \Delta y_n \right] + q_n y_n = 0, \quad n \geq n_0, \quad \alpha > 1, \quad (1) \]

where \( \Delta \) is the forward difference operator and \( \{ p_n \} \), \( \{ q_n \} \) are sequences of nonnegative real numbers with \( p_n > 0 \). The study of (1) has been initiated by Reháček in [1]. It is well known that there is a close similarity between (1) and the linear second order difference equation. Indeed, if \( \{ y_n \} \) is a solution of (1), then so is \( \{ cy_n \} \) for any constant \( c \). Thus, (1) has one half of linearity properties [2].

In the presence of damping, (1) has been extended further to the second order half-linear difference equation with damping term of the form

\[ \Delta \left[ p_n \Delta y_n \right] + q_n \Delta y_n + r_n y_{n+1} = 0, \quad n \geq n_0, \quad \alpha > 1, \quad (2) \]

where \( \{ q_n \} \) is a sequence of nonnegative real numbers. It is to be noted that neither (1) nor (2) has involved a delaying term. There are numerous numbers of oscillation criteria established in the literature for the solutions of (1) and (2). Most of these results were obtained by using certain efficient tools among them we name the Riccati transformation, variational principle, and some inequality techniques; see, for instance, the monograph [3] in which many contributions have been cited therein and to the recent papers [4–9].

Let \( Q : \mathbb{R} \to \mathbb{R} \) be defined by \( Q(s) = |s|^{\alpha-2} ; \alpha > 1 \) is a fixed real number and \( \mathbb{N}_{n_0} = \{ n_0, n_0 + 1, \ldots \} \). Consider the \( m \)-th order half-linear functional difference equation with damping term of the form

\[ \Delta \left[ p_n Q \left( \Delta^{m-1} y_n \right) \right] + q_n Q \left( \Delta^{m-1} y_n \right) + r_n Q \left( y_{n+1} \right) = 0, \quad n \in \mathbb{N}_{n_0}, \quad (3) \]

where \( m \) is even number, and

(H1) \( \{ p_n \} : \mathbb{N}_{n_0} \to \mathbb{R}^+ \) with \( \Delta p_n \geq 0 \) for all \( n \geq n_0 \);  
(H2) \( \{ q_n \} \) and \( \{ r_n \} : \mathbb{N}_{n_0} \to \mathbb{R} \) with \( q_n \geq 0 \) and \( r_n > 0 \);  
(H3) \( \{ r_n \} : \mathbb{N}_{n_0} \to \mathbb{Z} \) with \( r_n < n \) and \( \lim_{n \to \infty} r_n = \infty \).

For close results regarding the continuous counterparts of (1), (2), and (3), the reader is suggested to consult [10–14].

A primary purpose of this paper is to establish sufficient conditions that guarantee the oscillation of solutions of (3). Our main results are obtained via employing the generalized
Riccati transformation. In view of (3), one can easily figure out that it is formulated in more general form so that it includes some particular cases which have been studied in the literature; see [15–23] for more details. To the best of authors’ observation, however, no published result has been concerned with the investigation of oscillatory behavior of solutions of (3) or its continuous counterpart. Therefore, our paper is new and presents a new approach.

2. Main Results

We start by recalling the following standard definitions.

Definition 1. A nontrivial sequence $y_n$ is called a solution of (3) if it is defined for all $n \geq 0$ where $n \in \mathbb{Z}, \sigma = \min_{n \geq n_1} |r_1|$, and $p_n Q(\Delta^{m-1} y_n)$ is differenceable on $\mathbb{N}_{n_1}$ and satisfies (3) for all $n \in \mathbb{N}_{n_1}$.

Definition 2. A nontrivial solution $y_n$ of (3) is said to be oscillatory if the terms of the sequence $y_n$ are not eventually positive or not eventually negative. Otherwise, the sequence is called nonoscillatory. A difference equation is called oscillatory if all its solutions oscillate.

To obtain our main results, we need the following essential lemmas. The first of these is the discrete analogue of the well-known Kiguradze's lemma.

