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Research Article

Static Consensus in Passifiable Linear Networks

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Sufficient conditions of consensus (synchronization) in networks described by digraphs and consisting of identical deterministic SIMO systems are derived. Identical and nonidentical control gains (positive arc weights) are considered. Connection between admissible digraphs and nonsmooth hypersurfaces (sufficient gain boundary) is established. Necessary and sufficient conditions for static consensus by output feedback in networks consisting of certain class of double integrators are rediscovered. Scalability for circle digraph in terms of gain magnitudes is studied. Examples and results of numerical simulations are presented.

1. Introduction

Control of multiagent systems has attracted significant interest in last decade since it has a great technical importance [1–5] and relates to biological systems [6].

In consensus problems agents communicate via decentralized controllers using relative measurements with a final goal to achieve common behaviour (synchronization) which can evolve in time. Many approaches have been developed for different problem settings.

Laplace matrix, its spectrum, and eigenspace play crucial role in description and analysis of consensus problems. Laplace matrix has broad applications, for example, [7]. Not all possible digraph topologies can provide consensus over dynamical networks. Admissible digraph topologies and connection with algebraic properties of Laplace matrix have been found in [8]. Tree structure analysis and properties of Laplace matrix spectrum of digraphs are also studied by these authors. Work [9] contains examples of out-forests as well as useful graph theoretical concepts and can be recommended as an entry reading to the research of these authors on algebraic digraph theory and consensus problems.

Concept of synchronization region in complex plane for networks consisting of linear dynamical systems is introduced in [10]. In [11] this concept is used for analysis of synchronization with leader. Problem is solved using Linear Quadratic Regulator approach in cases when full state is available for measurement and when it is not. In the last case observers are constructed.

Analysis of consensus with scalar coupling strengths [10, 11] is fruitful in a sense that conditions on gains (which depend on connection topology and single agent properties) give more insight to problem. A lot of works on topic consider dynamic couplings; however, for certain type of connections it might happen that tunable parameters will exceed upper bound on possible control gains, that is, not meet physical limitations. So, necessary and sufficient conditions on consensus achievement for different connection types in terms of coupling strengths are needed.

Celebrated Kalman-Yakubovich-Popov Lemma (Positive Real Lemma) establishes important connection between passivity (positive-realness) of transfer function $\chi(s)$ and matrix relations on its minimal state-space realization (A, B, C); see [12, 13]. Positive Real Lemma is a basis for Passification Method [14, 15] ("Feedback Kalman-Yakubovich Lemma") which answers question when a linear system can be made passive, that is, strictly positive real (SPR) by static output feedback. Powerful idea of rendering system into passive by feedback has been also studied for nonlinear systems, for example, [13, 16, 17].

In consensus-type problems considering SPR agents with stable (Hurwitz) matrix A leads to a synchronous behaviour when all states go to zero. The latter is undesirable in essence, since such behaviour can be reached by local control without

communication. So, instead of SPR systems it is possible to consider passifiable systems, with an opportunity that a study is extendable to nonlinear systems. Also, Passification Method allows avoiding constructing observers for reaching full-state consensus by output feedback. Observers implementation increases dimension of overall phase space and complexity of a dynamic network. However, finding a passification matrix (vector g) is long standing open problem; still set of passifiable systems is nonempty.

In this paper Passification Method is used to synthesize a decentralized control law and to derive sufficient conditions of full-state synchronization by relative output feedbacks in networks described by digraphs with Linear Time Invariant dynamical nodes in continuous time. Assumptions made on network topology are minimal. Synchronous behaviour is described, including case of nonidentical gains. It is determined that boundary of sufficient gain region geometrically is a hypersurface in corresponding gain space. For certain three-node network this geometrical observation connects algebraic properties of Laplace matrix with constructed hypersurface. Namely, Jordan block appears in a direction of a cusp (nonsmooth) extremal point of the hypersurface.

Necessary and sufficient conditions for static consensus by output feedback in networks consisting of certain class of double integrators have been rediscovered. Conditions are given in terms of Laplace matrix spectrum.

Scalability in a circle digraphs in terms of gain (coupling strength) is studied. It is shown that common gain in large cycle digraphs consisting of double integrators should grow not slower than quadratically in number of agents.

Results of numerical simulations in 3- and 20-node double-integrator networks are presented.

Neighbouring papers, which influenced this work, are cited before Conclusions.

2. Theoretical Study

2.1. Preliminaries and Notations. Notations, some terms of graph theory, and Passification Lemma are listed in this section.

2.1.1. Notations. Notation $\|\cdot\|_2$ stands for Euclidian norm. For two symmetric matrices M_1, M_2 inequality $M_1 > M_2$ means that matrix $M_1 - M_2$ is positive definite. Notation $\mathrm{col}(v_1, \ldots, v_d)$ stands for vector $(v_1, \ldots, v_d)^{\mathrm{T}}$. Identity matrix of size d is denoted by I_d. Vector $\mathbf{1}_d = (1, 1, \ldots, 1)$ is vector of size d and consisting of ones. Vector $\mathbf{0}_d$ is defined similarly. Matrix $\mathrm{diag}(v_1, \ldots, v_d)$ is square matrix whose ith element on main diagonal is v_i, $i = 1, \ldots, d$; other entries are zeroes. Notation $\otimes$ stands for Kronecker product of matrices. Definition and properties of Kronecker product, including eigenvalues property, can be found in [18, 19]. Direct sum of matrices [20] is denoted by $\oplus$.

