Research Article

Numerical Solution of First-Order Linear Differential Equations in Fuzzy Environment by Runge-Kutta-Fehlberg Method and Its Application

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The numerical algorithm for solving “first-order linear differential equation in fuzzy environment” is discussed. A scheme, namely, “Runge-Kutta-Fehlberg method,” is described in detail for solving the said differential equation. The numerical solutions are compared with (i)-gH and (ii)-gH differential (exact solutions concepts) system. The method is also followed by complete error analysis. The method is illustrated by solving an example and an application.

1. Introduction

Fuzzy Differential Equation. In modeling of real natural phenomena, differential equations play an important role in many areas of discipline, exemplary in economics, biomathematics, science, and engineering. Many experts in such areas widely use differential equations in order to make some problems under study more comprehensible. In many cases, information about the physical phenomena related is always immanent with uncertainty.

Today, the study of differential equations with uncertainty is instantaneously growing as a new area in fuzzy analysis. The terms such as “fuzzy differential equation” and “fuzzy differential inclusion” are used interchangeably in mention to differential equations with fuzzy initial values or fuzzy boundary values or even differential equations dealing with functions on the space of fuzzy numbers. In the year 1987, the term “fuzzy differential equation (FDE)” was introduced by Kandel and Byatt [1]. There are different approaches to discuss the FDEs: (i) the Hukuhara derivative of a fuzzy number valued function is used, (ii) Hüllermeier [2] and Diamond and Watson [3–5] suggested a different formulation for the fuzzy initial value problems (FIVP) based on a family of differential inclusions, (iii) in [6, 7], Bede et al. defined generalized differentiability of the fuzzy number valued functions and studied FDE, and (iv) applying a parametric representation of fuzzy numbers, Chen et al. [8] established a new definition for the differentiation of a fuzzy valued function and used it in FDE.

Solution of Fuzzy Differential Equation by Numerical Techniques. Numerical methods are the methods by which we can find the solution of differential equation where the exact solution is critical to find. There exist various numerical methods for solving differential equation such as Setia et al. [9], Liu [10], and Setia et al. [11]. Our aim is to find the numerical techniques by which the solution of a linear or nonlinear first-order fuzzy differential equation comes easily and the solution is very close to the exact solution. There exist many techniques of numerical methods for finding the solution of fuzzy differential equation. Authors applied the method in certain types of fuzzy differential equation which shows that their techniques are best fit for that particular problem. The first paper on fuzzy differential equation and numerical analysis was published in 1999 by Ma et al. [12]. Allahviranloo et al. [13] apply the two-step method on fuzzy differential equations. Allahviranloo et al. [14] find the numerical solution by using predictor-corrector

**Solution of Fuzzy Differential Equation by Runge-Kutta Method.** Runge-Kutta method is well known for finding the approximate or numerical solution. In the last decade Runge-Kutta method is applied in fuzzy differential equation for finding the numerical solution. The researchers are giving various types of view to apply these methods. Someone changes the order and someone applies different types on FDE, a comparison of another method to Runge-Kutta method. The details of published work done in Runge-Kutta method are summarized below.

Numerical Solution of Fuzzy Differential Equations by Runge-Kutta method of order three is developed by Duraisamy and Usha [36]. Solution techniques for fourth-order Runge-Kutta method with higher order derivative approximations are developed by Nirmala and Chenthur Pandian [37]. Runge-Kutta method of order five is developed by Jayakumar and Kanagarajan [38]. The techniques extended Runge-Kutta-like formulae of order four are developed by Ghazanfari and Shakerami [39]. Third-order Runge-Kutta method is developed by Kanagarajan and Sambath [40].

**Application of Fuzzy Differential Equation.** Fuzzy differential equations play a significant role in the fields of biology, engineering, and physics as well as among other fields of science, for example, in population models [49], civil engineering [50], bioinformatics and computational biology [51], quantum optics and gravity [52], modeling hydraulic [53], HIV model [54], decay model [55], predator-prey model [56], population dynamics model [57], friction model [58], growth model [59], bacteria culture model [60], bank account and drug concentration problem [61], barometric pressure problem [62], concentration problem [63], weight loss and oil production model [64], arm race model [65], vibration of mass [66], and fractional predator-prey equation [67].

**Novelties.** Although some developments are done, some new interest and new work have been done by ourselves which are mentioned below:

(i) Differential equation is solved in fuzzy environment by numerical techniques; that is, coefficients and initial condition both are taken as fuzzy number of a differential equation and solved by numerical techniques.

(ii) The numerical solution is compared with the exact solution ((i)-gH and (ii)-gH both cases).

(iii) Runge-Kutta-Fehlberg method for solving fuzzy differential equation is used.

(iv) For application purpose a mixture problem is considered.

