Research Article

The Dynamical Analysis of a Prey-Predator Model with a Refuge-Stage Structure Prey Population

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1. Introduction

The development of the qualitative analysis of ordinary differential equations is deriving to study many problems in mathematical biology. The modeling for the population dynamics of a prey-predator system is one of the important and interesting goals in mathematical biology, which has received wide attention by several authors [1–6]. In the natural world many kinds of prey and predator species have a life history that is composed of at least two stages: immature and mature, and each stage has different behavioral properties. So, some works of stage structure prey-predator models have been provided in a good number of papers in the literatures [7–12]. Zhang et al. in [9] and Cui and Takeuchi in [11] proposed two mathematical models of prey stage structure; in these models the predator species consumes exclusively the immature prey. Indeed, there are many factors that impact the dynamics of prey-predator interactions such as disease, harvesting, prey refuge, delay, and many other factors. Several prey species have gone to extinction, and this extinction must be caused by external effects such as overutilization, overpredation, and environmental factors (pollution, famine). Prey may avoid becoming attacked by predators either by protecting themselves or by living in a refuge where it will be out of sight of predators. Some theoretical and empirical studies have shown and tested the effects of prey refuges and making an opinion that the refuges used by prey have a stabilizing influence on prey-predator interactions; also prey extinction can be prevented by the addition of refuges; for instance, we can refer to [13–23]. Therefore, it is important and worthwhile to study the effects of a refuge on the prey population with prey stage structure. Consequently, in this paper, we proposed and analyzed the prey-predator model involving a stage structure in prey population together with a prey’s refuge property as a defensive property against the predation.

2. Mathematical Model

In this section a prey-predator model with a refuge-stage structure prey population is proposed for study. Let \(x(T)\) represent the population size of the immature prey at time \(T\); \(y(T)\) represents the population size of the mature prey at time \(T\), while \(z(T)\) denotes the population size of the predator species at time \(T\). Therefore in order to describe the dynamics...
of this model mathematically the following hypotheses are adopted:

(1) The immature prey grows exponentially depending completely on its parents with growth rate \( r > 0 \). There is an intraspecific competition between their individuals with intraspecific competition rate \( \delta_1 > 0 \). The immature prey individual becomes mature with grownup rate \( \beta > 0 \) and faces natural death with a rate \( d_1 > 0 \).

(2) There is an intraspecific competition between the individuals of mature prey population with intraspecific competition rate \( \delta_2 > 0 \). Further the mature prey species faces natural death rate too with a rate \( d_2 > 0 \).

(3) The environment provides partial protection of prey species against the predation with a refuge rate \( 0 < m < 1 \); therefore there is \( 1 - m \) of prey species available for predation.

(4) There is an intraspecific competition between the individuals of predator population with intraspecific competition rate \( \delta_3 > 0 \). Further the predator species faces natural death rate too with a rate \( d_3 > 0 \).

The predator consumes the prey in both the compartments according to the mass action law represented by Lotka–Volterra type of functional response with conversion rates \( 0 < e_1 < 1 \) and \( 0 < e_2 < 1 \) for immature and mature prey, respectively. Consequently, the dynamics of this model can be represented mathematically with the following set of differential equations.

\[
\begin{align*}
\dot{x} &= ry - \delta_1 x^2 - d_1 x - \beta x - y_1 (1 - m) x z \\
\dot{y} &= \beta x - \delta_2 y^2 - d_2 y - y_2 (1 - m) y z \\
\dot{z} &= e_1 y_1 (1 - m) x z + e_2 y_2 (1 - m) y z - \delta_3 z^2 - d_3 z
\end{align*}
\]

with initial conditions, \( x(0) > 0 \), \( y(0) > 0 \), and \( z(0) > 0 \).

In order to simplify the analysis of system (1) the number of parameters in system (1) is reduced using the following dimensionless variables in system (1):

\[
\begin{align*}
y_1 &= \frac{e_1 y_1 (1 - m)}{d_2} x, \\
y_2 &= \frac{e_2 y_2 (1 - m)}{d_2} y, \\
y_3 &= \frac{\delta_3}{d_2} z, \\
t &= d_2 T.
\end{align*}
\]

Accordingly, the dimensionless form of system (1) can be written as

\[
\begin{align*}
\dot{y}_1 &= a_1 y_2 - a_2 y_2^2 - a_3 y_1 - a_4 y_1 y_3 \\
\dot{y}_2 &= b_1 y_1 - b_2 y_2^2 - y_2 - b_3 y_2 y_3 \\
\dot{y}_3 &= y_1 y_3 + y_2 y_3 - y_3^2 - b_4 y_3,
\end{align*}
\]

where the dimensionless parameters are given by

\[
\begin{align*}
a_1 &= \frac{e_1 y_1 r}{e_2 y_2 d_2}, \\
a_2 &= \frac{\delta_1}{e_1 y_1 (1 - m)}, \\
a_3 &= \frac{d_1 + \beta}{d_2}, \\
a_4 &= \frac{y_1 (1 - m)}{d_3}, \\
b_1 &= \frac{e_2 y_2 \beta}{e_1 y_1 d_2}, \\
b_2 &= \frac{\delta_2}{e_2 y_2 (1 - m)}, \\
b_3 &= \frac{y_2 (1 - m)}{d_3}, \\
b_4 &= \frac{d_3}{d_2}.
\end{align*}
\]

According to the equations given in system (3), all the interaction functions are continuous and have a continuous partial derivatives. Therefore they are Lipschitzian and hence the solution of system (3) exists and is unique. Moreover the solution of system (3) is bounded as shown in the following theorem.

