Linearization of Fifth-Order Ordinary Differential Equations by Generalized Sundman Transformations

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In this article, the linearization problem of fifth-order ordinary differential equation is presented by using the generalized Sundman transformation. The necessary and sufficient conditions which allow the nonlinear fifth-order ordinary differential equation to be transformed to the simplest linear equation are found. There is only one case in the part of sufficient conditions which is surprisingly less than the number of cases in the same part for order 2, 3, and 4. Moreover, the derivations of the explicit forms for the linearizing transformation are exhibited. Examples for the main results are included.

1. Introduction

Nonlinear problems are of interest to engineers, physicists, mathematicians and many other scientists since most equations are inherently nonlinear in nature. Although linear ordinary differential equations can be solved by a large number of methods but this situation does not hold for nonlinear equations. One common method to solve nonlinear ordinary differential equations is to change their unknowns by suitable variables so as to get linear ordinary differential equations.

The main tools used to solve the linearization problem are transformations such as point, contact, tangent, and generalized Sundman transformations.

It was recognized that Lie \cite{1} is the first person who solved linearization problem for ordinary differential equations in 1883. He discovered the linearization of second-order ordinary differential equations by point transformations. Later, Liouville \cite{2} and Tresse \cite{3} attacked the equivalence problems for second-order ordinary differential equations via group of point transformations. Moreover, Cartan \cite{4} approached the second-order ordinary differential equations by geometric structure of a certain form.

Mahomed and Leach \cite{5} indicated that the $n$th-order ($n > 3$) linear ordinary differential equation has exactly one of $n+1, n+2, n+4$ point symmetries. They suggested that the necessary and sufficient conditions for the $n$th-order ($n \geq 3$) to be linearizable by a point transformation must admit the $n$ dimensional Abelian algebra.

The linearization of third-order ordinary differential equations under point transformations was solved by Bocharov et al. \cite{6}, Grebot \cite{7}, and Ibragimov and Meleshko \cite{8}. Fourth-order ordinary differential equation was studied by Ibragimov et al. \cite{9}. They found the necessary and sufficient conditions for a complete linearization problem. The linearization problem of a fifth-order ordinary differential equation with respect to fiber preserving transformations was considered by Suksern and Pinyo \cite{10}.

In the series of articles \cite{8, 11-14} the linearization problem of a third-order ordinary differential equation via the contact transformations was solved. For a fourth-order ordinary differential equation, this problem was studied in \cite{15, 16}. The criteria of the linearization problem of fifth-order ordinary differential equations were discovered by Suksern \cite{17}.

The linearization problems of third-order and fourth-order ordinary differential equations by the tangent transformations are examined in \cite{18, 19}. These are the first application of tangent (essentially) transformations to the linearization problems of third-order and fourth-order ordinary
differential equations. Necessary and sufficient conditions for third-order and fourth-order ordinary differential equations to be linearizable are obtained there.

Sundman introduced the generalized Sundman transformations in 1992. Later on Duarte et al. [20] applied this method to transform second-order ordinary differential equations into free particle equations. In addition, Muriel and Romero [21] characterized the equations that can be linearized by means of generalized Sundman transformations in terms of first integral. A new characterization of linearizable equations in terms of the coefficients of ordinary differential equation and one auxiliary function was given by Mustafa et al. [22]. Moreover, Nakpim and Meleshko [23] pointed out that the solution given by Duarte et al. using the Laguerre form is not complete.

For the third-order ordinary differential equations, the linearization by the generalized Sundman transformation was investigated by [24] for the form \( X'''(T) = 0 \) and [25] for the Laguerre form. Some applications of the generalized Sundman transformation to ordinary differential equations can be found in [26]. More information of the generalized Sundman transformation are collected in the book [27].

The linearization problem of a fourth-order ordinary differential equation with respect to generalized Sundman transformations was studied in [28]. They found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation to be transformed to the simplest linear equation.

In this article, we intend to use the generalized Sundman transformations to linearize the fifth-order ordinary differential equations in some particular cases. We use computer algebra system Reduce to compute the necessary and sufficient conditions of the linearization. We provide some examples to illustrate the conditions that we have found and also obtain the linearizing transformations.

2. Necessary Conditions

We now concentrate on finding the fifth-order ordinary differential equations

\[
x^{(5)} = f \left( t, x, x', x'', x''', x^{(4)} \right),
\]

which can be transformed to the linear equation

\[
X^{(5)}(T) = 0,
\]

under the generalized Sundman transformation

\[
X = F(t, x),
\]

\[
dT = G(t, x) \, dt.
\]

It turns out that those equations must be in the form of the following theorem.

**Theorem 1.** Any linearizable fifth-order ordinary differential equations that can be transformed by a generalized Sundman transformation has to be in the form

\[
x^{(5)} + \left( A_1 x' + A_0 \right) x^{(4)}
+ \left( B_3 x''' + B_2 x'' + B_1 x' + B_0 \right) x'''
+ \left( C_1 x' + C_0 \right) x''
+ \left( D_3 x'^3 + D_2 x'^2 + D_1 x' + D_0 \right) x''
+ \left( H_4 x'^4 + H_3 x'^3 + H_2 x'^2 + H_1 x' + H_0 \right) = 0.
\]

Here \( A_1 = A_1(t, x), B_i = B_i(t, x), C_i = C_i(t, x), D_i = D_i(t, x), \) and \( H_i = H_i(t, x) \) are some functions of \( t \) and \( x \). Expressions of these coefficients are presented in the appendix.

**Proof.** By a generalized Sundman transformation (3), we have

\[
X'(T) = \frac{D_x F(t, x)}{D_t \int G(t, x) \, dt} = \frac{F_x + x'F_x}{G} = P(t, x, x'),
\]

\[
X''(T) = \frac{D_x P(t, x, x')}{D_t \int G(t, x) \, dt} = \frac{P_x + x'P_x + x''P_x}{G},
\]

\[
X'''(T) = \frac{D_x Q(t, x, x', x'')}{D_t \int G(t, x) \, dt} = \frac{Q_x + x'Q_x + x''Q_x + x'''Q_x}{G} = \frac{1}{G^2} \left( F_x G^2 \right) x'''
\]

\[
+ G \left( 3F_{xx}G - 4F_xG_x \right) x''
\]

\[
+ G \left( 3F_{xx}G - 6F_xG_x - 3F_xG_{xx} \right) x'' + \cdots \]

\[
= R(t, x, x', x'', x''').
\]

\[
X^{(4)}(T) = \frac{D_x R}{D_t \int G(t, x) \, dt} = \frac{R_x + x'R_x + x''R_x + x'''R_x + x''''R_x}{G} = \frac{1}{G^2} \left( F_x G^2 \right) x^{(4)}
\]

\[
+ G \left( (4F_{xxx}G - 7F_xG_x) x'' + \cdots \right)
\]