2.1.2. Terms of Graph Theory. A pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is set of arcs (ordered pairs) is called digraph (directed graph). Let $\mathcal{V}$ have N elements, $N \in \mathbb{N}$. It is assumed hereafter that graphs do not have self-loops, that is, for any vertex, $\alpha \in \mathcal{V}$ $\mathrm{arc}(\alpha, \alpha) \notin \mathcal{E}$.

Digraph is called directed tree if all its vertices except one (called root) have exactly one parent. Let us agree that in any $\mathrm{arc}(\alpha, \beta) \in \mathcal{E}$ vertex β is parent or neighbour. Directed spanning tree of a digraph $\mathcal{G}$ is a directed tree formed of all digraph $\mathcal{G}$ vertices and some of its arcs such that there exists path from any vertex to the root vertex in this tree. Existence of directed spanning tree has connection to principal achievement of synchronization in consensus-like problems.

A digraph is called weighted if to any pair of vertices $\alpha, \beta \in \mathcal{E}$ number $w(\alpha, \beta) \geq 0$ is assigned such that

$$\begin{aligned} w(\alpha, \beta) &> 0 \quad \text{if } (\alpha, \beta) \in \mathcal{E}, \\ w(\alpha, \beta) &= 0 \quad \text{if } (\alpha, \beta) \notin \mathcal{E}. \end{aligned} \tag{1}$$

A digraph in which all nonzero weights are equal to 1 will be referred to as unit weighted.

An adjacency matrix $\mathcal{A}(\mathcal{G})$ is $N \times N$ matrix whose ith, jth entry is equal to $w(\alpha_i, \alpha_j)$, $i, j = 1, \ldots, N$.

Laplace matrix of digraph $\mathcal{G}$ is defined as follows:

$$L(\mathcal{G}) = \mathrm{diag}\left(\mathcal{A}(\mathcal{G}) \cdot \mathbf{1}_N\right) - \mathcal{A}(\mathcal{G}). \tag{2}$$

Matrix $L(\mathcal{G})$ always has zero eigenvalue with corresponding right eigenvector $\mathbf{1}_N : L(\mathcal{G}) \cdot \mathbf{1}_N = 0 \cdot \mathbf{1}_N$. By construction and Gershgorin Circle Theorem all nonzero eigenvalues of L have positive real parts. Let us denote by $v(L) \in \mathbb{R}^N$ left eigenvector of L which is corresponding to zero eigenvalue and scaled such that $v(L)^{\mathrm{T}} \cdot \mathbf{1}_N = 1$. It is known that vector $v(L)$ describes synchronous behaviour if reached.

Suppose that a digraph has directed spanning tree. A set of digraph vertices is called Leading Set ("basic bicomponent" in terms of [9]) if subdigraph constructed of them is strongly connected and no vertex in this set has neighbours in the remaining part of digraph. Nonzero components of $v(L)$ and only them correspond to vertices of Leading Set. Definition of basic bicomponent is wider and applicable for digraphs with no directed spanning trees.

For illustration, by [21], there are 16 different types of digraphs which can be constructed on 3 nodes. 12 of them have directed spanning tree; among these, 5 digraphs have Leading Set with 3 nodes, 2 digraphs have Leading Set with 2 nodes, and 5 digraphs have Leading Set with 1 node.

2.1.3. Passification Lemma. Problem of linear system passification is a problem of finding static linear output feedback which is making initial system passive. It was solved in [14, 15] for nonsquare SIMO and MIMO systems including case of complex parameters. Brief outline of SIMO systems passification is given below.

Let A, B, C be real matrices of sizes $n \times n$, $n \times 1$, $n \times l$ accordingly. Denote by $\chi(s) = C^{\mathrm{T}}(sI - A)^{-1}B$, $s \in \mathbb{C}$. Let vector $g \in \mathbb{R}^l$. If numerator of function $g^{\mathrm{T}}\chi(s)$ is Hurwitz with degree $n - 1$ and has positive coefficients then function $g^{\mathrm{T}}\chi(s)$ is called hyper-minimum-phase.

Lemma 1 (Passification Lemma [14, 15]). *The following statements are equivalent.*

(1) *There exists vector $g \in \mathbb{R}^l$ such that function $g^{\mathrm{T}}\chi(s)$ is hyper-minimum-phase.*

(2) *Number $\varkappa_0 = \sup_{\omega\in\mathbb{R}^1}\mathrm{Re}(g^{\mathrm{T}}\chi(\mathrm{i}\omega))^{-1}$ is positive $\varkappa_0 > 0$ and for any $\varkappa > \varkappa_0$ there exists $n \times n$ real matrix $H = H^{\mathrm{T}} > 0$ satisfying the following matrix relations:*

$$\begin{aligned} HA_* + A_*^{\mathrm{T}}H &< 0, \\ HB &= Cg, \\ A_* &= A - \varkappa Bg^{\mathrm{T}}C^{\mathrm{T}}. \end{aligned} \tag{3}$$

2.2. Problem Statement and Assumptions. Consider a network consisting of N agents modelled as linear dynamical systems:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= C^{\mathrm{T}}x_i(t), \end{aligned} \tag{4}$$

where $i = 1, \ldots, N$, $x_i \in \mathbb{R}^n$ is state vector, $y_i \in \mathbb{R}^l$ is output or measurements vector, $u_i \in \mathbb{R}^1$ is input or control, and A, B, C are real matrices of appropriate size. By associating agents with N vertices of unit weighted digraph $\mathcal{G}$ and introducing set of arcs one can describe information flow in the network. For $i = 1, \ldots, N$ let us introduce notation for relative outputs

$$\overline{y}_i(t) = \sum_{j\in\mathcal{N}_i}\left(y_i(t) - y_j(t)\right), \tag{5}$$

where $\mathcal{N}_i$ is a set of ith agents neighbours.