Lemma 3 (see [24]). Let $y_n$ be defined for $n \geq n_0 \in \mathbb{N}$ and $y(n) > 0$ with $\Delta^m y_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer $l$, $0 \leq l \leq m$ with $(m + l)$ odd for $\Delta^m y_n \leq 0$ and $(m + l)$ even for $\Delta^m y(n) \geq 0$ such that

(i) $l \leq m - 1$ implies $(-1)^l \Delta^l y_n > 0$ for all $n \geq m$, $l \leq i \leq m - 1$, 
(ii) $l \geq 1$ implies $\Delta^l y_n > 0$ for all large $n \geq n_0$, $1 \leq i \leq l - 1$.

Lemma 4 (see [25]). Let $y_n$ be defined for $n \geq n_0$ and $y_n > 0$ with $\Delta^m y_n \leq 0$ for $n \geq n_0$ and not identically zero. Then, there exists a large integer $n_1 \geq n_0$ such that

$$y_n \geq \frac{1}{(m - 1)!} (n - n_1)^{m-1} \Delta^{m-1} y_{2^{m-1} n_1}, \quad n \geq n_1, \quad (4)$$

where $l$ is defined as in Lemma 3. Further, if $y_n$ is increasing, then

$$y_n \geq \frac{1}{(m - 1)!} \left( \frac{n}{2^{m-1}} \right)^{m-1} \Delta^{m-1} y_n, \quad n \geq 2^{m-1} n_1. \quad (5)$$

Lemma 5. Let $y_n$ satisfy conditions of Lemmas 3 and 4 and $\Delta^{m-1} y_n\Delta^m y_n \leq 0$ for $n \geq n_1 \geq n_0$. Further, if $y_n$ is increasing, then

$$\Delta y_{n-k} \geq Mn^{m-2} \Delta^{m-1} y_n, \quad n \geq n_1, \quad (6)$$

where $M = (1/((m - 1)!2^{(m-1)^2})) > 0$.

The proof of Lemma 5 is straightforward and it can be achieved by using the last inequality of Lemma 4.

Lemma 6. Let $y_n$ be an eventually positive solution of (3). If

$$\lim_{n \to \infty} \frac{1}{P_k} \sum_{k=n_1}^{n-1} \left( 1 - \frac{q_k}{P_k} \right) \Delta y_n \geq 0,$$

(14)

then $\Delta^{m-1} y_n > 0, \Delta^m y_n \leq 0$, and $\Delta y_n > 0$ for all $n \geq n_1 \geq n_0$.

Proof. The fact that $y_n$ is eventually positive solution of (3) implies $y_n > 0$ and $y_n > 0$ for all $n \geq n_1 \geq n_0$. In view of (3), we get

$$\Delta \left[ p_n Q(\Delta^{m-1} y_n) \right] + q_n Q(\Delta^{m-1} y_n) < 0, \quad (8)$$

which leads to

$$\Delta \left[ p_n Q(\Delta^{m-1} y_n) - \prod_{k=n_1}^{n-1} \left( 1 - \frac{q_k}{P_k} \right) p_n Q(\Delta^{m-1} y_n) \right] < 0. \quad (9)$$

Hence,

$$p_n Q(\Delta^{m-1} y_n) - \prod_{k=n_1}^{n-1} \left( 1 - \frac{q_k}{P_k} \right) p_n Q(\Delta^{m-1} y_n) \leq 0.$$ 

(10)

is decreasing and $\Delta^{m-1} y_n$ is eventually positive or eventually negative.

We claim that

$$\Delta^{m-1} y_n > 0, \quad n \geq n_1. \quad (11)$$

Assume, on the contrary, that $\Delta^{m-1} y_n < 0, n \geq n_1$. Then, from (10), we obtain

$$p_n \Delta^{m-1} y_n \geq \frac{n-1}{\prod_{k=n_1}^{n-1} (1 - q_k/ P_k)} p_n \Delta^{m-1} y_n = 0.$$ 

Therefore, from (12), we have

$$p_n \Delta^{m-1} y_n \geq \frac{n-1}{\prod_{k=n_1}^{n-1} (1 - q_k/ P_k)} p_n \Delta^{m-1} y_n = 0.$$ 

