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Research Article

Multiple Solutions for the Asymptotically Linear Kirchhoff Type Equations on $\mathbb{R}^N$

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The multiplicity of positive solutions for Kirchhoff type equations depending on a nonnegative parameter λ on $\mathbb{R}^N$ is proved by using variational method. We will show that if the nonlinearities are asymptotically linear at infinity and $\lambda > 0$ is sufficiently small, the Kirchhoff type equations have at least two positive solutions. For the perturbed problem, we give the result of existence of three positive solutions.

1. Introduction and Main Results

The purpose of this article is to investigate the multiplicity of positive solutions to the following nonlocal Kirchhoff type equations:

$$\begin{aligned}&\left(a+\lambda\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)dx\right)\right)(-\Delta u+V(x)u)\\&\quad=q(x)f(u)+h(x)\quad \text{in } \mathbb{R}^N,\end{aligned}\tag{1}$$

where $N \geq 3$, a is a positive constant, $\lambda > 0$ is a parameter, and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function.

In recent years, the following Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u)\quad \text{in } \mathbb{R}^N,\tag{2}$$

has been studied by many researchers under variant assumptions on V and f. Problem (2) is often referred to as nonlocal problem because of the appearance of the term $(\int_{\mathbb{R}^N}|\nabla u|^2dx)\Delta u$ which implies that (2) is no longer a pointwise identity. This causes some mathematical difficulties which make the study of (2) particularly interesting. Problem (2) arises in an interesting physical context. Indeed, replacing $\mathbb{R}^N$ by a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and setting $V(x) = 0$, then problem (2) becomes the following Kirchhoff type Dirichlet problem:

$$\begin{aligned}-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u&=f(x,u),\quad \text{in } \Omega,\\ u&=0,\quad \text{on } \partial\Omega,\end{aligned}\tag{3}$$

which is related to the stationary analogue of the Kirchhoff equation

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,u)\tag{4}$$

that was presented by Kirchhoff [1] as a generalization of the well-known d'Alembert's equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=f(x,u)\tag{5}$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. The readers can learn some early classical research of Kirchhoff equations from [2, 3]. However,

(4) received great attention only after Lions [4] proposed an abstract framework to the problem. Some interesting results for problem (4) can be found in [5–7] and the references therein. There have been many works about the existence and multiplicity of nontrivial solutions to problem (3) using variational methods (see [8–18] and the references therein). Nevertheless, the problems they studied were based on a bounded domain of $\Omega \subset \mathbb{R}^N$. Very recently, some authors had studied the Kirchhoff type equation on the whole space $\mathbb{R}^N$. Many solvability conditions with f near zero and infinity for problem (2) have been considered, such as the superlinear case (see [19–28]); the asymptotically linear case (see [29, 30]); the sublinear case (see [31–33]).

Particularly, the following Kirchhoff type problem has been studied widely by some authors under various conditions on f and V:

$$\left(a+\lambda\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)dx\right)\right)(-\Delta u+V(x)u) = f(x,u), \quad \text{in } \mathbb{R}^N. \tag{6}$$

When $f(x,u)=|u|^{p-2}u$, $p\in(2,2^*)$, Huang and Liu [34] considered (6) and studied existence and nonexistence of positive solution by variational methods; they also discussed the energy doubling property of nodal solutions by Nehari manifold; Wu et al. [35] gave a total description on the positive solutions to (6), and they make an observation on the sign-changing solutions. The results of [34], respectively, complement the corresponding results of [25, 36]. Li and Ye [25] showed that problem (6) has no nontrivial solution provided $f(x,u)=|u|^{p-2}u$, $p\in(2,3)$ when $\lambda>0$ is sufficiently large. If $V(x)=b$, Liu et al. [37] studied the existence of a positive solution for problem (6) involving subcritical growth, which unifies and sharply improves the results of [36]. Fan and Liu [38] studied (6) with concave-convex nonlinearities and showed that problem (6) has at least two positive solutions for $\lambda>0$ sufficiently small. When $f(x,u)$ is asymptotically linear with respect to u at infinity, Ye and Yin [39] studied (6) and proved the existence of positive solution for λ sufficiently small and the nonexistence result for λ sufficiently large. When $V(x)=V(|x|)$, $f(x,u)=f(u)$ is asymptotically linear with respect to u at infinity; Li and Sun [40] showed the existence, nonexistence, and multiplicity to (6) in radial space $H_r^1(\mathbb{R}^N)$. When the nonlinearities f is sublinear or local sublinear, [41, 42] considered the existence and multiplicity of nontrivial solutions to problem (6). Recently, some authors extend problem (6) to the p-Kirchhoff elliptic equations (see, e.g., [43–49] and the references therein). In all works for (6) mentioned above except for [39, 40], we found that the nonlinearities f are superlinear, sublinear, or local sublinear. To the best of our knowledge, there is little information on the multiplicity of solution for (6) with the nonlinearities f satisfying the asymptotically linear condition at infinity. In this paper, we will try to study multiplicity of positive solutions for problem (1) when f is asymptotically linear at infinity.