Problem is to design controllers which use relative outputs and ensure achievement of the state synchronization (consensus) of all agents:

$$\lim_{t\to\infty}\left(x_i(t) - x_j(t)\right) = 0, \quad i, j = 1, \ldots, N. \tag{6}$$

In the case of synchronization achievement asymptotical behaviour of all agents will be described by the same time-dependant consensus vector which is denoted hereafter by $c(t)$:

$$\lim_{t\to\infty}\left(x_i(t) - c(t)\right) = 0, \quad i = 1, \ldots, N. \tag{7}$$

Let us make the following assumption about dynamics of a single agent.

(A1) There exists vector $g \in \mathbb{R}^l$ such that transfer function $g^{\mathrm{T}}\chi(s) = g^{\mathrm{T}}C^{\mathrm{T}}(sI_n - A)^{-1}B$ is hyper-minimum-phase.

Now let us make an assumption on graph topology.

(A2) Digraph $\mathcal{G}$ has at least one directed spanning tree.

Zero eigenvalue of Laplace matrix L has unit multiplicity iff this assumption holds [8].

2.3. Static Identical Control. Denote

$$r(L) = \min_{\lambda_j\neq 0}\mathrm{Re}\,\lambda_i, \tag{8}$$

where λ_j are eigenvalues of L. Under assumption (A2) zero eigenvalue is simple. By properties of L other eigenvalues lie in open right half of complex plane, so $r(L)$ is positive number.

Suppose that assumption (A1) holds with known vector $g \in \mathbb{R}^l$. Consider the following static consensus controller with gain $k \in \mathbb{R}^1$, $k > 0$ which is the same for all agents:

$$u_i(t) = -kg^{\mathrm{T}}\overline{y}_i(t), \quad i = 1, \ldots, N, \tag{9}$$

where relative output $\overline{y}_i(t)$ has been defined in the previous section. Denote $x(t) = \mathrm{col}(x_1(t), \ldots, x_N(t))$.

Theorem 2. *Let assumptions (A1) and (A2) hold. Then for all k such that*

$$k > \frac{\varkappa_0}{r(L)} \tag{10}$$

controller (9) ensures achievement of goal (6) in dynamical network (4); asymptotical behaviour in the case of goal achievement is described by the following consensus vector:

$$c(t) = \exp(At)\left(v(L)^{\mathrm{T}} \otimes I_n\right)x(0). \tag{11}$$

Proof. Closed loop system (4) and (9) can be rewritten in the following form:

$$\dot{x}(t) = \left(I_N \otimes A - kL \otimes Bg^{\mathrm{T}}C^{\mathrm{T}}\right)x(t). \tag{12}$$

Consider nonsingular matrix P (real or complex) such that

$$\Lambda = \begin{pmatrix} 0 & \mathbf{0}_{N-1}^{\mathrm{T}} \\ \mathbf{0}_{N-1} & \Lambda_e \end{pmatrix} = P^{-1}LP, \tag{13}$$

where $\Lambda_e \in \mathbb{R}^{(N-1)\times(N-1)}$ or $\Lambda_e \in \mathbb{C}^{(N-1)\times(N-1)}$. All eigenvalues of Λ_e have positive real parts. By considering first (zero) columns of matrices $P\Lambda = LP$ and $(P^{\mathrm{T}})^{-1}\Lambda^{\mathrm{T}} = L^{\mathrm{T}}(P^{-1})^{\mathrm{T}}$ we can accept that first column of P is $\mathbf{1}_N$ and first row of P^{-1} is $v(L)^{\mathrm{T}}$.

Let us apply coordinate transformation $z(t) = (P^{-1} \otimes I_n)x(t)$ and rewrite (12):

$$\dot{z}_1(t) = Az_1(t), \tag{14}$$

$$\dot{z}_e(t) = \left(\left(I_{N-1} \otimes A\right) - k\left(\Lambda_e \otimes Bg^{\mathrm{T}}C^{\mathrm{T}}\right)\right)z_e(t), \tag{15}$$

where $z = \mathrm{col}(z_1, z_e)$, $z_1 \in \mathbb{R}^n$, or $z_1 \in \mathbb{C}^n$. Note that zero solution of (15) is globally asymptotically stable iff goal (6) is achieved.