(v) The solutions are found using different step length for better accuracy of the result.

(vi) The necessary algorithm for numerical solution is given.

**Structure of the Paper.** The paper is organized as follows: in Preliminary Concepts, the preliminary concepts and basic concepts on fuzzy number and fuzzy derivative are given. The method for finding the exact solution is discussed in Exact Solution of Fuzzy Differential Equation. In Numerical Solution of Fuzzy Differential Equation we proposed
Runge-Kutta-Fehlberg method in fuzzy environment. The convergence of the said method and algorithm for finding the numerical results are also discussed in this section. Numerical Example shows a numerical example. In Application an important application, namely, mixture problem, is illustrated in fuzzy environment. Finally conclusions and future research scope of this paper are drawn in last section, Conclusion.

2. Preliminary Concepts

Definition 1 (fuzzy set). A fuzzy set \( \tilde{A} \) is defined by \( \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0, 1]\} \). In the pair \((x, \mu_{\tilde{A}}(x))\) the first element \(x\) belongs to the classical set \(A\) and the second element \(\mu_{\tilde{A}}(x)\) belongs to the interval \([0, 1]\), called membership function.

Definition 2 (\(\alpha\)-cut of a fuzzy set). The \(\alpha\)-level set (or interval of confidence at level \(\alpha\) or \(\alpha\)-cut) of the fuzzy set \(\tilde{A}\) of \(X\) is a crisp set \(A_\alpha\) that contains all the elements of \(X\) that have membership values in \(A\) greater than or equal to \(\alpha\); that is, \(A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X, 0 < \alpha \leq 1\}\).

Definition 3 (fuzzy number). The basic definition of fuzzy number is as follows [30]: if we denote the set of all real numbers by \(\mathcal{R}\) and the set of all fuzzy numbers on \(\mathcal{R}\) is illustrated in fuzzy environment. Finally conclusions and future research scope of this paper are drawn in last section, Conclusion.

Definition 4 (parametric form of fuzzy number [31]). A fuzzy number is as follows [30]: if we denote the set of all real numbers by \(\mathcal{R}\), the numerical results are also discussed in this section. Numerical Example shows a numerical example. In Application an important application, namely, mixture problem, is illustrated in fuzzy environment. Finally conclusions and future research scope of this paper are drawn in last section, Conclusion.

Definition 5 (generalized Hukuhara difference [20]). The generalized Hukuhara difference of two fuzzy numbers \(u, v \in \mathcal{R}_{\tilde{A}}\) is defined as follows:

\[
u \oplus_{gH} w = w \iff \begin{cases} (i) & u = v \oplus w, \quad \text{or} \quad (ii) & v = u \oplus (-1)w. \end{cases} \]

Consider \([w]_\alpha = [u_1(\alpha), v_1(\alpha)];\) then \(u_1(\alpha) = \min\{u_1(\alpha) - v_1(\alpha), u_2(\alpha) - v_2(\alpha)\}\) and \(v_2(\alpha) = \max\{u_1(\alpha) - v_1(\alpha), u_2(\alpha) - v_2(\alpha)\}\).

Note. If \(u_1(\alpha) = u_2(\alpha) = \alpha\), then \(\alpha\) is a crisp number.

Definition 6 (generalized Hukuhara derivative for first order [20]). The generalized Hukuhara derivative of a fuzzy valued function \(f : (a, b) \to \mathcal{R}_{\tilde{A}}\) at \(t_0\) is defined as

\[
f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \oplus_{gH} f(t_0)}{h}. \tag{2}\]

If \(f'(t_0) \in \mathcal{R}_{\tilde{A}}\) satisfying (2) exists, we say that \(f\) is generalized Hukuhara differentiable at \(t_0\).

Also we say that \(f(t)\) is (i)-gH differentiable at \(t_0\) if

\[
[f'(t_0)]_a = \left[f'_1(t_0, \alpha), f'_2(t_0, \alpha)\right], \tag{3}\]

and \(f(t)\) is (ii)-gH differentiable at \(t_0\) if

\[
[f'(t_0)]_a = \left[f'_1(t_0, \alpha), f'_2(t_0, \alpha)\right]. \tag{4}\]

Definition 7 (see [6]). For arbitrary \(u = (u_1, u_2)\) and \(v = (v_1, v_2) \in \mathcal{E}^1\), the quantity

\[
D(u, v) = \left(\int_0^1 (u_1 - v_1)^2 + \int_0^1 (u_2 - v_2)^2\right)^{1/2} \tag{5}\]

is the distance between fuzzy numbers \(u\) and \(v\).