**Theorem 1.** All the solutions of system (3) that initiate in the positive octant are uniformly bounded.

**Proof.** Let \( W = y_1 + y_2 + y_3 \) be solutions of system (3) with initial conditions, \( y_1(0) > 0, y_2(0) > 0, \) and \( y_3(0) > 0 \). Then by differentiation \( W \) with respect to \( t \) we get

\[
\frac{dW}{dt} \leq a_1 y_2 \left( 1 - \frac{y_2}{a_1/b_2} \right) + b_1 y_1 \left( 1 - \frac{y_1}{b_1/a_2} \right) - \sigma_1 W
\]

\[
\leq \sigma_2 - \sigma_1 W;
\]

here \( \sigma_1 = \min \{a_1, b_1, b_2\} \) and \( \sigma_2 = a_1^2/4b_2 + b_1^2/4a_2 \). Now by using comparison theorem, we get

\[
0 < W \leq \left( W_0 e^{-\sigma_1 t} + \frac{\sigma_2}{\sigma_1} (1 - e^{-\sigma_1 t}) \right).
\]

Thus for \( t \to \infty \) we obtain \( 0 < W \leq \sigma_2/\sigma_1 \). Hence, all solutions of system (3) in \( R^3_+ \) are uniformly bounded and therefore we have finished the proof.

### 3. Local Stability Analysis

It is observed that system (3) has at most three biologically feasible equilibrium points; namely, \( E_i \), \( i = 0, 1, 2 \). The existence conditions for each of these equilibrium points are discussed below:
(1) The trivial equilibrium points $E_0 = (0,0,0)$ exist always.

(2) The predator free equilibrium point is denoted by $E_1 = (\tilde{y}_1, \tilde{y}_2, 0)$, where
\[
\tilde{y}_1 = \frac{1}{b_1} \left( b_2 \tilde{y}_2^2 + \tilde{y}_2 \right)
\] while $\tilde{y}_2$ is a positive root of the following third-order polynomial
\[
A_1 y_2^3 + A_2 y_2^2 + A_3 y_2 + A_4 = 0;
\]
here, $A_1 = a_2 b_2 / b_1^2$, $A_2 = 2a_2 b_2 / b_1^2$, $A_3 = (a_2 + a_4 b_2) / b_1$, and $A_4 = (a_4 - a_1 b_1) / b_1$ exist uniquely in the interior of $y_1, y_2$-plane if and only if the following condition holds:
\[
a_3 < a_1 b_1.
\]

(3) The interior (positive) equilibrium point is given by
\[
E_2 = (y_1^*, y_2^*, y_3^*),
\]
where
\[
y_1^* = \frac{[(b_2 + b_3) y_2^* + (1 - b_3 b_4)] y_2^*}{b_1 - b_3 y_2^*},
\]
\[
y_3^* = y_1^* + y_2^* - b_4
\]
while $y_2^*$ is a positive root of the following third-order polynomial:
\[
B_1 y_2^3 + B_2 y_2^2 + B_3 y_2 + B_4 = 0;
\]
here,
\[
B_1 = a_2 (b_2 + b_3)^2 + a_4 b_2 (b_2 + b_4) > 0,
\]
\[
B_2 = [2a_2 (b_2 + b_3) + 2a_4 b_2 + a_3 b_4] (1 - b_3 b_4)
\]
\[
- \left[ a_2 b_2 b_3 + b_3^2 (a_3 + a_1) \right],
\]
\[
B_3 = 2a_4 b_2 b_3 + (a_2 + a_4) (1 - b_3 b_4)^2
\]
\[
+ b_1 (a_4 - a_1 b_3) (b_2 + b_4)
\]
\[
+ [b_4 a_4 - b_3 (a_3 - a_1 b_2)] (1 - b_3 b_4),
\]
\[
B_4 = b_1 [a_4 - a_1 b_3 - a_4 b_4 b_2 - b_4 (1 - b_3 b_4)].
\]

Clearly, (11) has a unique positive root represented by $y_2^*$ if the following set of conditions hold:
\[
B_4 < 0 \quad \text{with} \quad (B_2 > 0 \text{ or } B_3 < 0)
\]

Therefore, $E_2$ exists uniquely in int. $R^3$ if in addition to condition (13) the following conditions are satisfied.
\[
y_1^* + y_2^* > b_4
\]
\[
\frac{b_4 b_3 - 1}{b_2 + b_3} < y_2^* < \frac{b_4}{b_3}
\]
or
\[
\frac{b_1}{b_3} < y_2^* < \frac{b_1 b_3 - 1}{b_2 + b_3}.
\]

Now to study the local stability of these equilibrium points, the Jacobian matrix $J(y_1, y_2, y_3)$ for the system (3) at any point $(y_1, y_2, y_3)$ is determined as
\[
\begin{pmatrix}
-2a_2 y_1 - a_3 y_2 & a_1 & -a_4 y_1 \\
b_1 & -2b_2 y_2 - 1 - b_3 y_3 & -b_3 y_2 \\
y_3 & y_3 & y_1 + y_2 - 2y_3 - b_4
\end{pmatrix}.
\]
Thus, system (3) has the following Jacobian matrix near $E_0 = (0,0,0)$.
\[
J(E_0) = \begin{pmatrix}
a_3 & a_1 & 0 \\
b_1 & -1 & 0 \\
0 & 0 & -b_4
\end{pmatrix}.
\]

Then the characteristic equation of $J(E_0)$ is given by
\[
(\lambda + b_4) \left[ \lambda^2 + (a_3 + 1) \lambda + a_3 - a_1 b_1 \right] = 0.
\]

Clearly, all roots of (17) have negative real parts if and only if the following condition holds:
\[
a_3 > a_1 b_1.
\]

So, $E_0$ is locally asymptotically stable under condition (18) and saddle point otherwise. Therefore, $E_0$ is locally asymptotically stable whenever $E_1$ does not exist and unstable whenever $E_2$ exists.