\[
= S(t, x, x', x'', x''', x^{(4)}),
\]

\[
X^{(5)}(T) = \frac{D_x S}{D_t \int G(t, x) \, dt} = \frac{S_x + x'S_x + x''S_x + x'''S_x + x''''S_x + x^{(4)}S_x}{G} = \frac{1}{G^2} \left( F_x G^2 \right) x^{(5)} + G^3 \left( 5F_{xxx}G - 11F_xG_x \right) x^{(4)}
\]
\[ G^3 (5F_{xx}G - F_xF_{xx} - 10F_xG_x) x^{(4)} + \cdots] \]

\[ = V \left( t, x, x', x'', x'''(4), x^{(5)} \right), \]

where \( D_v = \partial / \partial t + x' (\partial / \partial x) + x'' (\partial / \partial x') + x''' (\partial / \partial x'') + x^{(4)} (\partial / \partial x^{(4)}) + \cdots \) is a total derivative. Replacing \( X^{(5)}(T) \) in (2), we get that

\[ x^{(5)} + \left( \frac{5F_{xx}G - 11F_{x}G_x}{F_xG} \right) x' + \left( \frac{5F_{xx}G - F_xG_x - 10F_xG_x}{F_xG} \right) x^{(4)} + \left( \frac{5(2F_{xx}G - 3F_xG_x)}{F_xG} \right) x'' \]

\[ + \left( \frac{10F_{xxx}G^2 - 45F_xG_xG - 14F_xG_{xx}G + \cdots}{F_xG^2} \right) x^{(12)} \]

\[ + \left( \frac{20F_{xxx}G^2 - 50F_xG_xG - 4F_xG_{xx}G + 15F_xG_xG + 40F_xG_{xx}G + \cdots}{F_xG^2} \right) x^{(11)} \]

\[ + \left( \frac{15F_{xxx}G^2 - 60F_xG_xG_x + 18F_xG_{xx}G + 70F_xG_xG + \cdots}{F_xG^2} \right) x' \]

\[ + \left( \frac{15F_{xxx}G^2 - 30F_xG_xG_x + 10F_xG_xG + 30F_xG_{xx}G + \cdots}{F_xG^2} \right) x^{(12)} \]

\[ + \left( \frac{10F_{xxx}G^3 - 70F_xG_xG_{xx}G^2 + 45F_xG_xG_{xx}G^2 + 195F_xG_{xx}G_xG + \cdots}{F_xG^3} \right) x^{(13)} \]

\[ + \left( \frac{F_{xxxx}G^4 - 10F_xG_xG_{xx}G_xG + 10F_xG_xG_{xx}G_xG^3 + \cdots}{F_xG^4} \right) x^{(15)} + \cdots = 0. \]

Denoting \( A_1, B_1, C_1, D_1, \) and \( H_1 \) as (A.1)–(A.18), we obtain the necessary form (4). This proves the theorem.

**Theorem 2.** Equation (4) can be linearizable by the generalized Sundman transformation if its coefficients satisfy the following equations:

### 3. Sufficient Conditions and Linearizing Transformation

To get the sufficient conditions, we consider (A.1)–(A.18) appearing in the previous section. After using the compatibility theory to those equations, we derive the following results.

\[ S_{4x} = \left( \frac{7A_1S_4 + 49S_1 + 23S_2^2}{280} \right), \]

\[ S_{4t} = \left( -27720S_2S_8 - 8520S_6S_8 + 62S_6S_8 + 4928S_7 - 115S_4^2 \right) (27720S_4)^{-1}, \]

\[ S_{8x} = \left( 21A_1S_8 + 147S_1S_8 + 73920S_2S_8 - 22720S_3S_8 - 40S_6S_8 + 69S_8S_8 \right) (840S_4)^{-1}, \]

\[ B_{2xx} = \left( -219520B_2A_1 + 823200B_2S_4 - 548800S_{1xx} - 644840S_{1x}S_4 + 82320S_4S_8 + 10633A_3^3 S_4 - 10633A_1^3 S_4 \right) \]

\[ + 1519A_1^3 S_4 + 89180A_1B_2S_4 + 72716A_3S_1S_4 + 2604A_1S_4 - 548800B_2^2 - 260680B_2S_4 + 1176B_2S_4^2 + 5488000H_5 \]

\[ + 34218S_7^2 + 1176S_6S_8 + 558S_8^3 \right) (5488000)^{-1}, \]

\[ S_{1x} = \left( -78400B_2x - 637A_1^3 S_4 - 15680A_1B_2 - 5635A_1S_1 + 105A_1S_2 + 2940B_2S_4 + 78400D_3 + 294S_1S_4 + 30S_2^2 \right) \]