Definition 8 (triangular fuzzy number). A triangular fuzzy number (TFN) denoted by \(\tilde{A}\) is defined as \((a, b, c)\) where the membership function

\[
\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a, \\ \frac{x - a}{b - a}, & a \leq x \leq b, \\ 1, & x = b, \\ \frac{c - x}{c - b}, & b \leq x \leq c, \\ 0, & x \geq c. \end{cases} \tag{6}\]

Definition 9 (\(\alpha\)-cut of a fuzzy set \(\tilde{A}\)). The \(\alpha\)-cut of \(\tilde{A}\) is \((a, b, c)\) given by

\[
A_\alpha = [a + \alpha (b - a), c - \alpha (c - b)], \quad \forall \alpha \in [0, 1]. \tag{7}\]

Definition 10 (fuzzy ordinary differential equation (FODE)). Consider a simple 1st-order linear ordinary differential equation as follows:

\[
\frac{dx}{dt} = kx + x_0 \quad \text{with initial condition } x(t_0) = y. \tag{8}\]

The above ordinary differential equation is called fuzzy ordinary differential equation if any one of the following three cases holds:

(i) Only \(y\) is a fuzzy number (Type-I).
(ii) Only \(k\) is a fuzzy number (Type-II).
(iii) Both \(k\) and \(y\) are fuzzy numbers (Type-III).
3. Exact Solution of Fuzzy Differential Equation

Consider the fuzzy initial value problem

\[ y'(t) = f(t, y(t)), \quad t \in I = [0, T] \text{ with } y(0) = y_0, \]  

where \( f \) is a continuous mapping from \( R_n \times R \) into \( R \) and \( y_0 \in E \) with \( \tau \)-level sets

\[ \{y_0\}_\tau = \{y_1(0; \alpha), y_2(0; \alpha)\}, \quad \alpha \in (0, 1]. \]  

We write \( f(t, y) = (f_1(t, y), f_2(t, y)) \) and \( f_1(t, y) = F[t, y_1, y_2], f_2(t, y) = G[t, y_1, y_2]. \)

Because of \( y'(t) = f(t, y) \) we have the following. When \( y(t, y) \) is (i)-gH differentiable

\[ y'_1(t, \alpha) = F[t, y_1(t, \alpha), y_2(t, \alpha)], \]  

\[ y'_2(t, \alpha) = G[t, y_1(t, \alpha), y_2(t, \alpha)]. \]  

When \( y(t, y) \) is (ii)-gH differentiable

\[ y'_1(t, \alpha) = F[t, y_1(t, \alpha), y_2(t, \alpha)], \]  

\[ y'_2(t, \alpha) = G[t, y_1(t, \alpha), y_2(t, \alpha)]. \]  

where, by using extension principle, we have the membership function

\[ f(t; y(t))(s) = \text{Sup} \{ y(t)(\tau) \mid s = f(t, \tau), s \in R \}. \]  

So fuzzy number is \( f(t; y(t)). \) From this it follows that

\[ \{f(t; y(t))(\alpha)\}_{\alpha} = \{f_1(t, y(t); \alpha), f_2(t, y(t); \alpha)\}, \quad \alpha \in [0; 1], \]  

where

\[ f_1(t, y(t); \alpha) = F[t; y_1(t; \alpha), y_2(t; \alpha)] \]  

\[ = \text{min} \{ f(t, u) \mid u \in [y_1(t; \alpha), y_2(t; \alpha)] \}, \]  

\[ f_2(t, y(t); \alpha) = G[t; y_1(t; \alpha), y_2(t; \alpha)] \]  

\[ = \text{max} \{ f(t, u) \mid u \in [y_1(t; \alpha), y_2(t; \alpha)] \}. \]  

Note. (1) Both cases ((i)-gH and (ii)-gH) can be applied to a FDE for finding exact solution.

(2) After taking \( \alpha \)-cut of the given FDE, it transforms to system of ordinary differential equation.

4. Numerical Solution of Fuzzy Differential Equation

4.1. Runge-Kutta-Fehlberg Method for Ordinary (Crisp) Differential Equation. Consider the initial value problem \( y'(t) = f(t, y(t)); y(0) = y_0. \)

The Runge-Kutta-Fehlberg method (denoted as RKF45) is one way to try to resolve this problem.

The problem is to solve the initial value problem in above equation by means of Runge-Kutta methods of order 4 and order 5.