(13)
where $M_1 = P_{n_1}^{-1/(\alpha-1)} |\Delta^{m-1} y_{n_1}| > 0$. It follows that

\[
(\Delta^{m-1} y_n)^{\alpha-1} \geq \frac{M_1^{\alpha-1}}{P_n} \left( 1 - \prod_{k=n_1}^{n-1} \left( 1 - \frac{q_k}{P_k} \right) \right)
\]  

or

\[
\Delta^{m-1} y_n \leq -M_1 \left( \frac{1}{P_n} \left( 1 - \prod_{k=n_1}^{n-1} \left( 1 - \frac{q_k}{P_k} \right) \right) \right)^{1/(\alpha-1)}.
\]  

Consequently, we obtain

\[
\Delta^{m-2} y_n \leq \Delta^{m-2} y_n
\]  

\[
- M_1 \sum_{n_1}^{n-1} \left( \frac{1}{P_n} \left( 1 - \prod_{k=n_1}^{n-1} \left( 1 - \frac{q_k}{P_k} \right) \right) \right)^{1/(\alpha-1)}.
\]  

Letting $n \to \infty$ in the above inequality, one gets $\lim_{n \to \infty} \Delta^{m-2} y_n = -\infty$. Hence, $y_n$ is an eventually negative function which contradicts that $y_n > 0$. Therefore, inequality (11) holds.

From (3), we get

\[
\Delta \left[ p_n Q \left( \Delta^{m-1} y_n \right) \right] = p_{n+1} \Delta \left( \Delta^{m-1} y_n \right)^{\alpha-1} + \left( \Delta^{m-1} y_n \right)^{\alpha-1} \Delta p_n \leq 0
\]

from which it follows that

\[
\Delta \left( \Delta^{m-1} y_n \right)^{\alpha-1} \leq 0, \quad n \geq n_1 \geq n_0.
\]

The above inequality implies that $\Delta^{m-1} y_n^{\alpha-1}$ is nonincreasing. Therefore, we can write

\[
\Delta \left( \Delta^{m-1} y_n \right)^{\alpha-1} = \left( \Delta^{m-1} y_{n+1} \right)^{\alpha-1} - \left( \Delta^{m-1} y_n \right)^{\alpha-1}
\]

\[
= \left( \Delta^{m-1} y_{n+1} - \Delta^{m-1} y_n \right) \times \left[ \left( \Delta^{m-1} y_{n+1} \right)^{\alpha-2} + \left( \Delta^{m-1} y_{n+1} \right)^{\alpha-3} \left( \Delta^{m-1} y_n \right) \right]
\]

\[
+ \left( \Delta^{m-1} y_{n+1} \right)^{\alpha-4} \left( \Delta^{m-1} y_n \right)^2
\]

\[
+ \cdots + \left( \Delta^{m-1} y_n \right)^{\alpha-2} \leq 0.
\]

Since $\left( \Delta^{m-1} y_n \right)^{\alpha-1}$ is nonincreasing and positive, then from the above inequality, we have

\[
\Delta \left( \Delta^{m-1} y_n \right)^{\alpha-1} \leq \left[ \Delta^{m-1} y_{n+1} - \Delta^{m-1} y_n \right] (\alpha - 1) \left( \Delta^{m-1} y_n \right)^{\alpha-2}
\]

\[
= (\alpha - 1) \Delta \left( \Delta^{m-1} y_n \right) \left( \Delta^{m-1} y_n \right)^{\alpha-2}
\]

\[
\leq (\alpha - 1) \Delta^{m} y_n \left( \Delta^{m-1} y_n \right)^{\alpha-2} \leq 0
\]

by which we have

\[
\Delta^m y_n \leq 0.
\]

In virtue of (21) and Lemma 3, we deduce that since $m$ is even then $l$ is odd. Hence $\Delta y_n > 0$ for $n \geq n_1 \geq n_0$. The proof is complete.