For simplicity let P, Λ_e, and $z(t)$ be real till the end of proof. For any fixed k satisfying (10) there exists $0 < \varepsilon_s < 1$ such that

$$\varepsilon_s k > \frac{\varkappa_0}{r(L)}. \tag{16}$$

Eigenvalues of matrix $(\Lambda_e - \varepsilon_s r(L) I_{N-1})$ have positive real parts. Therefore, according to [22], there exists $(N-1)\times(N-1)$ real matrix $Q = Q^{\mathrm{T}} > 0$ such that the following Lyapunov inequality holds:

$$\left(\Lambda_e - \varepsilon_s r(L) I_{N-1}\right)^{\mathrm{T}} Q + Q\left(\Lambda_e - \varepsilon_s r(L) I_{N-1}\right) > 0. \quad (17)$$

We can rewrite the last inequality:

$$\Lambda_e^{\mathrm{T}} Q + Q\Lambda_e > 2\varepsilon_s r(L) Q. \quad (18)$$

By assumption (A1) there exists $H = H^{\mathrm{T}} > 0$ such that (3) is true with $\varkappa = \varepsilon_s k r(L)$, since $\varkappa > \varkappa_0$. Considering following Lyapunov function,

$$V\left(z_e(t)\right) = z_e^{\mathrm{T}}(t)(Q \otimes H) z_e(t), \quad (19)$$

and derivating it along the nonzero trajectories of (15), we obtain

$$\frac{d}{dt} V\left(z_e(t)\right) = z_e^{\mathrm{T}}(t) M z_e(t), \quad (20)$$

where

$$\begin{aligned} M &= Q \otimes \left(A^{\mathrm{T}} H + HA\right) - k\left(\Lambda_e^{\mathrm{T}} Q + Q\Lambda_e\right) \\ &\quad \otimes \left(Cgg^{\mathrm{T}} C^{\mathrm{T}}\right) \\ &\le Q \otimes \left(A^{\mathrm{T}} H + HA\right) - 2k\varepsilon_s r(L) Q \otimes \left(Cgg^{\mathrm{T}} C^{\mathrm{T}}\right) \\ &= Q \otimes \left(\left(A^{\mathrm{T}} - \varkappa C g B^{\mathrm{T}}\right) H + H\left(A - \varkappa B g^{\mathrm{T}} C^{\mathrm{T}}\right)\right) \\ &= Q \otimes \left(A_*^{\mathrm{T}} H + HA_*\right) < 0. \end{aligned} \quad (21)$$

Matrix relations (3) have been used here. Last inequality concludes the proof. □

Assumptions of this theorem are relaxed in comparison with Theorem 2 from [23]. Proof of Theorem 2 also provides the following auxiliary result.

Lemma 3. *Let assumption (A2) hold. Controller (9) ensures achievement of goal (6) in dynamical network (4) if and only if all eigenvalues of matrix*

$$R = \left(I_{N-1} \otimes A\right) - k\left(\Lambda_e \otimes B g^{\mathrm{T}} C^{\mathrm{T}}\right) \quad (22)$$

have negative real parts. In the case of goal achievement asymptotical behaviour is described by (11).

2.4. Nonidentical Control and Gain Region. Let the initial digraph $\mathcal{G}$ be unit weighted. Let us fix Laplace matrix L and consider static control with nonidentical gains $k_i > 0$:

$$u_i(t) = -k_i g^{\mathrm{T}} \overline{y}_i(t), \quad i = 1, \ldots, N. \quad (23)$$

Without loss of generality we can assume that network does not have a leader (formally: cardinality of Leading Set is more than 1), since in leader case we can reduce the following consideration of synchronization gain region to lower dimension $N-1$.

Let us denote by $\widehat{k} = (k_1, \ldots, k_N)$ and by $\widehat{k}' = (k_1', \ldots, k_N')$ point which is projection of point $\widehat{k}$ on unit sphere $\mathcal{S}$

$$\begin{aligned} &\widehat{k} = k \cdot \widehat{k}', \quad k > 0, \\ &\sum_{i=1}^{N} \left(k_i'\right)^2 = 1, \end{aligned} \quad (24)$$

where scalar common gain k is radius vector magnitude of point $\widehat{k}$. Points $\widehat{k}, \widehat{k}'$ lie in orthant $\mathcal{O} = \{(k_1, \ldots, k_N) \in \mathbb{R}^N \mid k_i > 0,\ i = 1, \ldots, N\}$.

Denote $K' = \mathrm{diag}(k_1', k_2', \ldots, k_N')$. Laplace matrices L and $K'L$ correspond to the same digraphs which differ only in arc weights. Equation for closed loop system (4) and (23) can be rewritten as follows:

$$\dot{x}(t) = \left(I_N \otimes A - k\left(K'L \otimes B g^{\mathrm{T}} C^{\mathrm{T}}\right)\right) x(t). \quad (25)$$

By repeating proof of Theorem 2 we can formulate the following result.

Theorem 4. *Let assumptions (A1) and (A2) hold. Then for all $k_i = k \cdot k_i'$ such that*

$$\sum_{i=1}^{N} \left(k_i'\right)^2 = 1, \quad k > \frac{\varkappa_0}{r(K'L)} \quad (26)$$

controller (23) ensures achievement of goal (6) in dynamical network (4); asymptotical behaviour in the case of goal achievement is described by the following consensus vector:

$$c(t) = \exp(At)\left(v\left(K'L\right)^{\mathrm{T}} \otimes I_n\right) x(0). \quad (27)$$

Let us denote by $\mathcal{K} \subset \mathcal{O}$ region in orthant such that for any point $(k_1, \ldots, k_N) \in \mathcal{K}$ in this region control (23) ensures achievement of the goal (6) in network (4), (23). Consider the following region:

$$\mathcal{K}_r = \left\{\widehat{k} \in \mathcal{O} \mid \widehat{k} = k \cdot \widehat{k}',\ \widehat{k}' \in \mathcal{S},\ k > \frac{\varkappa_0}{r(K'L)}\right\} \quad (28)$$

which is subset of $\mathcal{K} : \mathcal{K}_r \subset \mathcal{K}$. Let us consider closed part of unit sphere

$$\mathcal{S}_\varepsilon = \left\{\widehat{k}' \in \mathcal{S} \mid k_i' \ge \varepsilon,\ i = 1, \ldots, N\right\}, \quad \varepsilon > 0. \quad (29)$$