The Jacobian matrix of system (3) around the equilibrium point $E_1 = (\tilde{y}_1, \tilde{y}_2, 0)$ reduced to
\[
J(E_1) = \begin{pmatrix}
-2a_2 \tilde{y}_1 - a_3 & a_1 & -a_4 \tilde{y}_1 \\
b_1 & -2b_2 \tilde{y}_2 - 1 & -b_3 \tilde{y}_2 \\
0 & 0 & \tilde{y}_1 + \tilde{y}_2 - b_4
\end{pmatrix}.
\]

Then the characteristic equation of $J(E_1)$ is written by
\[
[\lambda - a_3] \left[ \lambda^2 - (a_{11} + a_{22}) \lambda + a_{11} a_{22} - a_{12} a_{21} \right] = 0.
\]

Straightforward computation shows that all roots of (20) have negative real part provided that
\[
\tilde{y}_1 + \tilde{y}_2 < b_4
\]
\[
(2a_2 \tilde{y}_1 + a_3) (2b_2 \tilde{y}_2 + 1) > a_1 b_1.
\]

So, $E_1$ is locally asymptotically stable if the above two conditions hold.

Finally the Jacobian matrix of system (3) around the interior equilibrium point $E_2 = (y_1^*, y_2^*, y_3^*)$ is written
\[
J(E_2) = \begin{pmatrix}
a_1 & a_1 & -a_4 y_1 \\
b_1 & -2b_2 y_2 - 1 & -b_3 y_2 \\
y_3 & y_3 & y_1 + y_2 - 2y_3 - b_4
\end{pmatrix}.
\]
here
\[
\begin{align*}
    b_{11} &= -(2a_2y_1^* + a_4y_3^* + a_3), \\
    b_{12} &= a_1, \\
    b_{13} &= -a_4y_1^*, \\
    b_{21} &= b_1, \\
    b_{23} &= -(2b_2y_1^* + b_3y_3^* + 1), \\
    b_{31} &= y_1^*, \\
    b_{32} &= y_3^*, \\
    b_{33} &= -y_3^*.
\end{align*}
\]

Hence, the characteristic equation of \( J(E_2) \) becomes
\[
\lambda^3 + D_1\lambda^2 + D_2\lambda + D_3 = 0
\]
with
\[
\begin{align*}
    D_1 &= -(b_{11} + b_{22} + b_{33}) > 0, \\
    D_2 &= b_{11}b_{22} - b_{21}b_{12} + b_{13}b_{31} - b_{13}b_{13} + b_{22}b_{33} \\
           &\quad - b_{23}b_{32}, \\
    D_3 &= b_{33}(b_{12}b_{21} - b_{11}b_{22}) + b_{11}b_{23}b_{32} - b_{12}b_{23}b_{31} \\
           &\quad + b_{22}b_{31}b_{31} - b_{21}b_{13}b_{31}.
\end{align*}
\]

Consequently, \( \Delta = D_1D_2 - D_3 \) can be written as
\[
\begin{align*}
    \Delta &= (b_{11} + b_{22})(b_{12}b_{21} - b_{11}b_{22}) - b_{11}b_{22}b_{33} \\
           &\quad - (b_{11} + b_{22})(b_{13}b_{31} - b_{13}b_{31}) \\
           &\quad - b_{22}b_{33} - b_{11}b_{33} - b_{22}b_{33} + b_{13}b_{31} + b_{21} \\
           &\quad + b_{22}b_{32} + b_{33} + b_{12}.
\end{align*}
\]

Since \( D_1 > 0 \), then according to Routh-Hurwitz criterion \( E_2 \) is locally asymptotically stable if and only if \( D_1 > 0 \) and \( \Delta = D_1D_2 - D_3 > 0 \). According to the form of \( D_3 \) and the signs of Jacobian elements the last four terms are positive, while the first term will be positive under the sufficient condition (28) below. However \( \Delta \) becomes positive if and only if in addition to condition (28) the second sufficient condition given by (29) holds.
\[
(2a_2y_1^* + a_4y_3^* + a_3)(2b_2y_1^* + b_3y_3^* + 1) > a_1b_1
\]
\[
y_1^* > b_1,
\]
\[
y_3^* > a_1.
\]

Therefore under these two sufficient conditions \( E_2 \) is locally asymptotically stable.

### 4. Global Stability

In this section the global stability for the equilibrium points of system (3) is investigated by using the Lyapunov method as shown in the following theorems.