**Theorem 7.** Let condition (A1) hold. Further, assume that there exists a constant $\lambda > \alpha - 1$ such that

\[
\lim \sup_{n \to \infty} \frac{1}{(n - n_1)\lambda} \sum_{k=n_1}^{n-1} \left( (n - k) \lambda \right) \leq \infty,
\]

where

\[
X_{nk} = \left( (n + 1 - k)^{\lambda} - (n - k)^{\lambda} \left( 1 + \frac{q_k}{P_{k+1}} \right) \right),
\]

\[
Y_{nk} = (\alpha - 1) M \tau^{-m-2} (n - k) \frac{P_k}{P_{k+1}} > 0,
\]

and $M$ is as in Lemma 5. Then, (3) is oscillatory.

**Proof.** For the sake of contradiction, assume that (1) has a nonoscillatory solution $y_n$. Without loss of generality, we assume that $y_n$ is eventually positive (the proof is similar when $y_n$ is eventually negative). That is, $y_n > 0, y_{\tau_n} > 0$ and $y_{\tau_n-k} > 0$ for all $n \geq n_1 \geq n_0$. By Lemma 6, we have $\Delta^{m-1} y_n > 0, \Delta^m y_n \leq 0$, and $\Delta y_n > 0$ for $n \geq n_1$. Consider the function

\[
\omega_n = \frac{p_n Q \left( \Delta^{m-1} y_n \right)}{Q \left( y_{\tau_n-k} \right)} = \frac{p_n \left( \Delta^{m-1} y_n \right)^{\alpha-1}}{\left( y_{\tau_n-k} \right)^{\alpha-1}} > 0, \quad n \geq n_1.
\]

Taking into account that $\Delta y_n > 0$ and $y_n$ is increasing and $\tau_{n-k} < \tau_n$, we deduce that $\Delta^m y_n \leq 0$ and $\Delta^{m-1} y_n$ is nonincreasing. Lemmas 3 and 4, (1), and (24) yield

\[
\Delta \omega_n = -\frac{q_n \left( \Delta^{m-1} y_n \right)^{\alpha-1} - r_n \left( y_{\tau_n-k} \right)^{\alpha-1}}{\left( y_{\tau_n-k} \right)^{\alpha-1}}
\]

\[
- p_n \frac{\Delta \left( \Delta^{m-1} y_n \right)^{\alpha-1}}{\left( y_{\tau_n-k} \right)^{\alpha-1}}
\]

\[
\leq -r_n - \frac{q_n}{p_{n+1}} \omega_{n+1}
\]

\[
= \frac{(\alpha - 1) p_n \left( \Delta^{m-1} y_n \right)^{\alpha-1} \Delta y_{\tau_n-k}}{\left( y_{\tau_n-k} \right)^{\alpha-1}}
\]

\[
= \frac{(\alpha - 1) \Delta^{m-1} y_n \left( \Delta^{m-1} y_n \right)^{\alpha-2}}{\left( y_{\tau_n-k} \right)^{\alpha-1}}
\]
\[\frac{1}{(n-n_1)\lambda} \sum_{k=n_1}^{n-1} (n-k)^\lambda r_k \leq \sum_{k=n_1}^{n-1} (n-k)^\lambda w_{n_k} - \sum_{k=n_1}^{n-1} \left( n+1-k \right)^\lambda \frac{q_k}{p_{k+1}} w_{k+1} \]

\[\leq (n-n_1)^\lambda w_{n_1} + \sum_{k=n_1}^{n-1} \left( n+1-k \right)^\lambda \frac{q_k}{p_{k+1}} w_{k+1} \]

\[\leq (n-n_1)^\lambda w_{n_1} + \sum_{k=n_1}^{n-1} \left( n+1-k \right)^\lambda \frac{q_k}{p_{k+1}} w_{k+1} \]

or

\[\frac{1}{(n-n_1)\lambda} \sum_{k=n_1}^{n-1} (n-k)^\lambda r_k - \sum_{k=n_1}^{n-1} \left( n+1-k \right)^\lambda \left( \frac{\Delta \delta_k}{\delta_k} (M_{r_k}^{m-2})^{1-\alpha} \right) \]

\[\leq w_{n_1}, \quad (27)\]

where

\[X_{nk} = (n+1-k)^\lambda \left( n-k \right)^\lambda \frac{1}{(n-k)^\lambda} \frac{q_k}{p_{k+1}} \frac{p_{k+1}}{p_k} \]

\[Y_{nk} = (n+1-k)^\lambda \left( n-k \right)^\lambda \frac{1}{(n-k)^\lambda} \frac{q_k}{p_{k+1}} \frac{p_{k+1}}{p_k} > 0. \]

Let

\[F(w_{k+1}) = X_{nk} w_{k+1} - Y_{nk} w_{k+1}^{\alpha/(n-k)} \]

Then, \(F\) has maximum value at \(w_{k+1} = ((n+1)/n)^{1-\alpha} X_{nk} Y_{nk} \).