Point on $\mathcal{S}_\varepsilon$ determines ray (half-line) in $\mathcal{O}$ with initial point at the origin. According to Theorem 4, by moving along this ray from origin, that is, increasing k, we will reach $\mathcal{K}_r$. Consider map

$$h : k' \longmapsto \frac{\varkappa_0}{r(K'L)} \cdot k', \quad k' \in \mathcal{S}_\varepsilon, \quad (30)$$

which is continuous as a composition of continuous maps ([20], continuous dependence of matrix eigenvalues on

parameters). Image of this map is a subset of boundary $\partial\mathscr{K}_r$; therefore, by continuity of map h, boundary $\partial\mathscr{K}_r$ is a hypersurface in $\mathbb{R}^N$. Further, let us consider induced map

$$h_\rho : k' \longmapsto \left\|h\left(k'\right)\right\|_2, \quad k' \in \mathscr{S}_\varepsilon. \tag{31}$$

Domain $\mathscr{S}_\varepsilon$ is compact, so we can apply Weierstrass Extreme Value Theorem and arrive at the following lemma.

Lemma 5. *Map $h : \mathscr{S}_\varepsilon \to h(\mathscr{S}_\varepsilon) \subset \partial\mathscr{K}_r$ is continuous. Map $h_\rho : \mathscr{S}_\varepsilon \to \mathbb{R}^1$ is continuous and has minimum and maximum.*

Generally, hypersurface $\partial\mathscr{K}_r$ is not smooth in all its points. Alternatively, part of a simplex of appropriate dimension can be taken instead of sphere part to serve as the domain for maps h and h_ρ.

Pairwise ratios of nonidentical gains and common gain define homogeneous coordinates in orthant. Common gain k coefficient relates to reachability of consensus and to speed of convergence but it does not influence consensus vector. Also, consensus vector can be changed only by gain ratios variation within Leading Set of agents; see [24].

3. Double-Integrator Networks

3.1. Agents Description. Suppose that each agent S_i in a network is modelled as follows:

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= C^{\mathrm{T}} x_i, \\ & \qquad\qquad i = 1, \ldots, N, \\ A &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ B &= \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \\ C &= \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}. \end{aligned} \tag{32}$$

For $g = 1$ transfer function $g^{\mathrm{T}}\chi(s) = C^{\mathrm{T}}(sI_2 - A)^{-1}B = (s+1)/s^2$ is hyper-minimum-phase. It can be shown that number $\varkappa_0 = 1$.

First and second components of x_i can describe (or can be interpreted as) velocity and position. Single system (32) can be viewed as double integrator with transfer function $1/s^2$ and proportionally differential (PD) control applied to it.

Since $g = 1$, static consensus controller (9) has the following form:

$$u_i(t) = -k\overline{y}_i(t), \quad i = 1, \ldots, N. \tag{33}$$

3.2. Necessary and Sufficient Conditions on Consensus. Let us denote by L_N^C Laplace matrix of unit weighted cycle digraph which is consisting of N nodes S_j with exactly N arcs:

$$\begin{aligned} &(S_1, S_2) \cup \cdots \cup \left(S_j, S_{j+1}\right) \cup \cdots \cup (S_{N-1}, S_N) \\ &\quad \cup (S_N, S_1). \end{aligned} \tag{34}$$

Eigenvalues of L_N^C are evenly located at circle in complex plane [25]:

$$\begin{aligned} \lambda_j &= 1 - \exp\left(\mathrm{i} \cdot j \cdot \frac{2\pi}{N}\right), \\ & \qquad j = 0, \ldots, N-1, \ \mathrm{i}^2 = -1. \end{aligned} \tag{35}$$

Theorem 6. *Controller (33) ensures achievement of goal (6) in dynamical network consisting of N double integrators (32) connected in directed cycle if and only if*

$$k > \frac{1}{2}\cot^2\frac{\pi}{N}. \tag{36}$$

Proof. Let us diagonalize L_N^C. Matrix R from Lemma 3 in our case is block diagonal:

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_{N-1}, \tag{37}$$

where

$$R_j = \begin{pmatrix} -k\lambda_j & -k\lambda_j \\ 1 & 0 \end{pmatrix}, \quad j = 1, \ldots, N-1. \tag{38}$$

So, matrix R is stable iff matrices R_j are stable for all $j = 1, \ldots, N-1$. Characteristic polynomial of R_j is

$$f_j(z) = z^2 + k\lambda_j z + k\lambda_j, \quad j = 1, \ldots, N-1. \tag{39}$$

Let $k\lambda_j = \alpha_j + \mathrm{i}\beta_j$, $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, \ldots, N-1$. Taking in account (35) we can obtain

$$\begin{aligned} \alpha_j &= 2k\sin^2\frac{j\pi}{N}, \\ \beta_j &= -2k\left(\sin\frac{j\pi}{N}\right)\left(\cos\frac{j\pi}{N}\right), \\ & \qquad j = 1, \ldots, N-1. \end{aligned} \tag{40}$$