In order to reduce our statements, we make the following assumptions:

(V) $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ satisfies $\inf_{\mathbb{R}^N}V(x)\ge V_0>0$, where V_0 is a constant.

(F_1) $f\in C(\mathbb{R},\mathbb{R}^+)$, $\lim_{t\to 0}(f(t)/t)=0$, and $f(t)\equiv 0$ for all $t\le 0$.

(F_2) There exists $l\in(0,+\infty)$ such that $\lim_{t\to\infty}(f(t)/t)=l$.

(F_3) There exists $\alpha\in(N/2,+\infty)$ such that $q(x)\in L^{\alpha}(\mathbb{R}^N)$.

(F_4) $l/a>\mu^* := \inf\{\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx : u\in H,\ \int_{\mathbb{R}^N}q(x)u^2dx=1\}$, where H will be given below.

(F_5) $0\le h\in L^2(\mathbb{R}^N)$.

Before stating our main results, we give several notations. Set

$$H^1\left(\mathbb{R}^N\right) := \left\{u\in L^2\left(\mathbb{R}^N\right) : |\nabla u|\in L^2\left(\mathbb{R}^N\right)\right\} \tag{7}$$

with the usual norm

$$\|u\|_{H^1}=\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2+u^2\right)dx\right)^{1/2}. \tag{8}$$

Let

$$H = \left\{u\in H^1\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)dx<+\infty\right\} \tag{9}$$

with the inner product and the norm

$$\langle u,v\rangle=\int_{\mathbb{R}^N}(\nabla u\cdot\nabla v+V(x)uv)\,dx, \qquad \|u\|=\langle u,u\rangle^{1/2}. \tag{10}$$

Since $V(x)$ satisfies (V), it is easy to see that $\|\cdot\|_{H^1}$ is equivalent to $\|\cdot\|$. Obviously, the embedding $H\hookrightarrow L^s(\mathbb{R}^N)$ is continuous for any $s\in[2,2^*]$. We denote by $\|\cdot\|_p$ the usual $L^p(\mathbb{R}^N)$ norm.

Define the functional $I_\lambda, J_\lambda : H\to\mathbb{R}$ by

$$I_\lambda(u)=\frac{1}{2}a\|u\|^2+\frac{1}{4}\lambda\|u\|^4-\int_{\mathbb{R}^N}q(x)F(u)\,dx, \qquad J_\lambda(u)=I_\lambda(u)-\int_{\mathbb{R}^N}h(x)u^+dx, \qquad u\in H, \tag{11}$$

where $F(t)=\int_0^t f(s)ds$. Clearly, by the assumptions imposed on f, g, and h, we know that $I_\lambda(u)$ and $J_\lambda(u)$ are well defined on H, and $I_\lambda, J_\lambda\in C^1(H,\mathbb{R})$ with the derivative given by

$$\left\langle I_\lambda'(u),v\right\rangle = \left(a+\lambda\|u\|^2\right)\int_{\mathbb{R}^N}(\nabla u\cdot\nabla v+V(x)uv)\,dx$$

$$
\begin{aligned}
& - \int_{\mathbb{R}^N} q(x) f(u) v \, dx, \\
\left\langle J_\lambda'(u), v \right\rangle &= \left\langle I_\lambda'(u), v \right\rangle - \int_{\mathbb{R}^N} h(x) v^+ dx, \\
& \qquad u, v \in H.
\end{aligned} \tag{12}
$$

It is standard to verify that the weak solutions of (1) correspond to the critical points of the functional J_λ. Our first result for (1) without $h(x)$ is as follows.

Theorem 1. *Assume that $N \geq 3$, a is a positive constant, and $\lambda > 0$ is a parameter. If the conditions (V) and (F_1)–(F_4) hold, then there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, problem (1) has at least two positive solutions.*

Remark 2. Compared with the works mentioned above except [39, 40], where the nonlinearities f are superlinear, sublinear, or local sublinear, here we consider problem (1) with asymptotically linear nonlinearities. So, our problem is different and extend the abovementioned results to some extent.

Remark 3. In [39], the authors only studied the existence of positive solutions. In this paper, we give multiplicity results when the potential V is different from the conditions of V in [39] and our method is simpler than that used in [39]. When $V(x) = V(|x|)$ satisfied some assumptions, Li and Sun [40] showed the existence, nonexistence, and multiplicity of radial solutions. Here, we get multiplicity results in nonradial space.