That is,

\[F_{\text{max}} = (n+1)^{1-\alpha} X_{nk} Y_{nk}^{1-\alpha} \]

(30)

Therefore, (27) can be rewritten as

\[\frac{1}{(n-n_1)\lambda} \sum_{k=n_1}^{n-1} (n-k)^\lambda r_k - \sum_{k=n_1}^{n-1} \left( n+1-k \right)^\lambda \frac{q_k}{p_{k+1}} w_{k+1} \]

\[\leq w_{n_1} \]

(31)

Hence, we have

\[\lim_{n \to \infty} \frac{1}{(n-n_1)\lambda} \sum_{k=n_1}^{n-1} (n-k)^\lambda r_k - \sum_{k=n_1}^{n-1} \left( n+1-k \right)^\lambda \frac{q_k}{p_{k+1}} w_{k+1} \]

\[\leq w_{n_1} \]

(32)

which contradicts condition (A2). The proof is complete. \(\square\)

**Theorem 8.** Let condition (A1) hold. Further, assume that there exists a function \(\delta_n : \mathbb{N} \to \mathbb{R}^+\) such that

\[(A3) \quad \lim_{n \to \infty} \sum_{k=n_1}^{n-1} \delta_k r_k - \frac{1}{\alpha^\alpha} \frac{\delta_k}{\delta_k} (M_{rk}^{m-2})^{1-\alpha} \]

\[= 0, \quad n_1 \geq n_0, \]

(33)

where \(M\) is as in Lemma 5. Then, (3) is oscillatory.

**Proof.** For the sake of contradiction, assume that (3) has a nonoscillatory solution \(y_n\). Without loss of generality, we assume that \(y_n\) is eventually positive (the proof is similar when \(y_n\) is eventually negative). That is, \(y_n > 0\), \(y_{r_n} > 0\) and \(y_{r_{n+1}} > 0\) for all \(n \geq n_1 \geq n_0\). By Lemma 6, we have
\[ \Delta^{m-1} y_n > 0, \Delta^m y_n \leq 0, \text{ and } \Delta y_n > 0 \text{ for } n \geq n_1. \] Consider the function
\[ w_n = \delta_n p_n \left( \frac{\Delta^{m-1} y_n}{y_{n+1}} \right)^{\alpha-1} > 0, \quad n \geq n_1. \] (34)
By utilizing the same approach as in the proof of Theorem 7, we arrive at
\[ \Delta w_n \leq -\delta_n r_n + \frac{1}{\alpha^2} \left( \frac{\Delta \delta}{\delta_k} \right)^\alpha \left( M r_n^{m-2} \right)^{1-\alpha}. \] (35)
Summing up (35) from \( n_1 \) to \( n - 1 \), we have
\[ \sum_{k=n_1}^{n-1} \left[ \delta_k r_k - \frac{1}{\alpha^2} \left( \frac{\Delta \delta}{\delta_k} \right)^\alpha \left( M r_k^{m-2} \right)^{1-\alpha} \right] \leq w_{n_1}. \] (36)
Letting \( n \to \infty \) in the above inequality and taking the upper limit, we get a contradiction to \( \Delta^3 \). The proof is complete.

**Remark 9.** In view of the statements of Theorems 7 and 8, one can easily deduce that condition \( (\Delta^3) \) is a generalization of \( (\Delta^2) \).

**Example 10.** Consider the fourth order half-linear functional difference equation with damping
\[ \Delta \left[ p_n \left( \Delta^3 y_n \right) + q_n \left( \Delta^2 y_n \right)^2 \right] + n \left( \Delta y_n \right)^2 = 0, \quad n \geq 2, \] (37)
where \( p_n = n, q_n = n, r_n = 1/n, \tau_n = n - 1, m = 4, \) and \( \alpha = 3 \). It is easy to see that conditions \( (H1)-(H3) \) are satisfied. It remains to check the validity of conditions \( A1 \) and \( A2 \).