\[ \cdot (5880)^{-1}, \]
\[ C_{\alpha x} = (-5488560 A_1 S_4^2 - 38419920 S_1 S_4 + 25347840 S_5 S_4 + 756000 S_6 S_4 - 18627840 A_1 C_0 S_4 - 6137208 A_1 S_2 S_4 + 6337320 A_1 S_4 S_5 + 16929 A_1 S_4 S_6 + 6573 A_1 S_4 S_8 + 23950080 C_0 S_4^2 + 46569600 D_2 S_4 + 2794176 S_4 S_2 - 5188848 S_5 S_4 \]
\[ - 12096 S_1 S_6 - 6237 S_8 S_6 - 5207664 S_2 S_8^2 + 2080288 S_3 S_6^2 + 3682 S_5 S_8^2 + 5925 S_7 S_8^2 \] (93139200S_4)^{-1},
\[ A_{11} = (-133056 A_1 S_2 S_4 + 168240 A_1 S_4 S_5 + 576 A_1 S_4 S_6 - 231 A_1 S_4 S_8 + 443520 C_0 S_4^2 + 25872 S_2 S_4 + 231 S_4 S_8 \]
\[ - 133056 S_3 S_4^2 + 21424 S_4 S_5 + 1445 S_5 S_6 - 215 S_6 S_8 \] (1330560 S_4)^{-1},
\[ S_{11} = (13587840 S_5 S_4 + 40320 S_6 S_4 - 1862784 A_1 S_4 S_5 + 2662464 A_1 S_4 S_6 + 7056 A_1 S_4 S_6 + 2541 A_1 S_4 S_8 + 6209280 C_0 S_4^2 \]
\[ + 1862784 S_1 S_2 - 3235008 S_1 S_6 - 8064 S_1 S_8 - 2541 S_1 S_8 - 1862784 S_2 S_4^2 + 639382 S_3 S_5 + 504 S_5 S_6 + 2165 S_6 S_8 \] (14)
\[ \cdot (1862784 S_4)^{-1}, \]
\[ S_{31} = (- (244640 (144 (3 (308 B_0 + 491 S_4) S_4 + 5 S_6) S_4 - 337 S_5 \times S_4) (112 S_5 + S_6) \]
\[ - (70209254400 S_5 + 33795200 S_5 S_6 - 26232800 S_5 S_6 + 614400 S_6 - 41624 S_6 S_6 + 2776064 S_5 + 5278 S_5 S_6) S_4 \]
\[ - 77 (149022720 S_6 S_4^2 - 23206400 S_6^2 - 62720 S_6 S_6 - 225680 S_5 S_6 - 224 S_6 S_6 + 21504 S_7 - 765 S_5^2) S_4 + 445320 (32 S_6 \]
\[ + 23 S_6 + 2744 S_6) S_4 S_6^2 + 77 (949760 S_5^2 + 8960 S_6 S_6 - 10000 S_6 S_6 + 416 S_6 S_6 + 21504 S_7 - 765 S_5^2 + 149022720 S_6 S_4^2 \]
\[ \cdot A_1 S_4^4) (5565999680 S_4)^{-1}, \]
\[ S_{51} = (-155232 A_1 S_4 S_5 - 1617 A_1 S_4 S_6 + 362208 S_1 S_5 + 1617 S_1 S_6 - 680064 S_2 S_4 + 69856 S_2 S_4 S_2 - 136 S_4 S_6 - 1449 S_4 S_6 \]
\[ \cdot (1034880 S_4)^{-1}, \]
\[ S_{71} = (77616000 S_6 S_4^2 - 64680 S_6 S_5 S_5 + 1617 A_1 S_4 S_6 + 206976 A_1 S_4 S_6 + 204906244000 B_0 S_4^4 + 11319 S_1 S_6 S_6 \]
\[ + 1448832 S_1 S_7 + 4719052800 S_2 S_6 S_4 S_6 + 2069760 S_2 S_6 S_4 S_6 + 17001600 S_2 S_6 S_4 S_6 + 182552382000 S_3 S_6 S_4 S_6 - 3786137600 S_5 S_6 S_6 \]
\[ - 1808576 S_5 S_6 S_6 - 52256000 S_5 S_6 S_6 - 30520 S_5 S_6 S_6 - 3887 S_5 S_6 S_6 - 1022336 S_7 S_7) (4139520 S_4)^{-1} \]
\[ C_1 = \frac{(9 A_1 S_4 + 60 B_2 + 3 S_1 + S_2)}{40}, \]
\[ S_8 = -41580 B_1 + 55440 C_0 - 8316 S_2 + 9176 S_5 \]
\[ + 27 S_6. \]

and (A.19), (A.20), (A.21), (A.22), (A.23), (A.24), (A.25) (moved to the appendix in order to avoid the huge expressions), where

\[ S_1 = -10 A_{1x} - 2 A_2^2 + 5 B_2, \]
\[ S_2 = -20 A_{2x} - 4 A_0 A_1 + 5 B_1, \]
\[ S_3 = 10 A_0 - 2 A_3^2 - 5 B_0, \]
\[ S_4 = -2 A_1 + B_3, \]
\[ S_5 = -80 S_4 + 2 A_0 S_4 + 7 S_2, \]
\[ S_6 = -462 A_0 S_4 + 231 S_2 - 337 S_5, \]
\[ S_7 = -13860 S_2 S_4 + 1386 S_2 S_5 + 8316 S_0 S_4^2 - 202 S_5^2 \]
\[ + S_5 S_6, \]
\[ S_8 = -41580 B_1 + 55440 C_0 - 8316 S_2 + 9176 S_5 \]
\[ + 27 S_6. \]

Proof. We start with the coefficients \( A_i, B_i, C_i, D_i, \) and \( H_i \) in Theorem 1 through the unknown functions \( F \) and \( G. \) From (A.1) and (A.2), we have the derivatives
\[ F_{xx} = \frac{F_x (11 G_x + A_1 G)}{(5G)} \]
\[ F_{tx} = \frac{(F G_x + 10 F_x G + F_x A_0 G)}{(5G)} \]
From (A.4), one obtains the derivative
\[ G_{xx} = \frac{(63 G_x^2 + G_x A_1 G + G^2 S_1)}{(40G)} \]
where
\[ S_1 = -10A_1x - 2A_1^2 + 5B_2. \]  
(23)

From (A.5), one gets the derivative
\[ G_{tx} = \frac{-9F_1G_x^2 + 135F_1G_xG_x + F_xG(2G_xA_0 + GS_2)}{(80F_xG)}, \]  
(24)
and the derivative
\[ F_{tt} = \frac{9F_1^2G_x^2 + 225F_1F_xG_x + F_xG(14G_xA_0 - GS_2) + 400F_x^2G_0G - 600F_x^2G_x^2 + 8F_x^2G_0^2S_3}{(120F_xG_xG)}, \]  
(26)

where
\[ S_3 = 10A_{0x} + 2A_0^2 - 5B_0. \]  
(27)

From (A.3), one obtains the derivative
\[ G_x = \frac{(GS_4)}{7}, \]  
(28)
where
\[ S_4 = -2A_1 + B_3. \]  
(29)

We note that, for the case \( G_x = 0 \), the generalized Sundman transformations are indeed the point transformations. We then suppose \( G_x \neq 0 \), which also implies \( S_4 \neq 0 \).

The relations \((G_x)_x = G_{xx}\) and \((G_x)_t = G_{tx}\) provide condition (7) and the derivative
\[ F_t = \frac{(385F_1G_xS_4 + 7F_1GS_4)}{(9GS_4^2)}, \]  
(30)
where
\[ S_5 = -80S_{4t} + 2A_0S_q + 7S_2. \]  
(31)

The relation \((F_t)_t = F_{tt}\) gives the derivative
\[ G_{tt} = \frac{-2156000G_x^2S_4^2 + 385G_xGS_4S_6 + 4G_x^2S_7}{(1386000GS_4^2)}, \]  
(32)
where
\[ S_6 = -462A_0S_4 + 231S_2 - 337S_5, \]
\[ S_7 = -13860S_5S_4 + 1386S_5S_5 + 8316S_3S_4^2 - 202S_5 \]
\[ + S_6S_5. \]  
(33)

Substituting \(A_0\) into \(A_{0x}\) and \(A_{0t}\), one obtains the conditions
\[ S_{2x} = (94360S_{5x}S_4 + 280S_{6x}S_4 - 11319A_1S_2S_4) \]
\[ + 16513A_1S_4S_5 + 49A_1S_4S_6 + 32340B_1S_4^2 \]
and
\[ S_2 = -20A_{0x} - 4A_0A_1 + 5B_1. \]  
(25)

From (A.6), one finds the derivative
\[ G_t = \frac{(GS_4)}{(6160S_4)}, \]  
(35)
where
\[ S_8 = -41580B_1 + 55440C_0 - 8316S_2 + 9176S_3 \]
\[ + 27S_6. \]  
(36)