Now let argument of $f_j(z)$ run on imaginary axis and let us decompose this polynomial on real and imaginary parts:

$$f_j(\mathrm{i}\omega) = \varphi_j(\omega) + \mathrm{i}\psi_j(\omega), \quad \omega \in \mathbb{R}^1, \tag{41}$$

where $j = 1, \ldots, N-1$ and

$$\begin{aligned} \varphi_j(\omega) &= -\omega^2 - \beta_j \cdot \omega + \alpha_j, \\ \psi_j(\omega) &= \alpha_j \cdot \omega + \beta_j. \end{aligned} \tag{42}$$

According to Hermite-Biehler Theorem, polynomial $f_j(z)$ is stable iff both of the following conditions are satisfied:

(i) roots of $\varphi_j(\omega)$ and $\psi_j(\omega)$ are interlacing;

(ii) Wronskian is positive

$$\varphi_j(\omega_0)\cdot\psi_j'(\omega_0)-\varphi_j'(\omega_0)\cdot\psi_j(\omega_0)>0 \tag{43}$$

for at least one value of argument ω_0.

Wronskian is positive for $\omega_0 = 0$, $j = 1,\dots,N-1$. Root interlacing property is equivalently transformable to

$$k>\frac{1}{2}\cot^2\frac{j\pi}{N},\quad j=1,\dots,N-1. \tag{44}$$

Right parts of these $N-1$ inequalities reach maximum when $j = 1$ (also when $j = N-1$) and this concludes proof. □

Therefore, for a large increasing number of agents N gain k should grow as N^2:

$$k\sim\frac{N^2}{2\pi^2},\quad N\longrightarrow\infty. \tag{45}$$

It is possible to conclude that consensus in large cycle digraphs is hard to achieve, at least for agents (32), since arbitrary high gains are not physically realizable.

On the other hand, it is worth noting that cycle digraph is the graph with minimal number of edges which is delivering average consensus among all its nodes; it is strongly connected.

Remark 7 (see [24]). Minimality in edges number causes simple relations on nonidentical gains and left eigenvector $v(KL_N^C)$ components for agents in form (4):

$$k_1v_1=k_2v_2=\cdots=k_Nv_N. \tag{46}$$

In other words, all pairs (k_j, v_j) lie on the same hyperbola.

The following result can be obtained by repeating proof of Theorem 6.

Theorem 8. *Consider network S consisting of N agents (32). Let a digraph, describing information flow, contain directed spanning tree. Let Laplace matrix of the digraph have real spectrum. Controller (33) ensures achievement of goal (6) in dynamical network consisting of N double integrators (32) if and only if*

$$k>0. \tag{47}$$

Proof. For diagonalizable Laplace matrix with real spectrum statement is following from the well-known fact that polynomial (39) with real coefficients is stable iff its coefficients are positive. For nondiagonalizable Laplace matrix L let us transform it to Jordan form. Expansion of matrix $R - zI$ determinant shows that only determinants $R_j - zI_2$ across main diagonal are forming (factorizing) characteristic polynomial of R. □

Note that undirected graphs (i.e., digraphs with symmetric L) have real spectrum and some class of digraphs have real spectrum too, for example, directed path graphs [26].

We can formulate similar result for general digraphs.

Theorem 9. *Consider network S consisting of N agents (32). Let digraph $\mathcal{G}$, describing information flow, contain at least one directed spanning tree. Let all nonzero eigenvalues of Laplace matrix $L(\mathcal{G})$ be denoted by λ_j, $1\le j\le N-1$. Controller (33) ensures achievement of goal (6) in dynamical network consisting of N double integrators (32) if and only if*

$$k>\max_{1\le j\le N-1}\frac{\sin^2\left(\arg\left(\lambda_j\right)\right)}{\operatorname{Re}\lambda_j}. \tag{48}$$

Proof. Using similar argumentation as in proofs of Theorems 8 and 6 we arrive at study of polynomial (39) stability with

$$\begin{aligned}\alpha_j&=k\cdot\operatorname{Re}\left(\lambda_j\right),\\ \beta_j&=k\cdot\operatorname{Im}\left(\lambda_j\right),\\ &\qquad j=1,\dots,N-1.\end{aligned} \tag{49}$$

Wronskian property does hold for $\omega_0 = 0$. For $j = 1,\dots,N-1$ root interlacing property is equivalent to trigonometric inequality

$$\alpha_j+\beta_j\tan\left(\arg\left(\lambda_j\right)\right)>\tan^2\left(\arg\left(\lambda_j\right)\right) \tag{50}$$

or

$$k\operatorname{Re}\left(\lambda_j\right)\left(1+\tan^2\left(\arg\left(\lambda_j\right)\right)\right)>\tan^2\left(\arg\left(\lambda_j\right)\right). \tag{51}$$

□

4. Examples and Numerical Simulations Results

4.1. Three-Node Digraph and Gain Region. Consider digraph shown in Figure 1 with dynamic nodes described in Section 3.1. By Lemma 5 distance from origin to $\partial\mathcal{K}_r$ reaches minimum. Let us draw $\partial\mathcal{K}_r$. First, let $\varepsilon,\delta\in\mathbb{R}$. Let $\varepsilon>0$ be small, and $\varepsilon\le\delta\le1-\varepsilon$, $k_2'=\delta$, $k_3'=1-\delta$. Let L be unit weighted. Eigenvalues of matrix $\operatorname{diag}(0,k_2',k_3')L$ are real: $\{0,\delta,2-2\delta\}$. Using Theorem 8 we conclude that $\mathcal{K}=\{k_2>0, k_3>0\}$. Any $\delta\in[\varepsilon,1-\varepsilon]$ determines angle