Remark 4. Indeed, it is not difficult to find some functions $f(t)$, $V(x)$, and $q(x)$ such that the conditions of Theorem 1 are satisfied. For example, for any fixed $R_0 > 0$, let

$$
\begin{aligned}
f(t) &= \begin{cases} \dfrac{R_0 t^2}{1+t}, & t > 0, \\ 0, & t \leq 0, \end{cases} \\
q(x) &= \begin{cases} \dfrac{1}{1+|x|}, & 0 \leq |x| \leq R_0, \\ \in L^\alpha \left(B_{R_0}^c (0) \right), & |x| \geq R_0. \end{cases}
\end{aligned} \tag{13}
$$

Choosing $V(x) = 1$, it is easy to know that (V) and (F_1)–(F_3) are satisfied for any $R_0 > 0$. Moreover, $l = R_0$ in (F_2) and $F(t) = R_0((t^2 - 2t)/2 + \ln(1+t))$. To verify the condition (F_4), we have to choose a special $R_0 > 0$. Indeed, for $R > 0$, take $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that

$$
\psi(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| \geq 2R, \end{cases} \tag{14}
$$

and $|\nabla \psi(x)| \leq C/R$ for all $x \in \mathbb{R}^N$, where $C > 0$ is a constant independent of x. Because of $\operatorname{supp} \psi \subset B_{2R}$, thus for $R_0 > 2R$, we have

$$
\begin{aligned}
& \frac{\int_{\mathbb{R}^N} \left(|\nabla \psi|^2 + V(x) \psi^2 \right) dx}{\int_{\mathbb{R}^N} q(x) \psi^2 dx} \\
&= \frac{\int_{|x| \leq 2R} \left(|\nabla \psi|^2 + V(x) \psi^2 \right) dx}{\int_{|x| \leq R} q(x) \psi^2 dx} \\
&\leq \frac{\int_{|x| \leq 2R} \left(1 + C^2/R^2 \right) dx}{\int_{|x| \leq R} (1/(1+|x|)) \, dx} \\
&\leq C'(R+1) + \frac{(R+1) C' C^2}{R^2},
\end{aligned} \tag{15}
$$

where C' is a constant independent of R. So, choosing R sufficiently large such that $(R+1)C^2/R^2 \leq 1$, the condition (F_4) holds for $R_0 = aC'(R+3)$.

In our second result, we consider the case of the perturbed Kirchhoff equations; that is, $h(x) \neq 0$, and we obtain the following result.

Theorem 5. *Assume that $N \geq 3$, a is a positive constant, and $\lambda > 0$ is a parameter. If the conditions (V) and $(F_1$–$F_5)$ hold, then there exist two constants $\tilde{\lambda} > 0$ and $m_0 > 0$ such that, for any $\lambda \in (0, \tilde{\lambda})$, problem (1) has at least three positive solutions when $\|h\|_2 < m_0$.*

Remark 6. In the aforementioned papers, the nonlinearities satisfy $f(x, 0) = 0$. Indeed, this condition is not necessary. Here, the nonlinearity may not be 0 at zero because of $f(x, 0) = h(x) \geq 0$.

In order to obtain our results, we have to overcome various difficulties. On the one hand, it is well known that Sobolev embedding $H \hookrightarrow L^p(\mathbb{R}^N)$ is continuous but not compact for $p \in [2, 2^*)$, and then it is usually difficult to prove that a minimizing sequence or a Palais-Smale sequence is strongly convergent if we seek solutions of problem (1) by variational methods. To overcome this difficulty, we make full use of integrability of potential function $q(x)$ and perturbation $h(x)$. On the other hand, as we all know, the (PS) sequence is bounded if the nonlinearity satisfies a variant of Ambrosetti-Rabinowitz type condition ((AR) in short) or 4-superlinearity. However, for the asymptotically linear case of problem (1), we can adopt a simple method to verify the boundedness of (PS) sequence. The conditions (F_1) and (F_2) are crucial to obtain the boundedness of (PS) sequence.

This paper is organized as follows. We give some previous results and prove Theorem 1 in Section 2. Section 3 is devoted to giving the proof of Theorem 5. Throughout this paper, C and C_i are used in various places to denote distinct constants.

2. Proof of Theorem 1

In the following, we give some lemmas which are important to prove our main result.