First we need some definitions:

\[ k_1 = hf(t_i, y_i), \]  

\[ k_2 = hf\left(t_i + \frac{1}{4}h, y_i + \frac{1}{4}k_1\right), \]  

\[ k_3 = hf\left(t_i + \frac{3}{8}h, y_i + \frac{3}{32}k_1 + \frac{9}{32}k_2\right), \]  

\[ k_4 = hf\left(t_i + \frac{12}{13}h, y_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right), \]  

\[ k_5 = hf\left(t_i + h, y_i + \frac{439}{216}k_1 - 8k_2 + \frac{2514}{768}k_3 + \frac{1351}{512}k_4\right), \]  

\[ k_6 = hf\left(t_i + h, y_i - \frac{8}{27}k_1 + 2k_2 - \frac{50}{27}k_3 + \frac{355}{128}k_4 - \frac{10}{21}k_5\right). \]  

Then an approximation to the solution of initial value problem is made using Runge-Kutta method of order 4:

\[ y_{i+1} = y_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4101}k_4 - \frac{1}{5}k_5. \]  

A better value for the solution is determined using a Runge-Kutta method of order 5:

\[ z_{i+1} = y_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28632}{56430}k_4 - \frac{9}{50}k_5 \]  

\[ + \frac{2}{55}k_6. \]  

The optimal step size \( s_h \) can be determined by multiplying the scalar \( s \) times the step size \( h. \) The scalar \( s \) is

\[ s = \left( \frac{eh}{2|z_{i+1} - y_{i+1}|} \right)^{1/4} \]  

\[ = 0.0840896 \left( \frac{eh}{|z_{i+1} - y_{i+1}|} \right)^{1/4}, \]  

where \( e \) is the specified error control tolerance.

Note that RK4 requires 4 function evaluations and RK5 requires 6 evaluations, that is, 10 for RK4 and RK5. Fehlberg devised a method to get RK4 and RK5 results using only 6 function evaluations by using some of \( K \) values in both methods.

4.2. Runge-Kutta-Fehlberg Method for Solving Fuzzy Differential Equations. Let \( Y = [Y_1, Y_2] \) be the exact solution and let
\[y = [y_1, y_2]\] be the approximated solution of the fuzzy initial value problem.

Let \([Y(t)]_{n} = [Y_1(t, \alpha), Y_2(t, \alpha)], \ [y(t)]_{n} = [y_1(t, \alpha), y_2(t, \alpha)].\]

Throughout this argument, the value of \(r\) is fixed. Then the exact and approximated solution at \(t_n\) are, respectively, denoted by

\[\begin{align*}
[Y(t_n)]_{\alpha} & = [Y_1(t_n, \alpha), Y_2(t_n, \alpha)], \\
[y(t_n)]_{\alpha} & = [y_1(t_n, \alpha), y_2(t_n, \alpha)].
\end{align*}\]  \(\text{(20)}\)

The grid points at which the solution is calculated are \(h = (T - t_0)/N, t_i = t_0 + ih, 0 \leq i \leq N.\)

Then we obtained

\[y_1(t_{n+1}, \alpha) = y_1(t_n, \alpha) + \frac{16}{135}K_1 + \frac{6656}{12,825}K_3 + \frac{28,561}{56,430}K_4 - \frac{9}{50}K_5 + \frac{2}{55}K_6,\]  \(\text{(21)}\)

where

\[\begin{align*}
K_1 & = hF(t_n, y_1(t_n, \alpha), y_2(t_n, \alpha)), \\
K_2 & = hF(t_n + \frac{1}{4}h, y_1(t_n, \alpha) + \frac{1}{4}K_1, y_2(t_n, \alpha) + \frac{1}{4}K_1), \\
K_3 & = hF(t_n + \frac{3}{8}h, y_1(t_n, \alpha) + \frac{3}{32}K_1 + \frac{9}{32}K_2, y_2(t_n, \alpha) + \frac{9}{32}K_2), \\
K_4 & = hF(t_n + \frac{12}{13}h, y_1(t_n, \alpha) + \frac{1932}{2197}K_1 - \frac{7200}{2197}K_2 + \frac{7296}{2197}K_3, y_2(t_n, \alpha) + \frac{1932}{2197}K_1 - \frac{7200}{2197}K_2 + \frac{7296}{2197}K_3), \\
K_5 & = hF(t_n + h, y_1(t_n, \alpha) + \frac{439}{216}K_1 - 8K_2 + \frac{3680}{513}K_3 - \frac{845}{4104}K_4, y_2(t_n, \alpha) + \frac{439}{216}K_1 - 8K_2 + \frac{3680}{513}K_3 - \frac{845}{4104}K_4), \\
K_6 & = hF(t_n + h, y_1(t_n, \alpha) - \frac{8}{27}K_1 + 2K_2 - \frac{3544}{2565}K_3 + \frac{1859}{4104}K_4 - \frac{11}{40}K_5, y_2(t_n, \alpha) - \frac{8}{27}K_1 + 2K_2 - \frac{3544}{2565}K_3 + \frac{1859}{4104}K_4 - \frac{11}{40}K_5).
\end{align*}\]  \(\text{(22)}\)