**Theorem 2.** Assume that the vanishing equilibrium point \( E_0 \) is locally asymptotically stable; then it is globally asymptotically stable in \( \mathbb{R}^3_+ \) if and only if the following condition holds:
\[
\frac{a_4b_1}{a_3} < b_3 < \frac{a_4}{a_1}.
\]

**Proof.** Consider the following positive definite real valued function:
\[
V_0(y_1, y_2, y_3) = \frac{1}{a_4}y_1 + \frac{1}{b_3}y_2 + y_3.
\]

Straightforward computation shows that the derivative of \( V_0 \) with respect to \( t \) is given by
\[
\frac{dV_0}{dt} < \left( \frac{a_4b_1 - a_4b_3}{a_4b_3} \right)y_1 + \left( \frac{a_4b_3 - a_4}{a_4b_3} \right)y_2 - b_3y_3.
\]

Therefore, by using condition (30), we obtain \( dV_0/\text{dt} \) which is negative definite in \( \mathbb{R}^3_+ \), and then \( V_0 \) is a Lyapunov function with respect to \( E_0 \). Hence \( E_0 \) is globally asymptotically stable in \( \mathbb{R}^3_+ \) and the proof is complete.

**Theorem 3.** Assume that the predator free equilibrium point \( E_1 = (\tilde{y}_1, \tilde{y}_2, 0) \) is locally asymptotically stable; then it is globally asymptotically stable in \( \mathbb{R}^3_+ \) if the following condition holds:
\[
\left( \frac{a_1}{a_4y_1} + \frac{b_1}{b_3y_2} \right)^2 < 4 \left( \frac{a_1\tilde{y}_2}{a_4\tilde{y}_1y_1} + \frac{a_2}{a_4} \right) \left( \frac{b_1\tilde{y}_1}{b_3y_2y_2} + \frac{b_2}{b_3} \right).
\]

**Proof.** Consider the following positive definite real valued function:
\[
V_1(y_1, y_2, y_3) = \frac{1}{a_4} \left( y_1 - \tilde{y}_1 - \tilde{y}_1 \ln \frac{y_1}{\tilde{y}_1} \right) \\
+ \frac{1}{b_3} \left( y_2 - \tilde{y}_2 - \tilde{y}_2 \ln \frac{y_2}{\tilde{y}_2} \right) + y_3.
\]

Straightforward computation shows that the derivative of \( V_1 \) with respect to \( t \) is given by
\[
\frac{dV_1}{dt} = - \left( \frac{a_1\tilde{y}_2}{a_4\tilde{y}_1y_1} + \frac{a_2}{a_4} \right)(y_1 - \tilde{y}_1)^2 \\
- \left( \frac{b_1\tilde{y}_1}{b_3y_2y_2} + \frac{b_2}{b_3} \right)(y_2 - \tilde{y}_2)^2 \\
+ \left( \frac{a_1}{a_4\tilde{y}_1} + \frac{b_1}{b_3y_2} \right)(y_1 - \tilde{y}_1)(y_2 - \tilde{y}_2) - y_3^2
\]
Now using condition (33) gives us that
\[
\frac{dV_1}{dt} < \left[ \frac{a_1 \tilde{y}_2}{a_1 y_1 \tilde{y}_1} + \frac{a_2}{a_4} (y_1 - \tilde{y}_1) \right] \\
- \sqrt{\frac{b_1 \tilde{y}_1 + b_2}{b_1 \tilde{y}_2}} (y_2 - \tilde{y}_2)^2 - (b_4 - \tilde{y}_1 - \tilde{y}_2) y_3.
\]
(36)

Clearly \(\frac{dV_1}{dt}\) is negative definite due to local stability condition (21). Hence \(V_1\) is a Lyapunov function with respect to \(E_1\), and then \(E_1\) is globally asymptotically stable, which completes the proof.

**Theorem 4.** Assume that the interior equilibrium point \(E_2 = (y_1^*, y_2^*, y_3^*)\) is locally asymptotically stable in \(R^3\); then it is globally asymptotically stable if and only if the following condition holds:
\[
\left( \frac{y_2^* a_1 b_3}{y_1^* b_1} - a_4 \right)^2 < 4 \left( \frac{y_2^* a_1 b_3}{y_1^* b_1} \right)^2.
\]
(37)

**Proof.** Consider the following positive definite real valued function around \(E_2\):
\[
V_1 (y_1, y_2, y_3) = \left( y_1 - y_1^* \right) \ln \frac{y_1}{y_1^*} + \left( y_2 - y_2^* \right) \ln \frac{y_2}{y_2^*} + \left( y_3 - y_3^* \right) \ln \frac{y_3}{y_3^*}.
\]
(38)

Our computation for the derivative of \(V_2\) with respect to \(t\) gives that
\[
\frac{dV_2}{dt} = \left( \frac{y_2^* a_1 b_3}{y_1^* b_1} - a_4 \right) (y_1 - y_1^*)^2
- \frac{y_2^* a_1 b_3}{y_1^* b_1} (y_2 - y_2^*)^2
+ \left( \frac{y_2^* a_1 b_3}{y_1^* b_1} - a_4 \right) (y_3 - y_3^*)^2.
\]
(39)

Now by using the condition (37) we obtain that
\[
\frac{dV_2}{dt} < \left( \frac{a_1 y_2}{y_1 y_3} (y_2 y_1^* - y_1 y_2^*)^2 - \frac{y_2^* a_1 b_3}{y_1^* b_1} (y_2 - y_2^*)^2 \right)
- \left[ \sqrt{a_2} (y_1 - y_1^*) - \frac{y_2^* a_1 b_3}{y_1^* b_1} (y_3 - y_3^*) \right]^2.
\]
(40)

According to the above inequality we have \(dV_2/dt\) which is negative definite; therefore, \(E_2\) is globally asymptotically stable in \(R^3\) and hence the proof is complete.