Lemma 7. *Suppose that (V) and (F_1)–(F_3) hold; then, $I_\lambda(u)$ is coercive on H.*

Proof. By (F_1) and (F_2), we see that $f(s)/s$ is bounded in $\mathbb{R}$. So, setting $L_0 = \sup_{s \in \mathbb{R}} (f(s)/s)$, thus $L_0 \in (0, +\infty)$ and for any $s \in \mathbb{R}$

$$
0 \leq \frac{f(s)}{s} \leq L_0. \tag{16}
$$

Then,

$$|F(s)| \le \frac{L_0}{2}|s|^2, \quad \forall s \in \mathbb{R}. \tag{17}$$

Because of $\alpha \in (N/2, +\infty)$, we have

$$2 < \frac{2\alpha}{\alpha - 1} < 2^*. \tag{18}$$

Furthermore, by (17), (18), (F_3), and the Hölder and Sobolev inequalities, we deduce that for any $u \in H$

$$\begin{aligned}\left|\int_{\mathbb{R}^N} q(x) F(u)\,dx\right| &\le \frac{L_0}{2}\int_{\mathbb{R}^N} q(x) u^2 dx \\ &\le \frac{L_0}{2}\left(\int_{\mathbb{R}^N} q^\alpha(x)\,dx\right)^{1/\alpha}\left(\int_{\mathbb{R}^N} u^{2\alpha/(\alpha-1)} dx\right)^{(\alpha-1)/\alpha} \\ &\le \frac{L_0}{2}\left(\int_{\mathbb{R}^N} q^\alpha(x)\,dx\right)^{1/\alpha}\|u\|^2_{2\alpha/(\alpha-1)} \\ &\le \frac{C_1 L_0}{2}\|u\|^2.\end{aligned} \tag{19}$$

Thus, we obtain

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \int_{\mathbb{R}^N} q(x) F(u)\,dx \\ &\ge \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \frac{C_1 L_0}{2}\|u\|^2, \end{aligned} \tag{20}$$

which shows that $I_\lambda(u)$ is coercive on H. □

It follows from Lemma 7 that I_λ is bounded from below on H and thus we may define $c_\lambda := \inf_H I_\lambda$.

Lemma 8. *Assume that* (V) *and* $(F_1$–$F_4)$ *are satisfied; then,* $I_\lambda(u)$ *satisfies the* (PS) *condition.*

Proof. Suppose that $\{u_n\} \subset H$ is the (PS) sequence for the functional I_λ; that is,

$$\begin{gathered}\{I_\lambda(u_n)\} \text{ is bounded}, \\ I'_\lambda(u_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{gathered} \tag{21}$$

By Lemma 7, the sequence $\{u_n\}$ is bounded in H. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in H for some $u \in H$. Now, we begin to prove $u_n \to u$ strongly in H. As we all know, it is sufficient to show that $\|u_n\| \to \|u\|$ as $n \to \infty$. By (21), we see that

$$\begin{aligned} o(1) &= \left\langle I'_\lambda(u_n), u_n - u\right\rangle \\ &= \left(a + \lambda\|u_n\|^2\right)\langle u_n, u_n - u\rangle \\ &\quad - \int_{\mathbb{R}^N} q(x) f(u_n)(u_n - u)\,dx. \end{aligned} \tag{22}$$

So, we have

$$\begin{aligned} &\left(a + \lambda\|u_n\|^2\right)\langle u_n, u_n - u\rangle \\ &\quad = o(1) + \int_{\mathbb{R}^N} q(x) f(u_n)(u_n - u)\,dx. \end{aligned} \tag{23}$$

Thus, to show that $\langle u_n, u_n - u\rangle = o(1)$ is equivalent to proving that

$$\int_{\mathbb{R}^N} q(x) f(u_n)(u_n - u)\,dx = o(1). \tag{24}$$

By (F_3), for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int_{|x|\ge R} q^\alpha(x)\,dx < \varepsilon^\alpha, \quad \forall R \ge R_\varepsilon. \tag{25}$$

By (16), (18), (25), (F_3), and the Sobolev and Hölder inequalities, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} q(x) f(u_n)(u_n - u)\,dx\right| \\ &\quad \le \int_{\mathbb{R}^N} |q(x)(u_n - u)|\,|f(u_n)|\,dx \\ &\quad \le L_0\int_{|x|\le R} |q(x)(u_n - u)|\,|u_n|\,dx \\ &\quad + L_0\int_{|x|\ge R} |q(x)(u_n - u)|\,|u_n|\,dx \\ &\quad \le L_0\|q\|_{L^\alpha(B_R(0))}\|u_n\|_{L^{2\alpha/(\alpha-1)}(B_R(0))} \\ &\quad \cdot \|u_n - u\|_{L^{2\alpha/(\alpha-1)}(B_R(0))} + L_0\|q\|_{L^\alpha(B_R^c(0))} \\ &\quad \cdot \|u_n - u\|_{L^{2\alpha/(\alpha-1)}(B_R^c(0))}\|u_n\|_{L^{2\alpha/(\alpha-1)}(B_R^c(0))} \le o(1) \\ &\quad + C_2\|q\|_{L^\alpha(B_R^c(0))}\|u_n - u\|\,\|u_n\| \le o(1) + C_3\varepsilon. \end{aligned} \tag{26}$$