4.3. Convergence of Fuzzy Runge-Kutta-Fehlberg Method. The solution is calculated by grid points at \(a = t_0 \leq t_1 \leq \cdots \leq t_N = b\) and \(h = (b - a)/N = t_{n+1} - t_n.\)

Therefore, we have

\[\begin{align*}
Y_1(t_{n+1}, \alpha) & = Y_1(t_n, \alpha) + F(t_n, Y_1(t_n, \alpha), Y_2(t_n, \alpha)), \\
Y_2(t_{n+1}, \alpha) & = Y_2(t_n, \alpha) + G(t_n, Y_1(t_n, \alpha), Y_2(t_n, \alpha)), \\
y_1(t_{n+1}, \alpha) & = y_1(t_n, \alpha) + F(t_n, y_1(t_n, \alpha), y_2(t_n, \alpha)), \\
y_2(t_{n+1}, \alpha) & = y_2(t_n, \alpha) + G(t_n, y_1(t_n, \alpha), y_2(t_n, \alpha)),
\end{align*}\]  \(\text{(25)}\)

where

\[\begin{align*}
L_1 & = hG(t_n, y_1(t_n, \alpha), y_2(t_n, \alpha)), \\
L_2 & = hG\left(t_n + \frac{1}{4}h, y_1(t_n, \alpha) + \frac{1}{4}L_1, y_2(t_n, \alpha) + \frac{1}{4}L_1\right), \\
L_3 & = hG\left(t_n + \frac{3}{8}h, y_1(t_n, \alpha) + \frac{3}{32}L_1 + \frac{9}{32}L_2\right), \\
L_4 & = hG\left(t_n + \frac{12}{13}h, y_1(t_n, \alpha) + \frac{1932}{2197}L_1 - \frac{7200}{2197}L_2 + \frac{7296}{2197}L_3, y_2(t_n, \alpha) + \frac{1932}{2197}L_1 - \frac{7200}{2197}L_2 + \frac{7296}{2197}L_3\right), \\
L_5 & = hG\left(t_n + h, y_1(t_n, \alpha) + \frac{439}{216}L_1 - 8L_2 + \frac{3680}{513}L_3 - \frac{845}{4104}L_4, y_2(t_n, \alpha) + \frac{439}{216}L_1 - 8L_2 + \frac{3680}{513}L_3 - \frac{845}{4104}L_4\right), \\
L_6 & = hG\left(t_n + h, y_1(t_n, \alpha) - \frac{8}{27}L_1 + 2L_2 - \frac{3544}{2565}L_3 + \frac{1859}{4104}L_4 - \frac{11}{40}L_5, y_2(t_n, \alpha) - \frac{8}{27}L_1 + 2L_2 - \frac{3544}{2565}L_3 + \frac{1859}{4104}L_4 - \frac{11}{40}L_5\right).
\end{align*}\]  \(\text{(24)}\)
Clearly, \(y_1(t, \alpha)\) and \(y_2(t, \alpha)\) converge to \(Y_1(t, \alpha)\) and \(Y_2(t, \alpha)\), respectively, whenever \(h \to 0\); that is,
\[
\lim_{h \to 0} y_1(t, \alpha) = Y_1(t, \alpha), \quad \lim_{h \to 0} y_2(t, \alpha) = Y_2(t, \alpha).
\] (26)

**Proof.** Before we go to the main proof we need to know some results.

**Lemma 11.** Let the sequence of numbers \(\{W_n\}_{n=0}^N\) satisfy
\[
|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N - 1,
\] (27)
for some given positive constants \(A\) and \(B\). Then
\[
|W_n| \leq A^n|W_0| + B \frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N.
\] (28)

**Lemma 12.** Let the sequence of numbers \(\{W\}_{n=0}^N\) and \(\{V\}_{n=0}^N\) satisfy
\[
|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B, \\
|V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B,
\] (29)
for some given positive constants \(A\) and \(B\), and denote
\[
U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N.
\] (30)
Then
\[
U_n \leq \bar{A}^nU_0 + B\frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \leq n \leq N,
\] (31)
where \(\bar{A} = 1 + 2A\) and \(\bar{B} = 2B\).