### 5. Local Bifurcation

In this section the local bifurcation near the equilibrium points of system (3) is investigated using Sotomayor’s theorem for local bifurcation [24]. It is well known that the existence of nonhyperbolic equilibrium point is a necessary but not sufficient condition for bifurcation to occur. Now rewrite system (3) in the form
\[
\dot{Y} = f (Y),
\]
(41)
where
\[
Y = (y_1, y_2, y_3)^T,
\]
\[
f = (f_1, f_2, f_3)^T.
\]
(42)

Then according to Jacobian matrix of system (3) given in (15), it is simple to verify that for any nonzero vector \(U = (u_1, u_2, u_3)^T\) we have
\[
D^2 f (y_1, y_2, y_3) (U, U) = \left( \begin{array}{c} -2a_2 u_1^2 - 2a_4 u_1 u_3 \\ -2b_2 u_2^2 - 2b_3 u_2 u_3 \\ 2u_1 u_3 + 2u_2 u_3 - 2u_3^2 \end{array} \right)
\]
(44)
and therefore
\[
D^3 f (y_1, y_2, y_3) (U, U, U) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right).
\]
(45)

Thus system (3) has no pitchfork bifurcation due to (45). Moreover, the local bifurcation near the equilibrium points is investigated in the following theorems:

**Theorem 5.** System (3) undergoes a transcritical bifurcation near the vanishing equilibrium point, but saddle node bifurcation cannot occur, when the parameter \(a_0\) passes through the bifurcation value \(a_0^* = a_1 b_1\).

**Proof.** According to the Jacobian matrix \(J(E_0)\) given by (16), system (3) at the equilibrium point \(E_0\) with \(a_0 = a_0^*\) has zero eigenvalue, say \(\lambda_0^* = 0\), and the Jacobian matrix \(J(E_0, a_0^*)\) becomes
\[
J (E_0, a_0^*) = J_0^* = \left( \begin{array}{ccc} -a_0^* & a_1 & 0 \\ b_1 & -1 & 0 \\ 0 & 0 & -b_3 \end{array} \right).
\]
(46)
Now let $U^{[0]} = (u_1^{[0]}, u_2^{[0]}, u_3^{[0]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda^*_1 = 0$. Thus $J_1^*U^{[0]} = 0$ gives $U^{[0]} = (u_1^{[0]}, b_1u_1^{[0]}, 0)^T$, where $u_1^{[0]}$ represents any nonzero real number. Also, let $W^{[0]} = (w_1^{[0]}, w_2^{[0]}, w_3^{[0]})^T$ represents the eigenvector corresponding to the eigenvalue $\lambda^*_3 = 0$ of $J_3^T$. Hence $f_0^*W^{[0]} = 0$ gives that $W^{[0]} = (w_1^{[0]}, a_1w_1^{[0]}, 0)^T$, where $w_1^{[0]}$ denotes any nonzero real number. Now, since
\[
\frac{df}{da_3} = f_{a_3}(Y, a_3) = \left(\frac{df_1}{da_3}, \frac{df_2}{da_3}, \frac{df_3}{da_3}\right)^T \quad (47)
\]
thus $f_{a_3}(E_0, a^*_3) = (0, 0, 0)^T$, which gives $(W^{[0]})^T f_{a_3}(E_0, a^*_3) = 0$. So, according to Sotomayor’s theorem for local bifurcation system (3) has no saddle node bifurcation at $a_3 = a^*_3$. Also, since
\[
Df_{a_3}(E_0, a^*_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (48)
\]
then,
\[
(W^{[0]})^T \left( Df_{a_3}(E_0, a^*_3) U^{[0]} \right) = (w_1^{[0]}, a_1w_1^{[0]}, 0)(-u_1^{[0]}, 0, 0)^T = -u_1^{[0]}w_1^{[0]} \neq 0. \quad (49)
\]
Moreover, by substituting $E_0, a^*_3$, and $U^{[0]}$ in (44) we get that
\[
D^2 f(E_0, a^*_3)(U^{[0]}, U^{[0]}) = \begin{pmatrix} -2a_1 & 0 & 2b_1 \\ 0 & -2b_2 & 0 \\ 2b_1 & 0 & -2b_2 \end{pmatrix}(u_1^{[0]}, u_1^{[0]}, 0)^T. \quad (50)
\]
Hence, it is obtain that
\[
(W^{[0]})^T D^2 f(E_0, a^*_3)(U^{[0]}, U^{[0]}) = -2(a_2 + a_1b_2b_1^2)(u_1^{[0]}, u_1^{[0]}, 0)^T \neq 0. \quad (51)
\]
Thus, according to Sotomayor’s theorem system (3) has a transcritical bifurcation at $E_0$ as the parameter $a_3$ passes through the value $a^*_3$; thus the proof is complete.

**Theorem 6.** Assume that condition (22) holds; then system (3) undergoes a transcritical bifurcation near the predator free equilibrium point $E_1$, but saddle node bifurcation cannot occur, when the parameter $b_4$ passes through the bifurcation value $b^*_4 = \bar{y}_1 + \bar{y}_2$.