This implies

$$\int_{\mathbb{R}^N} q(x) f(u_n)(u_n - u)\,dx = o(1). \tag{27}$$

So, $\langle u_n, u_n - u\rangle \to 0$. It is easy to see that $\langle u, u_n - u\rangle \to 0$. Hence, $\langle u_n - u, u_n - u\rangle \to 0$; that is, $u_n \to u$ strongly in H. □

Proof of Theorem 1. The proof of this theorem is divided into two steps.

Step 1. In this step, we will show that problem (1) has a mountain pass solution.

By (F_3), we see that $\alpha > N/2$, and thus $2^*(\alpha - 1)/\alpha > 2$. So, we may choose a constant $\beta \in (2, 2^*(\alpha - 1)/\alpha]$ such that $\beta\alpha/(\alpha - 1) \le 2^*$. For any $\varepsilon > 0$, it follows from (F_1) and (F_2) that there exists $C_\varepsilon > 0$ and $\beta \in (2, 2^*(\alpha - 1)/\alpha]$ such that

$$|f(s)| \le \varepsilon|s| + C_\varepsilon|s|^{\beta-1}, \quad \forall s \in \mathbb{R}, \tag{28}$$

and, then,

$$|F(s)| \le \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{\beta}|s|^\beta, \quad \forall s \in \mathbb{R}. \tag{29}$$

Using (29), (F_3), and Sobolev and Hölder inequalities, we deduce that for any $u \in H$

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} q(x) F(u)\,dx\right| &\le \frac{\varepsilon}{2}\int_{\mathbb{R}^N} q(x) u^2 dx + \frac{C_\varepsilon}{\beta}\int_{\mathbb{R}^N} q(x)|u|^\beta dx \\
&\le \frac{\varepsilon}{2}\left(\int_{\mathbb{R}^N} q^\alpha(x)\,dx\right)^{1/\alpha}\left(\int_{\mathbb{R}^N} u^{2\alpha/(\alpha-1)}dx\right)^{(\alpha-1)/\alpha} \\
&\quad + \frac{C_\varepsilon}{\beta}\left(\int_{\mathbb{R}^N} q^\alpha(x)\,dx\right)^{1/\alpha}\left(\int_{\mathbb{R}^N} u^{\beta\alpha/(\alpha-1)}dx\right)^{(\alpha-1)/\alpha} \\
&\le \frac{\varepsilon}{2}\left(\int_{\mathbb{R}^N} q^\alpha(x)\,dx\right)^{1/\alpha}\|u\|^2_{2\alpha/(\alpha-1)} \\
&\quad + \frac{C_\varepsilon}{\beta}\left(\int_{\mathbb{R}^N} q^\alpha(x)\,dx\right)^{1/\alpha}\|u\|^\beta_{\beta\alpha/(\alpha-1)} \le \frac{C_4\varepsilon}{2}\|u\|^2 \\
&\quad + \frac{C_5C_\varepsilon}{\beta}\|u\|^\beta .
\end{aligned} \tag{30}$$

Thus, we obtain

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \int_{\mathbb{R}^N} q(x) F(u)\,dx \\
&\ge \frac{a - C_4\varepsilon}{2}\|u\|^2 - \frac{C_5C_\varepsilon}{\beta}\|u\|^\beta .
\end{aligned} \tag{31}$$

So, fixing $\varepsilon \in (0, a/C_4)$ and letting $\|u\| = \rho > 0$ sufficiently small, it is easy to see that there exists a constant $\alpha > 0$ such that

$$I_\lambda(u)\big|_{\|u\|=\rho} \ge \alpha > 0. \tag{32}$$

By (F_4), there is $v \in H$ such that $v \ge 0$, $\int_{\mathbb{R}^N} q(x)v^2 dx = 1$, and $\mu^* \le \|v\|^2 < l/a$. Combining (F_2) with Fatou's lemma, we deduce that

$$\begin{aligned}
\lim_{t\to+\infty}\frac{I_0(tv)}{t^2} &= \frac{a}{2}\|v\|^2 - \lim_{t\to\infty}\int_{\mathbb{R}^N} q(x)\frac{F(tv)}{t^2}dx \\
&\le \frac{1}{2}\left(a\|v\|^2 - l\right) < 0,
\end{aligned} \tag{33}$$

which implies that there exist $e \in H$ with $\|e\| > \rho$ such that $I_0(e) < 0$. Since $I_\lambda(e) \to I_0(e)$ as $\lambda \to 0^+$, we see that there exists $\lambda^* > 0$ such that $I_{\lambda^*}(e) < 0$, and then

$$I_\lambda(e) < I_{\lambda^*}(e) < 0 \tag{34}$$

for all $\lambda \in (0, \lambda^*)$. From (32), (34), and Mountain Pass Theorem, there is a sequence $\{u_n\} \subset H$ such that

$$\begin{gathered}
\{I_\lambda(u_n)\} \text{ is bounded}, \\
I'_\lambda(u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{gathered} \tag{35}$$