Let \(F(t, u, v)\) and \(G(t, u, v)\) be obtained by substituting \([y_1(t, \alpha), y_2(t, \alpha)] = [u, v]\) in (21) and (23); that is,
\[
F(t, u, v) = \frac{16}{135}K_1(t, u, v) + \frac{6656}{12825}K_3(t, u, v)
\]
\[
+ \frac{28,561}{56,430}K_4(t, u, v) - \frac{9}{50}K_5(t, u, v)
\]
\[
+ \frac{2}{55}K_6(t, u, v),
\]
\[
G(t, u, v) = \frac{16}{135}L_1(t, u, v) + \frac{6656}{12825}L_3(t, u, v)
\]
\[
+ \frac{28,561}{56,430}L_4(t, u, v) - \frac{9}{50}L_5(t, u, v)
\]
\[
+ \frac{2}{55}L_6(t, u, v).
\] (32)

The domain where \(F\) and \(G\) are defined is as
\[
H = \{(t, u, v) \mid 0 \leq t \leq T, -\infty < v < \infty, -\infty < u \leq v\}.
\] (33)

**Theorem 13.** Let \(F(t, u, v)\) and \(G(t, u, v)\) belong to \(C^{p-1}(K)\) and let the partial derivative of \(F\) and \(G\) be bounded over \(K\). Then for arbitrary fixed \(0 \leq \alpha \leq 1\), the numerical solution of (9), \([y_1(t, \alpha), y_2(t, \alpha)]\) converges to the exact solution \([Y_1(t, \alpha), Y_2(t, \alpha)]\).

**Proof (see [46]).** By using Taylor’s theorem we get
\[
y_1(t, \alpha) = Y_1(t, \alpha) + hF(t, u, v) \frac{\partial}{\partial u} + \frac{h^{p+1}}{(p+1)!}Y_1^{(p+1)}(\xi_1),
\] (34)
\[
y_2(t, \alpha) = Y_2(t, \alpha) + hG(t, u, v) \frac{\partial}{\partial u} + \frac{h^{p+1}}{(p+1)!}Y_2^{(p+1)}(\xi_2),
\] (35)
where \(\xi_1, \xi_2 \in (t_n, t_{n+1})\).

Now if we denote
\[
W_n = Y_1(t, \alpha) - y_1(t, \alpha), \\
V_n = Y_2(t, \alpha) - y_2(t, \alpha),
\] (36)
then the above two expressions converted to
\[
W_{n+1} = W_n + h[F(t, u, v) + \frac{h^{p+1}}{(p+1)!}Y_1^{(p+1)}(\xi_1)],
\]
\[
V_{n+1} = V_n + h[G(t, u, v) + \frac{h^{p+1}}{(p+1)!}Y_2^{(p+1)}(\xi_2)],
\] (37)
where \(M = \max\{|Y_1^{(p+1)}(t, \alpha)|, \max\{|Y_2^{(p+1)}(t, \alpha)|\} for t \in [0, T]\) and \(L > 0\) is a bound for the partial derivative of \(F\) and \(G\).

Therefore we can write
\[
|W_{n+1}| \leq |W_n| + 2Lh \max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!}M, \\
|V_{n+1}| \leq |V_n| + 2Lh \max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!}M,
\] (38)
where \(|U_0| = |W_0| + |V_0|\).
In particular,
\[ |W_N| \leq (1 + 4Lh)^N |U_0| \]
\[ + \frac{2h^{p+1}}{(p+1)!} M (1 + 4Lh)^{T/h} - 1, \]
\[ |V_N| \leq (1 + 4Lh)^N |U_0| \]
\[ + \frac{2h^{p+1}}{(p+1)!} M (1 + 4Lh)^{T/h} - 1. \]  
(39)

Since \( W_0 = V_0 = 0 \), we have
\[ |W_N| \leq M \frac{e^{4LT} - 1}{2L (p + 1)!} h^p, \]
\[ |V_N| \leq M \frac{e^{4LT} - 1}{2L (p + 1)!} h^p. \]  
(40)

Thus, if \( h \to 0 \), we get \( W_N \to 0 \) and \( V_N \to 0 \), which completes the proof.

4.4. Algorithm for Finding the Numerical Solution

Step 1. \( F(t, y_1, y_2) \leftarrow \text{“Function to be supplied”} \)
\( G(t, y_1, y_2) \leftarrow \text{“Function to be supplied”} \)

Step 2. Read \( t(0), y_1(0), y_2(0) \), \( h \), limit.