For \( n \geq 2 \), we have
\[ \Gamma_1 := \sum_{s=n_1}^{n-1} \left( \frac{1}{p_s} \left( 1 - \prod_{v=s}^{s-1} \left( 1 - \frac{q_v}{p_v} \right) \right) \right)^{1/(\alpha-1)} = \sum_{s=2}^{n-1} \left( \frac{1}{s} \right)^{1/2}. \] (38)
It is clear that \( \Gamma_1 \to \infty \) as \( n \to \infty \). Therefore, condition \( A1 \) holds. For \( n \geq 2 \) and \( \lambda = 3 > \alpha - 1 = 2 \), we have
\[ \Gamma_2 := \frac{1}{(n-n_1)^3} \sum_{k=n_1}^{n-1} \left[ \frac{(n-k)^3}{k} - \frac{(\alpha - 1)^{\alpha-1}}{\alpha^\alpha} X_{nk}^{\alpha} Y_{nk}^{1-\alpha} \right] \]
\[ = \frac{1}{(n-2)^3} \sum_{k=2}^{n-1} \left[ \frac{(n-k)^3}{k} - \frac{4}{27} X_{nk}^{3} Y_{nk}^{2} \right], \] (39)
where
\[ X_{nk}^{3} = \left( \frac{(k+1) (n+1-k)^3 - (2k+1) (n-k)^3}{k+1} \right)^3, \] \[ Y_{nk}^{2} = \frac{(k+1)^2}{4M^2 k^2 (k-2)^3 (n-k)^6}. \] (40)
It is clear that \( \Gamma_2 \to \infty \) as \( n \to \infty \). Then, condition \( A2 \) holds. Thus, by the conclusion of Theorem 7, (37) is oscillatory.

**Example II.** Consider the sixth order half-linear functional difference equation with damping
\[ \Delta \left[ n \left( \Delta^3 y_n \right) + q_n \left( \Delta^2 y_n \right)^2 \right] + n^2 \left( \Delta y_n \right)^2 = 0, \quad n \geq 2, \] (41)
where \( p_n = n, q_n = n, r_n = n^2, \tau_n = n - 1, m = 6, \) and \( \alpha = 3 \). It is easy to see that conditions \( (H1)-(H3) \) are satisfied. In Example 10, we have seen that \( (\Delta^3) \) is satisfied. It remains to check the validity of condition \( (\Delta^2) \).

For \( n \geq 2 \) and \( \delta_n = n \), we have
\[ \Gamma_3 := \sum_{k=n_1}^{n-1} \left[ \delta_k r_k - \frac{1}{\alpha^2} \left( \frac{\Delta \delta}{\delta_k} \right)^\alpha \left( M r_k^{m-2} \right)^{1-\alpha} \right] \]
\[ = \sum_{k=2}^{n-1} \left[ k^2 (k+1) - \frac{1}{27M^2 k^6 (k-1)^8} \right]. \] (42)
\[ = \sum_{k=2}^{n-1} \left[ 27M^2 k^5 (k+1) (k-1)^8 - 1 \right] \frac{1}{27M^2 k^6 (k-1)^8}. \]
It is clear that \( \Gamma_3 \to \infty \) as \( n \to \infty \). Then, condition \( (\Delta^2) \) holds. Thus, by the conclusion of Theorem 8, (41) is oscillatory.

**Remark 12.** It is not possible to decide the oscillatory behavior of solutions of (37) and (41) by using any of the results reported in [12, 13]. This implies that the results of our paper extend and generalize some known theorems.

**Remark 13.** The main results of this paper remain valid for nondelay difference equations of the form
\[ \Delta \left[ p_n Q \left( \Delta^{m-1} y_n \right) + q_n Q \left( \Delta^m y_n \right) + r_n Q \left( y_n \right) = 0, \quad n \in \mathbb{N}_{n_1}, \] (43)

**Conflict of Interests**
The authors declare that there is no conflict of interests regarding the publication of this paper.

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