Using Lemma 8, we know I_λ satisfies (PS)-condition. So, by Theorem 2.2 in [50], I_λ possess a critical point v_0 with $I_\lambda(v_0) \ge \alpha > 0$. Setting $v^- = \max\{0, -v\}$, since $(a + \lambda\|v_0\|^2)\langle v_0, v_0^-\rangle - \int_{\mathbb{R}^N} q(x) f(v_0) v_0^- dx = 0$, then by (F_1) we have

$$\|v_0^-\| = 0, \tag{36}$$

which implies $v_0 \ge 0$ a.e. in $\mathbb{R}^N$. By the strong maximum principle, v_0 is positive on H and $I_\lambda(v_0) > 0$.

Step 2. Problem (1) has a global minimum; that is, there exists a positive function $u_0 \in H$ such that $I'_\lambda(u_0) = 0$ and $c_\lambda = \inf_H I_\lambda = I_\lambda(u_0) < 0$.

From Lemmas 7 and 8, we know that $I_\lambda(u)$ is bounded from below and satisfies (PS)-condition, and then by Theorem 4.4 in [51], $c_\lambda = \inf_H I_\lambda$ is a critical value of I_λ; that is, there exists a function $u_0 \in H$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) = c_\lambda$. In view of (34), we know $c_\lambda = I_\lambda(u_0) \le I_\lambda(e) < 0$, which implies that $u_0 \ne 0$, and using the same arguments as in Step 1, it is easy to know that u_0 is positive.

Because of $I_\lambda(v_0) > 0 > I_\lambda(u_0)$, we get two different critical points $v_0, u_0 > 0$; that is, problem (1) has two positive solutions, and then the proof of Theorem 1 is completed. □

3. Proof of Theorem 5

First, we need the following lemmas which are important to prove Theorem 5.

Lemma 9. *Suppose that (V) and (F_1)–(F_5) hold; then, $J_\lambda(u)$ is coercive on H.*

Proof. The proof is similar to the proof of Lemma 7, so we omit it here. □

Lemma 10. *Assume that (V) and (F_1)–(F_5) are satisfied; then, $J_\lambda(u)$ satisfies the (PS) condition.*

Proof. By Lemma 8, we only need to show $\int_{\mathbb{R}^N} h(x)(u_n - u)dx = o(1)$. By (F_5), for the above-given $\varepsilon > 0$, there exists $R'_\varepsilon > 0$ such that

$$\int_{|x|\ge R} h^2(x)\,dx < \varepsilon^2, \quad \forall R \ge R'_\varepsilon. \tag{37}$$

By (37) and the Sobolev and Hölder inequalities, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} h(x)(u_n - u)\,dx\right| \\
&\quad \le \int_{|x|\le R} \left|h(x)(u_n - u)\right| dx
\end{aligned}$$

$$+\int_{|x|\geq R}|h(x)(u_n-u)|\,dx$$
$$\leq \|h\|_{L^2(B_R(0))}\|u_n-u\|_{L^2(B_R(0))}$$
$$+\|h\|_{L^2(B_R^c(0))}\|u_n-u\|_{L^2(B_R^c(0))}\leq o(1)+C'\varepsilon; \tag{38}$$

that is

$$\int_{\mathbb{R}^N}h(x)(u_n-u)\,dx=o(1). \tag{39}$$

Using (27) and (39), we obtain

$$\int_{\mathbb{R}^N}q(x)f(u_n)(u_n-u)\,dx+\int_{\mathbb{R}^N}h(x)(u_n-u)\,dx = o(1). \tag{40}$$

So, $\langle u_n,u_n-u\rangle\to 0$. It is easy to see that $\langle u,u_n-u\rangle\to 0$. Hence, $\langle u_n-u,u_n-u\rangle\to 0$; that is, $u_n\to u$ strongly in H. □

Proof of Theorem 5. The proof of this theorem is divided into four steps.

Step 1. In this step, we will show that problem (1) has a positive mountain pass solution.