The relations \((G_t)_x = G_{tt}\) and \((G_t)_t = G_{tx}\) provide conditions (8) and (9). From (A.18), (A.15), (A.17), (A.13), and (A.11), we obtain conditions (A.19)–(A.21), (10), (A.22). Substituting the relation \(C_{0t}\) into \(C_{0x}\), one obtains condition (A.23). Equations (A.9), (A.10), and (A.12) provide conditions (11), (12), (A.24). Comparing the mixed derivatives \(F_{xxx} = (F_{xx})_t\), \(G_{xxx} = (G_{xx})_t\), \((G_{tt})_x = (G_{x})_t\), \((F_t)_x = F_{tx}\), we obtain conditions (13)–(16). Substituting the relation \(S_{4x}\) into \(S_{xx}\), one obtains condition (A.25). Comparing the mixed derivative \((G_{tt})_x = (G_{tx})_t\), one arrives at condition (17). From (A.7), one obtains condition (18). This proves the theorem.

\[ \blacksquare \]

**Corollary 3.** Under the sufficient conditions in Theorem 2, the transformation (3) mapping equation (4) to a linear equation (2) can be solved by the compatible system of (20), (28), (30), and (35).

**Remark 4.** In the part of sufficient conditions for second-order, there are 2 cases in [20] and 3 cases in [23]. For the third-order, there are 3 cases in [24] and 4 cases in [25]. For the fourth-order, there are 2 cases in [28]. But for the fifth-order there is only one case.
4. Examples

Example 1. For the fifth-order ordinary differential equation

\[ x(5)x^4 - 11x'(4)x^3 - 15x''x^3 + 60x'^{2}x^2 + 70x'^{2}x^2 - 210x'^{3}x + 105x^5 = 0, \]  

we can verify that this equation cannot be linearized by a point transformation [10] or contact transformation [17]. However, (37) is in fact the form (4) in Theorem 1 with the coefficients

\[
\begin{align*}
A_1 &= -\frac{11}{x}, \\
A_0 &= 0, \\
B_3 &= -\frac{15}{x}, \\
B_2 &= \frac{60}{x^2}, \\
B_1 &= 0, \\
B_0 &= 0, \\
C_4 &= \frac{70}{x^2}, \\
C_0 &= 0, \\
D_3 &= -\frac{210}{x^3}, \\
D_2 &= 0, \\
D_1 &= 0, \\
D_0 &= 0, \\
H_5 &= \frac{105}{x^4}, \\
H_4 &= 0, \\
H_3 &= 0, \\
H_2 &= 0, \\
H_1 &= 0, \\
H_0 &= 0, \\
S_5 &= -\frac{52}{x^2}, \\
S_4 &= 0, \\
S_3 &= 0, \\
S_2 &= \frac{7}{x}, \\
S_1 &= 0, \\
S_0 &= 0.
\end{align*}
\]

Moreover, these coefficients also satisfy the conditions in Theorem 2. We now conclude that (37) is linearizable by a generalized Sundman transformation. Corollary 3 yields the linearizing transformation by solving the following equations:

\[
\begin{align*}
F_{xx} &= 0, \\
G_x &= \frac{G}{x}, \\
G_t &= 0, \\
F_t &= 0.
\end{align*}
\]

Considering (40), one arrives at

\[ G = xK_1(t). \]  

Considering (41), one obtains

\[ G = K_2(x). \]  

From (43) and (44), one can choose \( K_1(t) = 1 \) and \( K_2(x) = x \); then we have

\[ G = x. \]  

Considering (39), one gets

\[ F = K_3(t)x + K_4(t). \]  

Considering (42), one arrives at

\[ F = K_5(x). \]  

From (46) and (47), one can choose \( K_3(t) = 1, K_4(t) = 0, \) and \( K_5(x) = x; \) then we obtain

\[ F = x. \]  

So the linearizing transformation is

\[ X = x, \quad dT = xdt. \]  

Hence, by (49), (37) becomes

\[ X^{(5)} = 0. \]  

The general solution of (50) is

\[ X = \frac{\xi_1}{24}T^4 + \frac{\xi_2}{6}T^3 + \frac{\xi_3}{2}T^2 + \xi_4T + \xi_5, \]

where \( \xi_1, \xi_2, \xi_3, \xi_4, \) and \( \xi_5 \) are constants.
where $c_1, c_2, c_3, c_4,$ and $c_5$ are arbitrary constants. Substituting (49) into (51), the general solution of (37) is

$$x(t) = \frac{c_1}{24} \phi(t)^4 + \frac{c_2}{6} \phi(t)^3 + \frac{c_3}{2} \phi(t)^2 + c_4 \phi(t) + c_5,$$

(52)

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \frac{c_1}{24} T^4 + \frac{c_2}{6} T^3 + \frac{c_3}{2} T^2 + c_4 T + c_5.$$  

(53)

**Example 2.** For the fifth-order ordinary differential equation

$$x^{(5)} tx^4 - 22x' x^{(4)} tx^3 + 3x^{(4)} x^4 - 30x'' x''' tx^3$$

$$+ 212x''^2 x''x^2 - 48x' x''' x^3 + 244x' x''^2 tx^2$$

$$- 26x''^2 x^3 - 1180x'^5 x't + 320x'^4 x^2$$

$$+ 880 x^5 t - 320 x^4 x^2 = 0,$$

(54)

we can verify that this equation cannot be linearized by a point transformation [10] or contact transformation [17]. However, (54) is in fact the form (4) in Theorem 1 with the coefficients

$$A_1 = \frac{-22}{x},$$

$$A_0 = \frac{3}{t},$$

$$B_3 = \frac{-30}{x},$$

$$B_2 = \frac{212}{x^2},$$

$$B_1 = \frac{-48}{tx},$$

$$B_0 = 0, $$

$$C_1 = \frac{244}{x^3},$$

$$C_0 = \frac{-26}{tx},$$

$$D_3 = \frac{-1180}{x^3},$$

$$D_2 = \frac{320}{tx^2},$$

$$D_1 = 0, $$

$$D_0 = 0,$$

$$H_5 = \frac{880}{x^4},$$

$$H_4 = \frac{-320}{tx^3},$$

Moreover, these coefficients also satisfy the conditions in Theorem 2. We now conclude that (54) is linearizable by a generalized Sundman transformation. Corollary 3 yields the linearizing transformation by solving the following equations:

$$F_{xx} = 0,$$

(56)

$$G_x = \frac{(2G)}{x},$$

(57)

$$G_t = 0,$$

(58)

$$F_t = \frac{(F_{xx})}{t}.$$  

(59)

Considering (57), one arrives at

$$G = K_1(t) x^2.$$  

(60)

Considering (58), one obtains

$$G = K_2(x).$$  

(61)

From (60) and (61), one can choose $K_1(t) = 1$ and $K_2(x) = x^2$; then we obtain

$$G = x^2.$$  

(62)