$$\gamma(\delta)=\arctan\frac{1-\delta}{\delta}=\arctan\frac{k_3'}{k_2'}=\arctan\frac{k_3}{k_2}, \tag{52}$$

and radius vector $\rho(\delta)=1/\min_{\delta\in[\varepsilon,1-\varepsilon]}\{\delta,2-2\delta\}$. Note that pair $(\rho(\delta),\gamma(\delta))$ is polar coordinates of boundary $\partial\mathcal{K}_r$. We can conclude that minimum on $\rho(\delta)$ is realized on a point for which $\delta=2/3$ and $k_2:k_3=2:1$. Boundary $\partial\mathcal{K}_r$ is presented in Figure 2. For all $(k_2,k_3)=k\cdot(2,1)$ matrix $K'L$ is similar to

$$0\oplus\begin{pmatrix}2k & 1\\ 0 & 2k\end{pmatrix}. \tag{53}$$

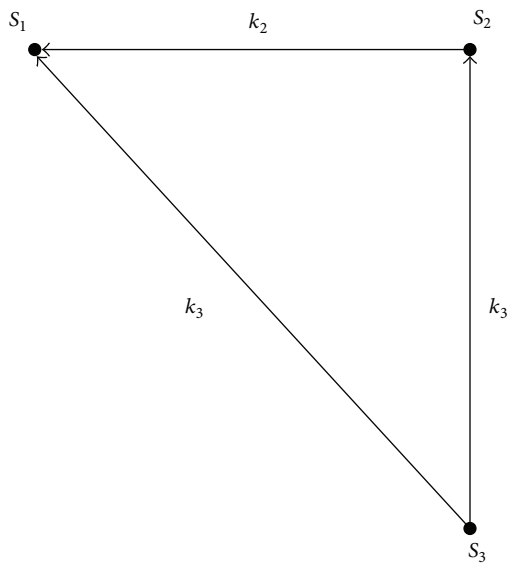

Figure 1: Digraph of 3 nodes.

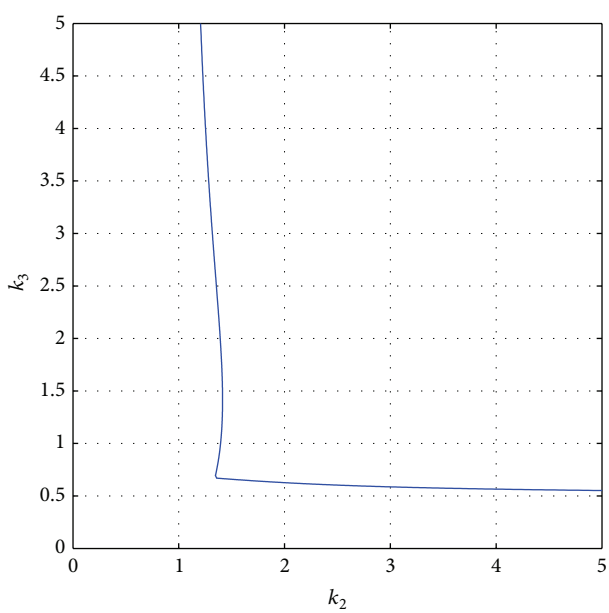

Figure 2: Gain region boundary $\partial\mathscr{K}_r$.

Let us consider two cases:

(1) $\delta = 1/2$, identical gains $k_2^{(1)} = k_3^{(1)} = 0.527 \cdot k$;

(2) $\delta = 2/3$, nonidentical gains $k_2^{(2)} = (2/3) \cdot k$, $k_3^{(2)} = (1/3) \cdot k$.

By Theorem 4 common gain k is as follows: $k = 3/2 = \varkappa_0/r(K^{(2)}L)$, $K^{(2)} = \mathrm{diag}(0, k_2^{(2)}, k_3^{(2)})$. Identical gains are chosen such that $\|(k_2^{(1)}, k_3^{(1)})\|_2 \approx \|(k_2^{(2)}, k_3^{(2)})\|_2$.

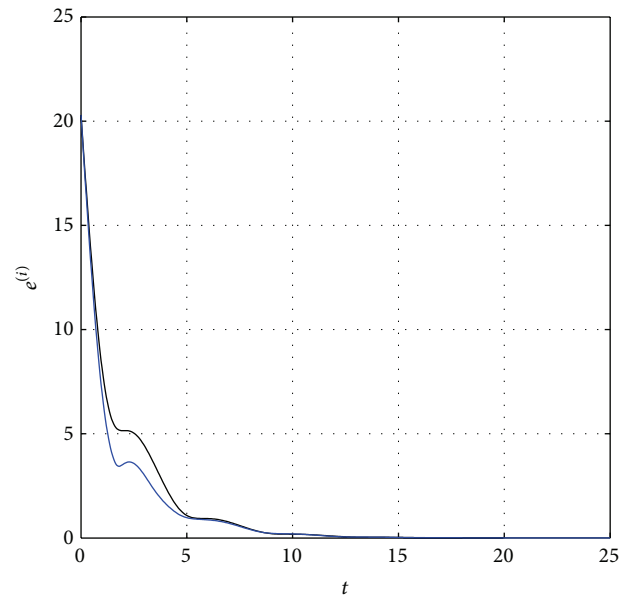

Figure 3: Performance with identical ($e^{(1)}(t)$; black line) and nonidentical ($e^{(2)}(t)$; blue line) gains.