Step 3. For \( i = 0(1) \) limit
\[ K_1 \leftarrow hF(t_i + 1/4h, y_1(t_i, r), y_2(t_i, r)) \]
\[ K_2 \leftarrow hF(t_i + 3/8h, y_1(t_i, r) + 3(32/32)K_1, y_2(t_i, r) + 3(32/32)K_2) \]
\[ K_3 \leftarrow hF(t_i + 12/13h, y_1(t_i, r) + (1932/2197)K_1 - (7200/2197)K_2 + (7296/2197)K_3, y_2(t_i, r) + (1932/2197)K_1 - (7200/2197)K_2 + (7296/2197)K_3) \]
\[ K_4 \leftarrow hF(t_i + h, y_1(t_i, r) + (439/216)L_1 - 8K_2 + (3680/513)K_3 - (845/4104)K_4, y_2(t_i, r) + (439/216)L_1 - 8K_2 + (3680/513)K_3 - (845/4104)K_4) \]
\[ K_5 \leftarrow hF(t_i + h, y_1(t_i, r) - (8/27)K_1 + 2K_2 - (3544/2565)K_3 - (11/40)K_3 - y_2(t_i, r) - (8/27)K_1 + 2K_2 - (3544/2565)K_3 - (11/40)K_3) \]
\[ K_6 \leftarrow hG(t_i + 1/4h, y_1(t_i, r), y_2(t_i, r)) \]
\[ L_1 \leftarrow hG(t_i + 1/4h, y_1(t_i, r), y_2(t_i, r)) \]
\[ L_2 \leftarrow hG(t_i + (3/8)h, y_1(t_i, r) + (3/32)L_1 + (9/32)L_2, y_2(t_i, r) + (3/32)L_1 + (9/32)L_2) \]
\[ L_3 \leftarrow hG(t_i + (12/13)h, y_1(t_i, r) + (1932/2197)L_1 - (28/2197)L_2, y_2(t_i, r) + (1932/2197)L_1 - (28/2197)L_2) \]

Step 4. \( y_1(t_{i+1}, r) = y_1(t_i, r) + (16/135)K_1 + (6656/2197)K_5 + (28,561/56,430)K_6 \)

Step 5. \( y_2(t_{i+1}, r) = y_2(t_i, r) + (16/135)K_1 + (6656/2197)K_5 + (28,561/56,430)K_6 \)

Step 6. \( t_{i+1} = t_i + h \). Write \( y_1(t_{i+1}, r), y_2(t_{i+1}, r), t_{i+1} \).

Step 7. Next \( i \)

Step 8. End.

5. Numerical Example

Example. Solve \( y' = -y + t + 1 \) with initial condition \( y(0) = (0.96, 1.01) \). Then find the solution at \( t = 0.1 \).

Solution. For (i)-gH differentiable case the exact solution is
\[ y_1 (t, r) = t + (0.96 + 0.04r) e^{-r}, \]
\[ y_2 (t, r) = t + (1.01 - 0.01r) e^{-r}. \]  
(41)

and for (ii)-gH differentiable case the exact solution is
\[ y_1 (t, r) = 1 + t + (-0.04 + 0.04r) e^{-r}, \]
\[ y_2 (t, r) = 1 + t + (0.01 - 0.01r) e^{-r}. \]  
(42)

Remark 14. From Figure 1 and Table 1 we conclude that the lower exact solution ((i)-gH case) is approximately equal to the numerical solution when we take the step length \( h = 0.01 \) (for \( h = 0.001 \) is nearly equal), whereas the lower exact solution ((ii)-gH case) is approximately equal to the numerical solution when we take the step length \( h = 0.1 \).

Remark 15. From Figure 2 and Table 1 we conclude that the upper exact solution ((i)-gH case) is approximately equal to the numerical solution when we take the step length \( h = 0.001 \) (for \( h = 0.01 \) is nearly equal), whereas the upper exact solution ((ii)-gH case) is approximately equal to the numerical solution when we take the step length \( h = 0.1 \).

6. Application

Problem. A tank initially contains 300 gals of brine which has dissolved in \( c \) lbs of salt. Coming into the tank at 3 gals/min
Figure 1: Comparison of lower exact solutions and numerical solution for different step lengths at $t = 0.1$.  

Figure 2: Comparison of upper exact solutions and numerical solution for different step lengths at $t = 0.1$.  

- **Lower exact solution for (i)-gH case**
- **Lower exact solution for (ii)-gH case**
- **Lower numerical solution for step length 0.1**
- **Lower numerical solution for step length 0.01**
- **Lower numerical solution for step length 0.001**