Equation (59) becomes

$$tF_t - xF_x = 0,$$

(63)

and by Cauchy method, one arrives at

$$F = tx.$$  

(64)
This solution satisfies (56), so the linearizing transformation is
\[ X = tx, \]
\[ dT = x^2 dt. \tag{65} \]
Hence, by (65), (54) becomes
\[ x^{(5)} = 0. \tag{66} \]
The general solution of (66) is
\[ X = \frac{c_1}{24} T^4 + \frac{c_2}{6} T^3 + \frac{c_3}{2} T^2 + c_4 T + c_5, \tag{67} \]
where \( c_1, c_2, c_3, c_4 \), and \( c_5 \) are arbitrary constants. Substituting (65) into (67), the general solution of (54) is
\[ x(t) = \left( \frac{c_1}{24} \phi(t) \right)^4 + \left( \frac{c_2}{6} \phi(t) \right)^3 + \left( \frac{c_3}{2} \phi(t) \right)^2 + c_4 \phi(t) + c_5, \tag{68} \]
where the function \( T = \phi(t) \) is a solution of the equation
\[ \frac{dT}{dt} = \left( \frac{c_1}{24} T^4 + \frac{c_2}{6} T^3 + \frac{c_3}{2} T^2 + c_4 T + c_5 \right)^2. \tag{69} \]

**Appendix**

**Equations for Theorem 1 in Section 2**

\[
A_1 = \frac{(5F_{xx}G - 11F_xG_x)}{(F_xG)}, \tag{A.1}
\]
\[
A_0 = \frac{(5F_{xx}G - F_xG_x - 10F_xG_x)}{(F_xG)}, \tag{A.2}
\]
\[
B_3 = \frac{5(2F_{xx}G - 3F_xG_x)}{(F_xG)}, \tag{A.3}
\]
\[
B_2 = \frac{(10F_{xxx}G^2 - 45F_{xx}G_xG - 14F_xG_{xx}G + 60F_xG_{xx}^2)}{(F_xG)} \tag{A.4}
\]
\[
B_1 = \frac{(20F_{xxx}G^2 - 50F_{xx}G_xG - 4F_xG_{xx}G + 15F_xG_{xx}^2)}{(F_xG)} - 40F_{xx}G_xG - 24F_{xx}G_g + 105F_xG_xG_x \tag{A.5}
\]
\[
B_0 = \frac{(-40F_{xx}G_x + 10F_{xxx}G^2 - 5F_xG_xG - 4F_xG_{xx}G + 15F_xG_{xx}G_x - 10F_xG_{xx}G_x + 45F_xG_{xx}^2)}{(F_xG)} \tag{A.6}
\]

\[
C_1 = \frac{(15F_{xxx}G^2 - 60F_{xx}G_xG - 18F_xG_{xx}G + 70F_xG_{xx}^2)}{(F_xG)} \tag{A.7}
\]
\[
C_0 = \frac{(15F_{xxx}G^2 - 30F_xG_xG_xG - 3F_xG_{xx}G + 10F_xG_{xx}^2)}{(F_xG)} - 30F_xG_xG - 15F_xG_{xx}G_x + 60F_xG_{xx}G \tag{A.8}
\]
\[
D_3 = \frac{(10F_{xxx}G^2 - 70F_{xx}G_xG_xG - 45F_xG_{xx}G - 45F_xG_{xx}^2 + 195F_xG_{xx}G_xG + 125F_xG_{xx}G_{xx}G)}{(F_xG)} \tag{A.9}
\]
\[
D_2 = \frac{(30F_{xxx}G^2 - 150F_{xx}G_xG_xG - 60F_xG_{xx}G - 210F_xG_{xx}G_xG + 205F_xG_{xx}G_{xx}G)}{(F_xG)} \tag{A.10}
\]
\[
D_1 = \frac{(-120F_{xxx}G^2 - 90F_xG_xG_xG + 390F_xG_{xx}G)}{(F_xG)} + 30F_{xxx}G^2 - 90F_{xx}G_xG_xG - 15F_xG_{xx}G \tag{A.11}
\]
\[
D_0 = \frac{(-30F_xG_{xx}G^2 + 135F_{xx}G_xG^2 + 10F_{xxx}G_{xx}G^2 - 10F_{xx}G_{xx}G^2 - 60F_xG_{xx}G_xG - 6F_{xxx}G_{xx}G^2 + 205F_xG_{xx}G_{xx}G)}{(F_xG)} + 60F_xG_{xx}G + 45F_xG_{xx}G_xG - 6F_{xxx}G_{xx}G \tag{A.12}
\]
\[
H_5 = \frac{(F_{xxxG}G^4 - 10F_{xxxG}G_{xx}G^2 - 10F_{xxG}G_{xx}G)}{(F_xG)} + 45F_{xx}G_{xx}G_x^2 + 60F_xG_{xx}G_x \tag{A.13}
\]
\[
H_4 = \frac{(5F_{xxxG}G^4 - 40F_{xxxG}G_{xx}G^3 - 30F_xG_{xx}G_xG^3)}{(F_xG)} + 135F_{xx}G_{xx}G_x^3 - 10F_{xxxG}G_{xx}G_x^3 + 120F_xG_{xx}G_xG_{xx}G + 210F_xG_{xx}G_{xx}^2 - 210F_xG_{xx}G - 15F_xG_{xx}G_xG \tag{A.14}
\]
\[ H_3 = \left( -40x_{xxxx}G^3 - 60x_{xxx}G^3 + 270x_{xxx}G_xG^2 \right) \]

\[ + \left( 30x_{xxx}G^3 + 240x_{xx}G_xG^2 + 120x_{xx}G_xG_xG^2 \right) \]

\[ - 630x_{xxxx}G^3 + 10x_{xxxx}G^3 - 60x_{xxxx}G^3 \]

\[ - 30x_{xxxx}G^3 + 135x_{xxxx}G_xG^2 - 5x_{xxxx}G_xG_xG^2 \]

\[ + 60x_{xxxx}G_xG_xG^2 - 105x_{xxxx}G_xG^2 - 40x_{xxxx}G_xG^2 \]

\[ + 15x_{xxxx}G_xG_xG^2 \]

\[ + 210x_{xxxx}G_xG_xG + 420x_{xxxx}G_xG_xG \]

\[ = \left( 23856004085760000000S_{x_6}S_7^2S_8 \right) \]

\[ + 852000145920000000S_{x_6}S_8^2S_9 \]

\[ - 45181825920000000S_6S_7^2S_8 \]

\[ + 12263683640000S_6S_8^2S_9 \]

\[ - 81792014008320000S_7_{xxxx}S_9^2 \]

\[ + 1214611408023552000000H_{05}S_9^2 \]

\[ + 12882242063104000S_5^2S_6^2S_7^2S_8 \]

\[ + 33433269227520000S_5^2S_6^2S_8^2 \]