**Proof.** According to the Jacobian matrix $J(E_1)$ given by (19), system (3) at the equilibrium point $E_1$ with $b_4 = b^*_4$ has zero eigenvalue, say $\lambda^*_1 = 0$, and the Jacobian matrix $J(E_1, b^*_4)$ becomes
\[
J(E_1, b^*_4) = J_1^* = (a^*_j)_{3 \times 3}, \quad (52)
\]
where $a^*_j = a_j \forall i, j = 1, 2, 3$, with $a^*_3 = 0$. Now let $U^{[1]} = (u_1^{[1]}, u_2^{[1]}, u_3^{[1]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda^*_1 = 0$. Thus $J_1^*U^{[1]} = 0$ gives $U^{[1]} = (u_1^{[1]}, b_1u_1^{[1]}, 0)^T$, where $u_1^{[1]}$ represents any nonzero real number. Also, let $W^{[1]} = (w_1^{[1]}, w_2^{[1]}, w_3^{[1]})^T$ represents the eigenvector corresponding to the eigenvalue $\lambda^*_3 = 0$ of $J_3^T$. Hence $f_0^*W^{[1]} = 0$ gives that $W^{[1]} = (w_1^{[1]}, a_1w_1^{[1]}, 0)^T$, where $w_1^{[1]}$ denotes any nonzero real number. Now, since
\[
\frac{df}{db_4} = f_{b_4}(Y, a_3) = \left(\frac{df_1}{db_4}, \frac{df_2}{db_4}, \frac{df_3}{db_4}\right)^T \quad (53)
\]
thus $f_{b_4}(E_0, b^*_4) = (0, 0, 0)^T$, which gives $(W^{[1]})^T f_{b_4}(E_0, b^*_4) = 0$. So, according to Sotomayor’s theorem for local bifurcation system (3) has no saddle node bifurcation at $b_4 = b^*_4$. Also, since
\[
Df_{b_4}(E_1, b^*_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (54)
\]
then, we can have
\[
(W^{[1]})^T \left( Df_{b_4}(E_1, b^*_4) U^{[1]} \right) = (0, 0, 0, -u_1^{[1]}, 0, 0, 0, -u_1^{[1]}, 0, 0, 0, 0)^T = -u_1^{[1]}w_1^{[1]} \neq 0. \quad (55)
\]
Moreover, substituting $E_1, b^*_4$, and $U^{[1]}$ in (44) gives
\[
D^2 f(E_1, b^*_4)(U^{[1]}, U^{[1]}) = 2(u_1^{[1]}, u_1^{[1]}, 0)^T \cdot (-2a_1 - 2b_1b_1^2)(u_1^{[1]}, u_1^{[1]}, 0)^T. \quad (56)
\]
Hence, it is obtain that
\[
(W^{[1]})^T D^2 f(E_1, b^*_4)(U^{[1]}, U^{[1]}) = 2(a_1u_1^{[1]} + 2b_1u_1^{[1]} + 2b_1^2u_1^{[1]})(u_1^{[1]}, u_1^{[1]}, 0)^T \neq 0. \quad (57)
\]
Thus, according to Sotomayor’s theorem system (3) has a transcritical bifurcation at $E_1$ as the parameter $b_4$ passes through the value $b^*_4$; thus the proof is complete.

**Theorem 7.** Assume that condition (21) holds; then system (3) undergoes a saddle node bifurcation near the predator free equilibrium point $E_1$, when the parameter $a_1$ passes through the bifurcation value $a^*_1 = 2a_1\bar{y}_1 + a_3(2b_2\bar{y}_2 + 1)/b_4$.

**Proof.** According to the Jacobian matrix $J(E_1)$ given by (19), system (3) at the equilibrium point $E_1$ with $a_1 = a^*_1$ has zero eigenvalue, say $\lambda^*_1 = 0$, and the Jacobian matrix $J(E_1, a^*_1)$ becomes
\[
J(E_1, a^*_1) = J_1^* = (a^*_j)_{3 \times 3}, \quad (58)
\]
where \( a_{ij}^* = a_{ij} \forall i, j = 1, 2, 3 \), with \( a_{12}^* = a_1^* \). Now let \( U^{[1]} = (u_1^{[1]}, u_2^{[1]}, u_3^{[1]})^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_1^* = 0 \). Thus \( J^{[1]}U^{[1]} = 0 \) gives \( U^{[1]} = (-a_1^*/a_{11}^*)u_2^{[1]}_2, u_2^{[1]}_1, u_2^{[1]}_0)^T \), where \( u_2^{[1]}_i \) represents any nonzero real numbers. Also, let \( W^{[1]} = (w_1^{[1]}, w_2^{[1]}, w_3^{[1]})^T \) represent the eigenvector corresponding to eigenvalue \( \lambda_1^* = 0 \) of \( J^{[1]}^*T \). Hence \( J^{[1]}^*TW^{[1]} = 0 \) gives that \( W^{[1]} = (-a_{11})u_2^{[1]}_1, w_2^{[1]}_1, (s_i/a_{13}^*)u_2^{[1]}_1)^T \), where \( s_i = a_{13}^*a_{21}^* - a_{11}^*a_{32}^* \) is negative due to the sign of the Jacobian elements and \( w_2^{[1]}_1 \) denotes any nonzero real numbers. Now, since
\[
df{df}{da_i} = f_{ai}(Y, a_i) = \left( \frac{df_1}{da_i}, \frac{df_2}{da_i}, \frac{df_3}{da_i} \right)^T = (y_2, 0, 0)^T
\] (59)
thus \( f_{a_i}(E_1, a_i^*) = (\bar{y}_2, 0, 0)^T \); hence \( (W^{[1]})^* f_{a_i}(E_1, a_i^*) = (-a_{11}^*/a_{11})\bar{y}_2 \neq 0 \). So, according to Sotomayor’s theorem for local bifurcation the first condition of saddle node bifurcation is satisfied in system (3) at \( a_1 = a_1^* \). Moreover, substituting \( E_1, a_1^* \), and \( U^{[1]} \) in (44) gives
\[
D^2 f(E_1, a_1^*)(U^{[1]}, U^{[1]}) = \left( u_2^{[1]}_1 \right)^2 \left( \frac{-2a_2a_3^*}{a_{11}^*}, -2b_2, 0 \right)^T.
\] (60)
Hence, it is obtained that
\[
(W^{[1]})^* D^2 f(E_1, a_1^*)(U^{[1]}, U^{[1]}) = 2 \left( \frac{a_2a_3^*}{a_{11}^*} - b_2 \right) \left( u_2^{[1]}_1 \right)^2 w_2^{[1]} \neq 0.
\] (61)
Thus, according to Sotomayor’s theorem system (3) has a saddle node bifurcation at \( E_1 \) as the parameter \( a_1 \) passes through the value \( a_1^* \); thus the proof is complete.