- **Upper exact solution for (i)-gH case**
- **Upper exact solution for (ii)-gH case**
- **Upper numerical solution for step length 0.1**
- **Upper numerical solution for step length 0.01**
- **Upper numerical solution for step length 0.001**
Table 1: Comparison of the exact solutions and numerical solutions for different step lengths at $t = 0.1$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>Exact solution for (i)-$gH$ differentiable case</th>
<th>Exact solution for (ii)-$gH$ differentiable case</th>
<th>Numerical solution for $h = 0.1$ by RKF method</th>
<th>Numerical solution for $h = 0.01$ by RKF method</th>
<th>Numerical solution for $h = 0.001$ by RKF method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Y_1$</td>
<td>$Y_2$</td>
<td>$Y_1$</td>
<td>$Y_2$</td>
<td>$Y_1$</td>
</tr>
<tr>
<td>0</td>
<td>0.9686</td>
<td>1.0139</td>
<td>1.0558</td>
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<tr>
<td>0.1</td>
<td>0.9723</td>
<td>1.0130</td>
<td>1.0602</td>
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<tr>
<td>0.2</td>
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<tr>
<td>0.3</td>
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<td>1.1077</td>
<td>1.0730</td>
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<td>0.4</td>
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<td>1.0867</td>
<td>1.1033</td>
<td>1.0906</td>
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<td>1.0066</td>
<td>1.0912</td>
<td>1.1022</td>
<td>1.0951</td>
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<tr>
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<td>1.0048</td>
<td>1.1000</td>
<td>1.1000</td>
<td>1.1039</td>
</tr>
</tbody>
</table>
is brine with concentration $k$ lbs salt/gals and the well stirred mixture leaves at the rate 3 gals/min. Let $y(x)$ lbs be the salt in the tank at any time $t \geq 0$. Then $dy(x)/dx + (1/100)y(x) = k$, $x \in [0,0.5]$ with $y(0) = c$, if the initial condition is being modeled as fuzzy numbers $c = (1, 2, 3)$ and $k = (1, 2, 4)$. Find solution at $x = 0.4$.

**Solution.** For (i)-gH differentiable case the exact solution is

$$y_1 (x, \alpha) = \frac{(1 + \alpha)}{100} \left(1 + 99e^{-\frac{1}{100}x}\right),$$

$$y_2 (x, \alpha) = \frac{(2 - \alpha)}{50} + \frac{(148 - 49\alpha)}{50} e^{-\frac{1}{100}x}. \quad (43)$$

For (ii)-gH differentiable case the exact solution is

$$y_1 (x, \alpha) = (149 - 149\alpha) e^{\frac{1}{100}x}$$

$$+ (-248 + 50\alpha) e^{-\frac{1}{100}x}$$

$$+ (100 + 100\alpha),$$

$$y_2 (x, \alpha) = - (149 - 149\alpha) e^{\frac{1}{100}x}$$

$$+ (-248 + 50\alpha) e^{-\frac{1}{100}x}$$

$$+ (400 - 200\alpha). \quad (44)$$

**Remark 16.** From Figure 3 and Table 2 we conclude that the lower exact solution ((i)-gH case) is approximately equal to the numerical solution when we take the step length $h = 0.001$ (for $h = 0.01$ is nearly equal), whereas the lower exact solution ((ii)-gH case) is not equal to any numerical solution.

**Remark 17.** From Figure 4 and Table 2 we conclude that the upper exact solution ((i)-gH case) is approximately equal to the numerical solution when we take the step length of $h = 0.01$ and $h = 0.001$. For $h = 0.1$ it is nearly equal, whereas the upper exact solution ((ii)-gH case) is not equal to any numerical solution.

### 7. Conclusion

In this paper we applied Runge-Kutta-Fehlberg method for finding the numerical solution of first-order linear differential equation in fuzzy environment. The numerical solution is compared with the exact solution ((i)-gH and (ii)-gH both cases). The results presented in the contribution show that Runge-Kutta-Fehlberg method is a powerful mathematical tool for solving first-order linear differential equation in fuzzy environment. The convergence of Runge-Kutta-Fehlberg method has been discussed. The process method is applied to a mechanical problem in fuzzy environment which shows that it is a promising method to solve the said types of differential equation. In the future we can apply these methods for solving higher order linear and nonlinear differential equation in fuzzy environment.
Table 2: Comparison of the exact solutions and numerical solutions for different step lengths at $x = 0.4$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Exact solution for (i)-gH differentiable case $Y_1$</th>
<th>Exact solution for (ii)-gH differentiable case $Y_2$</th>
<th>Numerical solution for $h = 0.1$ by RKF method $Y_1$</th>
<th>Numerical solution for $h = 0.01$ by RKF method $Y_2$</th>
<th>Numerical solution for $h = 0.001$ by RKF method $Y_3$</th>
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<td>2.6075</td>
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<td>1.2143</td>
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<td>2.6279</td>
<td>3.2723</td>
<td>1.3247</td>
</tr>
<tr>
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<td>1.2949</td>
<td>2.6894</td>
<td>2.6482</td>
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</tr>
<tr>
<td>0.4</td>
<td>1.3945</td>
<td>2.5897</td>
<td>2.6685</td>
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<tr>
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</tr>
<tr>
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<td>3.0314</td>
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<tr>
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</tr>
</tbody>
</table>

Note: The table continues with more values for $\alpha = 0.4$ to $\alpha = 1$. Each row represents the step length $h$ and the corresponding numerical solutions. The exact solutions are given for comparison.
Figure 4: Comparison of upper exact solutions and numerical solution for different step lengths at $t = 0.4$.

Competing Interests
The authors declare that there are no competing interests.

References