\[ + 20224245888000S_6S_8^2S_9^2 \]

\[ + 1247633114112000S_5^2S_6^2S_9^2 \]

\[ - 398786229338112000S_5^2S_7^2 \]

\[ + 2142294739200S_5^2S_8^2S_9^2 \]

\[ + 169957431705600S_6S_7^2S_8^2 \]

\[ - 1775000304000S_8^2S_9^2 \]

\[ - 147532492800000S_7S_8 \]

\[ + 4876539017112000S_5S_6^2S_7S_8 \]

\[ + 2581818624000000S_5S_6^2S_8^2 \]

\[ + 394760676096000S_5S_7S_8^2 \]

\[ + 394760676096000S_5S_7S_9^2 \]

\[ \text{(A.15)} \]

\[ \text{Equations for Theorem 2 in Section 3} \]
\[-1004306790187008000 S_6 S_5 S_4 S_7 - 3325546312704000000 S_2 S_3 S_6 S_6 - 14291403033600000 S_3 S_5 S_6 S_8 - 12335121469440000 S_2 S_5 S_6 S_8 + 3534379287183360000 S_2 S_6 S_7 - 28314316800000 S_2 S_3 S_6^3 - 6606314668800 S_2 S_3 S_6 S_8 + 11763248910336000 S_3 S_5 S_6 S_8 + 2489610816000 S_2 S_5 S_6 S_8 + 1496046182400000 S_2 S_5 S_6 S_8 - 1086668520 S_2 S_3 S_8 + 2607397309440 S_2 S_6 S_8^2 + 4354257600 S_2 S_6 S_8^2 + 130235212800 S_3 S_6 S_7 S_8 - 35904330000 S_3 S_6 S_8^2 + 1765382750208000 S_6 S_7 S_8^2 - 5122656000000 S_2 S_5 S_6 S_8^3 + 159828655472640000000 S_3 S_4 S_6 S_8^3 + 21846724791091200000 S_3 S_4 S_5 S_6 S_8 + 67229513809920000000 S_4 S_5 S_6 S_8^2 + 107085863808000000 S_4 S_5 S_6 S_8^2 + 10429690363020000 S_3 S_5 S_6 S_8 S_8 + 33758438650675200000 S_3 S_5 S_6 S_8^2 + 52496978688000000 S_3 S_5 S_6 S_8^2 + 1488364416000000 S_3 S_4 S_6 S_8^2 + 44090234496000 S_3 S_4 S_6 S_8^2 - 740940963840000 S_3 S_4 S_6 S_8 S_7 + 4756386096000 S_3 S_4 S_6 S_8^2 + 2916374630400000 S_3 S_4 S_6 S_8 S_8 + 216798120000000 S_3 S_4 S_6 S_8^2 - 55249411176832000000 S_3 S_4 S_6 S_8^2 + 6828947736576000000 S_3 S_6 S_8 - 7047118848000000 S_3 S_6 S_8 - 2911104000000 S_3 S_6 S_8^3 + 22696119552000 S_2 S_3 S_6 S_8 S_8 - 4028136099840000 S_3 S_6 S_8 S_7 - 5178872160000 S_4 S_5 S_6 S_8^2 - 571560960000000 S_4 S_6 S_8 S_7 - 2773848000000 S_4 S_6 S_8 S_7 - 1049440000 S_5 S_6 S_8^4 + 6279324480 S_5 S_6 S_8 S_8 - 650224460800 S_5 S_6 S_8 S_8 S_7 - 97072668000 S_5 S_6 S_8 S_8 S_7 - 6378262272000 S_5 S_6 S_8 S_8 S_7 - 2195490000000 S_5 S_6 S_8 S_8 S_7 + 49374324 S_5 S_6 S_8 S_8 + 896097408 S_5 S_8 - 14788620 S_5 S_6 S_8^2 - 11033920000 S_6 S_8 S_6 S_6 - 1631850 S_6^2 S_6 S_8^3 + 151872430080 S_6 S_6 S_7 S_8^2 - 5366592000 S_6 S_7 S_8 S_8 + 32170875 S_6^3 S_8 S_6 - 12909568000 S_6 S_8 S_8 S_7 + 8487600000 S_6 S_8 S_8 S_7 + 7683984000 (155232000 S_2 S_3 S_6 S_6 + 5544000 S_2 S_5 S_6 S_8 - 170311680 S_2 S_7 S_8 - 8382528000 S_3 S_4 S_6 S_6 - 357952000 S_3 S_4 S_6 S_6 S_6 + 11200000 S_4 S_6 S_6 - 1648000 S_6 S_8 S_8 + 48762880 S_5 S_7 - 272 S_6 S_8 + 960000 S_7 S_7 + 1155 S_6 S_6 S_7 + 64000 S_7 S_8 ) B_0 S_4^2 + 17740800 ( 276623424000 B_0 S_4^2 - 829870272000 S_3^2 S_8 + 69144768000 S_3 S_5
\[-59209920S_2S_6 + 55440000S_2S_8 \]
\[+ 264608467200S_6S_4^2 \]
\[-49174841600S_5^2 - 232872800S_6S_8 \]
\[-1724000S_8S_6 \]
\[-514314S_6^2 - 1160S_6S_8 - 49754880S_7 \]
\[+ 144375S_8^2 S_2S_4 \]
\[+ 25818186240000 \left( 4158000S_6S_4 \right) \]
\[-4158000S_2S_6 + 1096381440S_3S_4^2 \]
\[-129382400S_5^2 \]
\[-741880S_5S_6 - 268800S_5S_8 \]
\[-1280S_6^2 - 684S_6S_8 \]
\[-4992S_7 - 825S_8^2 \right) S_3S_4^3 \]
\[+ 1022400175104000000 \left( 77S_6 + 20S_8 \right) \]
\[+ 23968S_7 \right) S_4S_6 \]
\[-619636469760000 \left( 24213S_6 \right) \]
\[+ 4880S_8 + 4866176S_5S_4 \]
\[+ 768000 \left( 417386935S_6 \right) \]
\[+ 422682482S_8 \right) S_6 \]
\[-88 \left( 1252844464S_7 + 2443875S_8^2 \right) \right) S_5^3 \]
\[-511200087552000 \left( S_6S_8 \right) \]
\[-384S_7 + 280S_7S_8 \right) S_2 \]
\[-8520001459200000 \left( S_6S_8 \right) \]
\[-192S_7 + 280S_7S_8 \right) B_{00}S_4^2 \]
\[+ 19670999040000 \left( 327S_6 - 125S_8 \right) \]
\[+ 26080S_7 + 249480S_8 \right) S_7S_4 \]
\[-76839840000 \left( 332640 \right) \left( 280S_5 \right) \]
\[+ S_6 \right) S_2 - 2520S_3S_4^2 \]
\[+ 54790400S_5^2 + 380800S_6S_8 \]
\[+ 158720S_5S_8 + 688S_6S_8 \]
\[-112896S_7 + 1155S_8^2 \right) S_{00}S_4 \]
\[-277200 \left( 6147187200 \right) \left( 3 \right) \left( 5B_0S_4 \right) \]
\[-S_2^2 \left( 280S_5 + S_6 \right) \]
\[-16S_7S_4 \right) - \left( 5438522880000S_5^3 \right) \]
\[+ 30283456000S_7S_8 \]
\[+ 18086822400S_6S_8 + 96768000S_5S_8^4 \]
\[+ 99126880S_5S_6S_8 \]
\[+ 6218168320S_5S_8 + 3603600S_5S_8^4 \]
\[+ 211536S_7S_8 \]
\[-29151360S_6S_7 + 120120S_6S_8^2 \]
\[+ 10810240S_7S_8 + 129525S_5 \]
\[+ 93139200 \left( 3160S_6 + 1397S_8 \right) \]
\[+ 879840S_5 \right) S_5S_8^2 \]
\[+ 55440 \left( 81088000S_5^2 - 565600S_6S_8 \right) \]
\[+ 326560S_7S_8 - 844S_5S_8 + 467712S_7 \]
\[-1155S_8^2 + 838252800S_5S_8^4 \right) S_5S_6S_8 \]
\[\left( 56681865707765760000000S_5^4 \right) \]
\[(A.19)\]