Set

$$I_{\lambda^*}(u)=\frac{1}{2}a\|u\|^2+\frac{1}{4}\lambda^*\|u\|^4-\int_{\mathbb{R}^N}q(x)F(u)\,dx, \tag{41}$$

where λ^* is given in Theorem 1. By Lemma 7, we known that $I_{\lambda^*}(u)$ is coercive on H. So we can define

$$c_{\lambda^*}:=\inf_H I_{\lambda^*}. \tag{42}$$

Using (34), we have $c_\lambda\leq c_{\lambda^*}<0$. By (31), we know

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{2}a\|u\|^2+\frac{1}{4}\lambda\|u\|^4-\int_{\mathbb{R}^N}q(x)F(u)\,dx \\
&\quad -\int_{\mathbb{R}^N}h(x)u^+dx \\
&\geq \frac{a-C_4\varepsilon}{2}\|u\|^2-\frac{C_5C_\varepsilon}{\beta}\|u\|^\beta-\frac{1}{\sqrt{V_0}}\|h\|_2\|u\| \\
&\geq \|u\|\left(\frac{a-C_4\varepsilon}{2}\|u\|-\frac{C_5C_\varepsilon}{\beta}\|u\|^{\beta-1}-\frac{1}{\sqrt{V_0}}\|h\|_2\right).
\end{aligned} \tag{43}$$

So, choosing $\varepsilon=a/2C_4$ and setting

$$g(t)=\frac{a}{4}t-\frac{C_5C_\varepsilon}{\beta}t^{\beta-1} \tag{44}$$

for $t\geq 0$, we see that there exists a constant $\rho_1>0$ sufficiently small such that $0<\rho_1<\delta:=\min\{\rho,\sqrt{(\sqrt{a^2-\lambda^*c_\lambda^*}-a)/\lambda^*}\}$ and $g(\rho_1)>0$, where $\beta\in(2,2^*(\alpha-1)/\alpha]$ and ρ is given by (32). Taking $m_1:=(\sqrt{V_0}/2)g(\rho_1)$, it then follows that there exists a constant $\alpha_1:=(1/2)g(\rho_1)\rho_1>0$ such that

$$J_\lambda(u)\big|_{\|u\|=\rho_1}\geq\alpha_1 \tag{45}$$

for all h satisfying $\|h\|_2<m_1$.

Using the similar proof of (34), we can obtain that there exists a constant $\widetilde{\lambda}>0$ and a function $e\in H$ with $\|e\|>\rho_1$ such that

$$J_\lambda(e)<0 \tag{46}$$

for all $\lambda\in(0,\widetilde{\lambda})$.

From (45), (46), and Mountain Pass Theorem, there is a sequence $\{u_n\}\subset H$ such that

$$\begin{gathered}
\{J_\lambda(u_n)\}\text{ is bounded,}\\
J'_\lambda(u_n)\longrightarrow 0\quad\text{as } n\longrightarrow\infty.
\end{gathered} \tag{47}$$

It follows from Lemma 10 that J_λ satisfies (PS)-condition. So, using Theorem 2.2 in [50], J_λ possess a critical point u_1 with $J_\lambda(u_1)\geq\alpha>0$ when $\|h\|_2<m_1$. Let $u^-=\max\{0,-u\}$. Since $(a+\lambda\|u_1\|^2)\langle u_1,u_1^-\rangle-\int_{\mathbb{R}^N}q(x)f(u_1)u_1^-dx-\int_{\mathbb{R}^N}h(x)(u_1^-)^+dx=0$, then by (F_1) and (F_5) we have

$$\|u_1^-\|=0, \tag{48}$$

which implies $u_1\geq 0$ a.e. in $\mathbb{R}^N$. So, by the strong maximum principle, u_1 is positive on H. □

Step 2. In this step, we prove the existence of local minimum solution for problem (1).

Since $h\in L^2(\mathbb{R}^N)$ and $h\geq 0$, we can choose a function $\psi\in H$ such that

$$\int_{\mathbb{R}^N}h(x)\psi^+dx>0. \tag{49}$$

Hence, we obtain

$$\begin{aligned}
\lim_{t\to 0}\frac{J_\lambda(t\psi)}{t} &= \lim_{t\to 0}\frac{(t^2/2)a\|\psi\|^2+(t^4/4)\lambda\|\psi\|^4-\int_{\mathbb{R}^N}q(x)F(t\psi)\,dx-\int_{\mathbb{R}^N}h(x)t\psi^+dx}{t} \\
&\leq \lim_{t\to 0}\left(\frac{t}{2}a\|\psi\|^2+\frac{t^3}{4}\lambda\|\psi\|^4-\int_{\mathbb{R}^N}h(x)\psi^+dx\right)=-\int_{\mathbb{R}^N}h(x)\psi^+dx<0.
\end{aligned} \tag{50}$$