**Theorem 8.** Assume that
\[
(2a_2^*y_1^* + a_4y_3^* + a_3^*)(2b_2y_2^* + b_3y_3^* + 1) < a_1b_1
\] (62)
y_3^* < \frac{a_1}{a_4}.
\] (63)
Then system (3) undergoes a saddle node bifurcation near the interior equilibrium point \( E_2 \), as the parameter \( b_1 \) passes through the bifurcation value \( b_1^* = \Gamma_1/\Gamma_2 \), where \( \Gamma_1 \) and \( \Gamma_2 \) are given in the proof.

**Proof.** According to the determinant of the Jacobian matrix \( J(E_2) \) given by \( D_3 \) in (25), condition (62) represents a necessary condition to have nonpositive determinant for \( J(E_2) \). Now rewrite the form of the determinant as follows:
\[
D_3 = \Gamma_1 - b_1 \Gamma_2.
\] (64)
Here \( \Gamma_1 = b_1b_2b_3 + b_2b_3b_1 - b_1b_3b_2 = \Gamma_2 = b_1b_2b_3 - b_2b_3b_1 \). Obviously, \( \Gamma_1 \) is positive always, while \( \Gamma_2 \) is positive under the condition (63). Thus it is easy to verify that
\[
D_3 = 0 \text{ and hence } J(E_2) \text{ has zero eigenvalue, say } \lambda_2^* = 0, \text{ as } b_1 \text{ passes through the value } b_1^* = \Gamma_1/\Gamma_2, \text{ which means that } E_2 \text{ becomes a nonhyperbolic point. Now let the Jacobian matrix of system (3) at } E_2 \text{ with } b_1 = b_1^* \text{ be given by}
\]
\[
J(E_2, b_1^*) = J_2^* = (b_1^* \Phi_2)_{b_3^*},
\] (65)
where \( b_1^* = b_j \ \forall j = 1, 2, 3 \), and \( b_2^* = b_2^* \).

Let \( U^{[1]} = (u_1^{[1]}, u_2^{[1]}, u_3^{[1]})^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_2^* = 0 \). Thus \( J_2^*U^{[1]} = 0 \) gives \( U^{[1]} = (\Phi_1 u_1^{[1]}, \Phi_2 u_2^{[1]}, u_3^{[1]})^T \), where \( \Phi_1 = (b_1b_2b_3 - b_2b_3b_1)(b_1b_2 - b_2b_3)(b_3b_1 - b_1b_2)/(b_1b_2 + b_2b_3) \) and \( \Phi_2 = (b_1b_3 - b_1b_2)/(b_1b_3 - b_2b_1) \). The \( \Phi_i \) are positive due to the Jacobian elements and \( u_3^{[1]} \) represents any nonzero real number. Also, let \( W^{[1]} = (w_1^{[1]}, w_2^{[1]}, w_3^{[1]})^T \) represent the eigenvector corresponding to eigenvalue \( \lambda_2^* = 0 \) of \( J_2^*T \). Hence \( J_2^*T W^{[1]} = 0 \) gives that \( W^{[1]} = (-a_{11})u_2^{[1]}_1, w_2^{[1]}_1, (s_i/a_{13})u_2^{[1]}_1)^T \), where \( s_i = a_{13}a_{21} - a_{11}a_{32} \) is negative due to the sign of the Jacobian elements and \( w_2^{[1]}_1 \) denotes any nonzero real numbers. Now, since
\[
\frac{df}{db_i} = f_{b_i}(Y, b_i) = \left( \frac{df_1}{db_i}, \frac{df_2}{db_i}, \frac{df_3}{db_i} \right)^T = (0, y_1, 0)^T
\] (66)
thus \( f_{b_i}(E_2, b_1^*) = (0, y_1^*, 0)^T \), which gives \( (W^{[1]})^* f_{b_i}(E_2, b_1^*) = (\Theta_1 w_1^{[1]}, \Theta_2 w_2^{[1]}, w_3^{[1]})^T \), where \( \Theta_1 = (b_1b_2b_3 - b_2b_3b_1)/(b_1b_2 - b_2b_3) \) and \( \Theta_2 = (b_1b_3 - b_1b_2)/(b_1b_3 - b_2b_1) \). The \( \Theta_i \) are positive due to the Jacobian elements and \( w_3^{[1]} \) denotes any nonzero real numbers. Now, since
\[
D^2 f(E_2, b_1^*)(U^{[1]}, U^{[1]}) = 2 \left( u_3^{[1]} \right)^2 \left( -a_2 \Phi_1^* - a_3 \Phi_1, -b_2 \Phi_2^* + b_3 \Phi_2, \Phi_1 + \Phi_2 - 1 \right)^T.
\] (67)
Hence, it is obtained that
\[
(W^{[1]})^* D^2 f(E_2, b_1^*)(U^{[1]}, U^{[1]}) = 2 \left( u_3^{[1]} \right)^2 \left[ -a_2 \Phi_1^* - a_3 \Phi_1, -b_2 \Phi_2^* + b_3 \Phi_2, \Phi_1 + \Phi_2 - 1 \right] \neq 0.
\] (68)
So, according to Sotomayor’s theorem, system (3) has a saddle node bifurcation as \( b_1 \) passes through the value \( b_1^* \) and hence the proof is complete.