\[S_{txt} = \left( \left( 2 \right) \left( 0.3020 \right) \left( 2 \right) \left( 0.99 \right) \left( 259952S_5^2 \right) \right) \]
\[= \left( 2 \right) \left( 40320 \right) \left( 2 \right) \left( 0.99 \right) \left( 259952S_5^2 \right) \]
\[= 436299S_5S_8^2 - 19404000H_3 \]
\[+ 17248000C_4 \right) S_4 \]
\[+ 24736S_7S_8 - 285959520S_6S_8^2 \]
\[+ 672348600S_4S_6 \]
\[+ 1280664000C_0S_2S_4 \]
\[= 384199200B_{00}S_4^2 \]
\[+ 96049800A_{11}S_4^2 \right) S_4 \]
\[-1334025 \left( 112S_7 + 8S_8 \right) S_{11} \]
\[-\left( 48428561920S_5^2 \right) \]
\[= 654609600S_6S_8 \]
\[+ 600371360S_5S_8 \]
\[+ 1929680S_8^2 + 9171338S_6S_8 \]
\[+ 552802176S_7 \]
\[-10786905S_8^2 \right) S_5^2 \]
\[-1890 \left( 819624960 S_2^2 \right) + 413090979840 \left( A_1 \right) - S_1 \right) C_{0\sigma} S_4^2 \\
- 2209320960 S_2 S_5 \\
- 7096320 S_2 S_6 \\
+ 1626240 S_2 S_8 \\
+ 204906240 S_3 S_4^2 \\
+ 1478702080 S_5^2 \\
+ 9564160 S_5 S_6 \\
- 2372480 S_5 S_8 \\
+ 15360 S_2^2 - 10274 S_6 S_8 \\
- 206976 S_7 + 5775 S_5^2 \right) B_2 \\
+ 189 \left( 13113999360 S_2^2 \right) \\
- 3571341120 S_2 S_5 \\
- 113541120 S_2 S_6 \\
+ 22767360 S_5 S_8 \\
+ 1434343680 S_3 S_4^2 \\
+ 2419066800 S_5^2 \\
+ 154603520 S_5 S_6 \\
- 33214720 S_5 S_8 \\
+ 245760 S_6^2 - 95326 S_6 S_8 \\
+ 206976 S_7 - 5775 S_5^2 \right) A_1^2 \\
- 42 \left( 29506498560 S_2^2 \right) \\
- 79535554560 S_2 S_5 \\
- 255467520 S_2 S_6 \\
+ 105114240 S_2 S_8 \\
- 101056032000 S_3 S_4^2 \\
+ 31943398080 S_5^2 \\
+ 253961600 S_5 S_6 \\
- 60480320 S_5 S_8 + 552960 S_6^2 \\
- 70036250 S_6 S_8 \\
- 15044736 S_7 + 976015 S_5^2 \right) S_1 \\
- 663896217600 \left( 21 S_1 \right) \\
- 11 S_4^2 \right) B_0 S_4^2 \\
+ 3098182348800 B_0 S_4^2 \\
+ 41309097984000 \left( A_1 \right) - S_1 \right) C_{0\sigma} S_4^2 \\
+ 34927200 \left( 8624 A_1 S_5 + 77 A_1 S_8 \right) \\
- 487872 S_2 S_4 + 265760 S_4 S_5 \\
+ 2112 S_4 S_6 - 3298 S_4 S_8 \right) S_{11} \\
- 73920 \left( 176576 S_6 + 4499 S_8 \right) \\
+ 51980272 S_5 S_2 S_4^2 \right) S_4 \\
- 582120 \left( 18627840 S_3 S_4^2 \right) \\
+ 636160 S_5^2 + 1120 S_6 S_8 \\
+ 5680 S_5 S_8 + 52 S_5 S_8 \\
+ 2688 S_7 - 75 S_8^2 \\
- 18480 \left( 112 S_5 + S_8 \right) S_2 S_{11} \\
- 3725568000 \left( 48 S_6 - 11 S_8 \right) \\
+ 14944 S_5 - 11088 S_5 \right) C_{0\sigma} S_4^2 \\
+ 698544000 \left( 192 S_6 - 11 S_8 \right) \\
+ 63472 S_5 - 44352 S_2 \right) B_2 S_4^2 \\
+ 4300800 \left( 17555 S_8 - 2693 S_8 \right) \\
+ 4417608 S_5 - 55800 S_3 S_6 \right) S_{11} S_4^2 \\
+ 13440 \left( 18360 S_6 - 2657 S_8 \right) \\
+ 379780 S_5 \\
- 4241160 S_2 \right) S_{11} S_4^2 \\
+ 8870400 \left( 1617 \left( 112 S_5 \right) \\
+ S_6 \right) S_1 - 1728 S_2 S_5 \right) \\
+ 8064 S_6 - 4841 S_8 \\
+ 1852816 S_5 S_2 S_4^2 \right) C_{0\sigma} S_4 \\
- 3 \left( 4851 \left( 18627840 S_3 S_4^2 \right) \\
- 48767160 S_5 \\
- 142240 S_5 S_6 - 425680 S_5 S_8 \\
- 1228 S_6 S_8 + 2688 S_7 - 75 S_8^2 \\
+ 277200 \left( 112 S_5 + S_8 \right) S_2 S_4 \right) S_1 \\
- \left( 3098182348800 B_0 S_4^2 \right)
\[
\begin{align*}
S_{72t} &= \left( 8520000145920000000B_{0t}S^4_4 ight. \notag \\
&- 852000145920000000S_{3tt}S^4_4 \notag \\
&+ 21515520000000S_{0t}S^4_4 S_5 \notag \\
&+ 30735936000000S_{06}S^4_4 S_5 \notag \\
&+ 852000145920000000B^4_{0t} \notag \\
&- 170400029184000000H_{1}S^4_4 \notag \\
&+ 55537778728960000S^2_4S^2_5 \notag \\
&- 18646678640000S^2_5S^2_8 \notag \\
&+ 368312320000S^2_5S^2_8 \notag \\
&- 17617797120000S^2_5 \notag \\
&- 3671754240000S^5_5S^2_6 \notag \\
&- 471905280000S^2_5S^4_8 \notag \\
&- 172480000S^2_5S^2_8 \notag \\
&- 11718537600S^5_6S_8 \notag \\
&+ 4367800729600S^5_5S_7 \notag \\
&- 2550240000S^2_5S^2_8 \notag \\
&- 18098080S^2_5S^2_8 \notag \\
&+ 256650240S^2_5S_8 \notag \\
&+ 46985400S^2_5S^2_8 \notag \\
&+ 2848384000S_2S_5S_8 \notag \\
&- 78540000S^3_8 \notag \\
&+ 39966552299520000S^2_3S^4_4 \notag \\
&- 10295681679360000S^3_4S^2_5 \notag \\
&- 7164267264000S^2_4S^2_5S_8 \notag \\
&- 19314408960000S^5_5S^3_7S_8 \notag \\
&- 109438560000S^3_5S^2_5S_8 \notag \\
&- 549920448000S^3_5S_8S_5 \notag \\
&+ 112482349056000S^3_5S_7 \notag \\
&- 48898080000S^3_5S^2_8 \notag \\
&+ 24367472640000S^5_4 \notag \\
&+ 339808896000S^6_5S_6 \notag \\
&+ 71705088000S^5_5S^2_8 + 86176S^3_5S^3_8 \notag \\
&- 29701056S^5_5S^3_8 - 56120S^3_5S^2_8 \notag \\
&+ 4396160S^3_5S_8S_5 - 267075S^5_6S_8 \notag \\
&- 1886760960S^5_5S_6S_7 \notag \\
&+ 628125S^4_8 \notag \\
&- 277200(8624000S^3_5S_7 \notag \\
&+ 252560S^3_5S_8 - 1397088000S^3_5S^2_8 \notag \\
&+ 7280000S^2_5 + 672000S^2_5S_6 \notag \\
&- 56160S^2_5S_8 \notag \\
&+ 876S^2_5S_8 - 276864S^2_5 - 455S^2_8S_6S_8}
\end{align*}
\]
\[ C_{0t} = -\left( (332640 \cdot 4 \cdot 45S_6S_4 \\
- 27720B_0S_4^2 - 92400D_1S_4 \\
- 5544S_6^2 + 20110S_3S_4^2)S_4 \\
- 385 \left( 112S_5 + S_6 \right) S_4 \\
- \left( 7120943360S_5^2 \\
+ 38913440S_4S_6 \\
- 10308880S_4S_8 + 60480S_6^2 \\
- 64408S_6S_8 - 769152S_7 \\
+ 5785S_8S_6^2 \right)S_4 \\
- 9 \left( 819624960S_2^2 \\
- 2299320960S_2S_3 \\
- 7096320S_2S_5 \\
+ 1626240S_2S_6 \\
+ 204906240S_3S_4 \\
+ 1478702080S_5^2 \right) \\
+ 9564160S_5S_6 \\
- 2372480S_6S_8 + 15360S_6^2 \\
- 10274S_6S_8 - 206976S_7 \\
+ 5775S_8S_6 \right) A_1 \\
+ 55440 (828S_6 - 65S_8) \\
+ 330832S_3S_7S_4S_6 \\
- 693 \left( 18627840S_3S_7^2 \\
+ 636160S_5^2 \\
+ 1120S_7S_8 + 5680S_8S_8 \\
+ 52S_7S_8 + 2688S_7 \\
- 75S_8^2 - 18480 (112S_5 + S_6) S_7 \right) S_8 \\
- 4435200 (48S_6 - 11S_8) \\
+ 14944S_5 - 11088S_2) C_0S_4^2 \\
+ 166320 (192S_6 - 11S_8) \\
+ 63472S_5 - 44352S_4) A_{1t}S_3^2 \right) \\
+ (245887488000S_4^3)^{-1} . \]