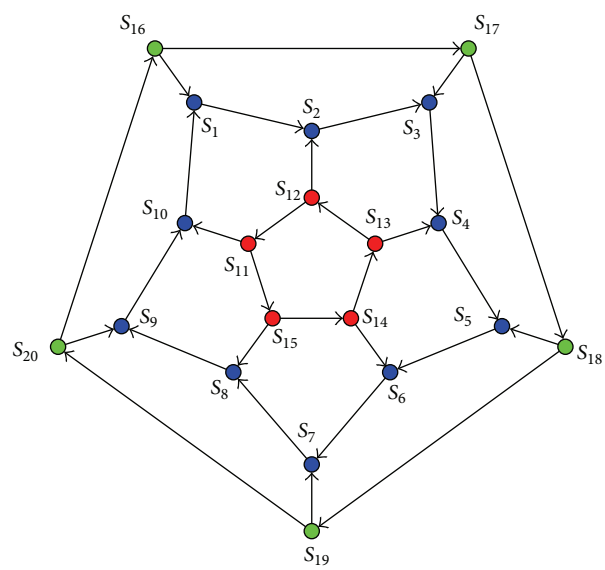

Figure 4: Digraph of 20 agents.

Let us choose the following initial conditions:

$$\begin{aligned} x_1(0) &= \mathrm{col}(2, -2), \\ x_2(0) &= \mathrm{col}(-7, 3), \\ x_3(0) &= \mathrm{col}(1, -3). \end{aligned} \tag{54}$$

Denote by $e(t) = \sum_{i=1}^{2} \|x_i(t) - x_{i+1}(t)\|_2$ sum of error norms or disagreement measure: $e^{(1)}(t)$ error in the first case and $e^{(2)}(t)$ error in the second case. Results of 25 sec. simulations are shown in Figure 3.

Note that consensus vector (27) does not change for all $(k_2, k_3) \in \mathscr{K}$ since subsystem S_1 is leader.

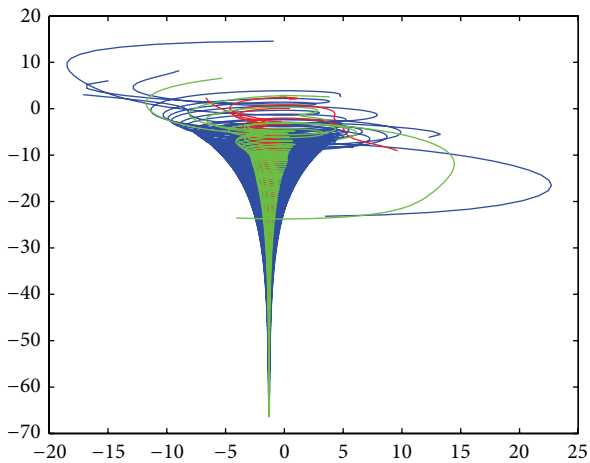

Figure 5: Trajectories of 20 agents on the same phase plane. Trajectories of $S_1, \ldots, S_{10}$ colored blue, $S_{11}, \ldots, S_{15}$ colored red, and $S_{16}, \ldots, S_{20}$ colored green.

4.2. Twenty-Node Digraph and Nonidentical Control. Let us consider digraph shown in Figure 4 consisting of 20 agents $S_1, \ldots, S_{20}$ described in Section 3.1. This dodecahedron-like digraph has Leading Set consisting of dynamic nodes $S_1, \ldots, S_{10}$ which are connected in directed circle. Let us choose $\nu = k_1 = k_2 = \cdots = k_{10}$ and $\mu = k_{11} = k_{12} = \cdots = k_{20}$. According to Theorem 6 gain ν should be chosen $\nu > 0.5 \cdot \cot^2(\pi/10) \approx 4.74$. For faster convergence ν let us choose $\nu = 5.5$. Simulations show that μ can be chosen considerably less than ν. Let us choose $\mu = 1$, and let agents have different initial conditions. Results of numerical simulation show that such nonidentical gain choice provides achievement of consensus. All trajectories of 20 agents on the same phase plane are shown in Figure 5.

Numerical simulations also show that by choosing $\mu = \nu$ and applying Theorem 9 for resulting Laplace matrix one can obtain lower bound approximation $\mu = \nu > 4.74$.

5. Reference Remarks

Assumptions of Theorem 2 are relaxed in comparison with Theorem 2 from [23]. Proof of these results uses coordinate transformation as in [27]. Lemma 3 partially succeeds Theorem 3 from [25]. Theorem 9 is a trigonometric form of Theorem 1 from [28] with different proof, which is potentially extendable to higher orders of state space.

6. Conclusions

By means of Passification Method sufficient conditions of consensus with identical and nonidentical gains are derived. Synchronous behaviour (consensus vector) is described; it can be affected by nonidentical gains (nonidentity in actuation) within Leading Set. Gain asymptote in growing cycle digraphs which have lowest communication cost for reaching average consensus and consisting of double integrators is studied.

It is rediscovered that cycle digraph connection with nonidentical actuation of nodes causes nonidentical impact on synchronous behaviour. Reachability of synchronization corresponds to positive scalar—common gain. By constructing boundary of sufficient gain region in 3-node digraph it is found that Jordan block of Laplace matrix (which affects transient dynamics) appears in a direction of extremal point. Comparison of dynamics is a subject of a future study. Geometrical interpretations which might be useful in theory and applications were developed.

Competing Interests

The author declares that he has no competing interests.

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