**6. Numerical Simulations**

In this section, the global dynamics of system (3) is investigated numerically. The objectives first confirm our obtained analytical results and second specify the control set of parameters that control the dynamics of the system. Consequently, system (3) is solved numerically using the following biologically feasible set of hypothetical parameters with different sets
of initial points and then the resulting trajectories are drawn in the form of phase portrait and time series figures.

\begin{align*}
a_1 &= 2, \\
a_2 &= 0.1, \\
a_3 &= 0.4, \\
a_4 &= 0.5, \\
b_1 &= 0.4, \\
b_2 &= 0.1, \\
b_3 &= 0.5, \\
b_4 &= 0.2. \\
\end{align*}

(69)

Clearly, Figure 1 shows the asymptotic approach of the solutions, which started from different initial points to a positive equilibrium point (0.46, 0.15, 0.42), for the data given by (69). This confirms our obtained result regarding the existence of globally asymptotically stable positive point of system (3) provided that certain conditions hold.

Now in order to discuss the effect of the parameters values of system (3) on the dynamical behavior of the system, the system is solved numerically for the data given in (69) with varying one parameter each time. It is observed that varying parameters values \(a_2, a_4, b_2, b_3,\) and \(b_4\) have no qualitative effect on the dynamical behavior of system (3) and the system still approaches to a positive equilibrium point. On the other hand, when \(a_1\) decreases in the range \((a_1 \leq 1.05)\) keeping other parameters fixed as given in (69) the dynamical behavior of system (3) approaches asymptotically to the vanished equilibrium point as shown in the typical figure given by Figure 2. Similar observations have been obtained on the behavior of system (3) in case of increasing the parameter \(a_3\) in the range \((a_3 \geq 0.8)\) or decreasing the parameter \(b_1\) in the range \((b_1 \leq 0.2)\), with keeping other parameters fixed as given in (69), and then the solution of system (3) is depicted in Figures 3 and 4, respectively. Finally, for the parameters \(a_2 = 0.9\) and \(b_4 = 0.7\) with other parameters fixed as given in (69), the solution of system (3) approaches asymptotically to the predator free equilibrium point as shown in the Figure 5.

7. Discussion

In this paper, a model that describes the prey-predator system having a refuge and stage structure properties in the prey population has been proposed and studied analytically as well as numerically. Sufficient conditions which ensure the
Figure 4: Time series of system (3) approaches asymptotically to the vanishing equilibrium point for $b_1 = 0.15$ with other parameters given by (69).

Figure 5: Time series of system (3) approaches asymptotically to the predator free equilibrium point $E_1 = (0.42, 0.16, 0)$ for $a_2 = 0.9$ and $b_4 = 0.7$ with other parameters given by (69).

stability of equilibria and the existence of local bifurcation are obtained. The effect of each parameter on the dynamical behavior of system (3) is studied numerically and the trajectories of the system are drowned in the typical figures. According to these figures, which represent the solution of system (3) for the data given by (69), the following conclusions are obtained.

(1) System (3) has no periodic dynamics rather than the fact that the system approaches asymptotically to one of their equilibrium points depending on the set of parameter data and the stability conditions that are satisfied.

(2) Although the position of the positive equilibrium point in the interior of $R^3_+$ changed as varying in the parameters values $a_2, a_4, b_2, b_3,$ and $b_4$, there is no qualitative change in the dynamical behavior of system (3) and the system still approaches to a positive equilibrium point. Accordingly adding the refuge factor which is included implicitly in these parameters plays a vital role in the stabilizing of the system at the positive equilibrium point.

(3) Decreasing in the value of growth rate of immature prey or in the value of conversion rate from immature prey to mature prey keeping the rest of parameter as in (69) leads to destabilizing of the positive equilibrium point and the system approaches asymptotically to the vanishing equilibrium point, which means losing the persistence of system (3).

(4) Increasing in the value of grownup rate of immature prey keeping the rest of parameter as in (69) leads to destabilizing of the positive equilibrium point and the system approaches asymptotically to the vanishing equilibrium point too, which means losing the persistence of system (3).

(5) Finally, for the data given by (69), increasing intraspecific competition of immature prey and natural death rate of the predator leads to destabilizing of the positive equilibrium point and the solution approaches instead asymptotically to the predator free equilibrium point, which confirm our obtained analytical results represented by conditions (21)-(22).

Competing Interests

The authors declare that they have no competing interests.

References