Thus, we have

$$c_1 := \inf\left\{J_\lambda(u) : u \in \overline{B}_{\rho_1}\right\} < 0, \tag{51}$$

where ρ_1 is given by (45) and $B_{\rho_1} = \{u \in H : \|u\| < \rho_1\}$. By Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$\begin{gathered} c_1 \le J_\lambda(u_n) < c_1 + \frac{1}{n}, \\ J_\lambda(v) \ge J_\lambda(u_n) - \frac{1}{n}\|v - u_n\| \end{gathered} \tag{52}$$

for all $v \in \overline{B}_{\rho_1}$. Then, by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of J_λ. Therefore, Lemma 10 implies that there exists a function $u_2 \in B_{\rho_1}$ such that $J_\lambda(u_2) = c_1 < 0$ and $J'_\lambda(u_2) = 0$. Similarly, $u_2 > 0$.

Step 3. Problem (1) has a global minimum.

It follows from Lemmas 9 and 10 that $J_\lambda(u)$ is bounded from below and satisfies the (PS) condition, so we may define $c_2 := \inf_H J_\lambda$. Using Theorem 4.4 in [51], $c_2 = \inf_H J_\lambda$ is a critical value of J_λ; that is, there exists a critical point $u_3 \in H$ such that $J'_\lambda(u_3) = 0$ and $J_\lambda(u_3) = c_2$. By (46), $c_2 = J_\lambda(u_3) = \inf_H J_\lambda < 0$, which implies $u_3 \ne 0$. Similarly, $u_3 > 0$.

Step 4. u_1, u_2, and u_3 are different from each other; that is, problem (1) has three positive solutions.

Since $J_\lambda(u_1) > 0 > J_\lambda(u_2)$ and $J_\lambda(u_1) > 0 > J_\lambda(u_3)$, thus $u_1 \ne u_2$ and $u_1 \ne u_3$. Next, we claim that $u_2 \not\equiv u_3$ and then the proof of Theorem 5 is completed. Using the proof of Theorem 1, we know $I_\lambda(u_0) = c_\lambda < 0$ and $I_\lambda(0) = 0$. Using (31), $I_\lambda(u) > 0$ for all $\|u\| < \rho$. Then,

$$\begin{aligned} \left|I_\lambda(u)\right| &= \left|\frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \int_{\mathbb{R}^N} q(x)F(u)\,dx\right| \\ &= \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \int_{\mathbb{R}^N} q(x)F(u)\,dx \\ &\le \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda^*\|u\|^4 < -\frac{1}{4}c_{\lambda^*} \end{aligned} \tag{53}$$

uniformly in λ when $\|u\| < \delta := \min\{\rho, \sqrt{(\sqrt{a^2 - \lambda^* c_\lambda^*} - a)/\lambda^*}\}$. So, when $\|u\| < \delta$ and $\|h\|_2 < m_2 := -\sqrt{V_0}c_{\lambda^*}/4\delta$, we deduce that

$$\begin{aligned} J_\lambda(u) &= I_\lambda(u) - \int_{\mathbb{R}^N} h(x)u^+\,dx \\ &\ge \frac{1}{4}c_{\lambda^*} - \frac{1}{\sqrt{V_0}}\|h\|_2\|u\| \ge \frac{1}{4}c_{\lambda^*} - \frac{1}{\sqrt{V_0}}\|h\|_2\delta \\ &> \frac{1}{2}c_{\lambda^*}. \end{aligned} \tag{54}$$

Since $\rho_1 < \delta$, using (54), we obtain

$$\begin{aligned} c_1 &= \inf\left\{J_\lambda(u) : u \in \overline{B}_{\rho_1}\right\} = J_\lambda(u_2) > \frac{1}{2}c_{\lambda^*}, \\ J_\lambda(u_0) &= I_\lambda(u_0) - \int_{\mathbb{R}^N} h(x)u_0^+\,dx \le I_\lambda(u_0) = c_\lambda \\ &\le c_{\lambda^*}, \end{aligned} \tag{55}$$

when $\|h\|_2 < m_2$. Thus, we have

$$J_\lambda(u_0) \le c_{\lambda^*} < \frac{1}{2}c_{\lambda^*} < J_\lambda(u_2). \tag{56}$$

So, $c_2 = J_\lambda(u_3) = \inf_H J_\lambda \le J_\lambda(u_0) < J_\lambda(u_2)$; that is, $u_2 \not\equiv u_3$. Set $m_0 := \min\{m_1, m_2\}$. From the discussion above, we can obtain that problem (1) has three positive solutions when $\|h\|_2 < m_0$.

Competing Interests

The authors declare that they have no competing interests.

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