\[ S_{6tt} = -\left( (36960 \cdot 15 \cdot 3049200 (672 \cdot 10 \cdot D_{1t} \\
- 3H_2) - 3B_{1t}S_4) S_6^2 \\
- \left( A_{1t}S_4 + 7S_{1tt} \left( 112S_5 + S_6 \right) \right) S_4 \\
+ (86240A_{1t}S_7S_5 + 2387A_{1t}S_4S_8 \\
+ 1509200S_1S_5 \\
+ 5390S_6S_8 + 3326400S_2S_4^2 \\
- 3437680S_2S_6^2 - 14400S_2S_6 \\
+ 7604S_5S_6 \right) S_{6t} \\
+ 1848 \left( 72S_6 + 91S_8 + 104944S_5 \right) S_2S_4 \\
- 24 \left( 330129S_6 - 20551S_8 \\
+ 94386224S_5 \right) S_3S_4 \\
+ 231 \left( 2383360S_5^2 + 7840S_5S_6 \\
+ 21280S_5S_8 + 28S_6S_8 + 2688S_7 \\
+ 75S_8^2 - 18627840S_3S_7^2 \right) S_1 \right) \\
\]
\begin{align*}
&\left(-3544441600S_5^2 + 15097240S_3S_6ight) + 703120S_5S_6 \\
&+ 17280S_5^2 + 3970S_6S_8 \\
&+ 3094464S_7 + 13785S_8^2 \\
&- 34379099280S_3S_4^2 S_5S_4 \\
&- 22880 \left(2695S_1 - 7099S_4^2 \right) + 385A_1S_4 S_3S_5^2 \\
&- 2880 \left(2695S_1 - 3183S_4^2 - 539A_1S_4 \right) S_7 \\
&+ 5940 \left(896S_2 - 25S_6^2 \right) \\
&+ 14S_5S_6 + 2069760S_3S_4^2 C_9S_4 \\
&- 5544000 \left(24S_6 - 115S_8 ight) \\
&+ 6856S_7 - 5544S_2 S_5^2 \\
&+ \left(160 \left(462688S_6 - 4733737S_8 \right) S_6 ight) + 293259136S_7 + 5053935S_8^2 S_6 \\
&+ 13742136217600S_5^3 \\
&+ 15059819200S_5^2 S_6 \\
&- 63989446400S_5^2 S_8 \\
&+ 1382400S_5^3 - 828024S_5^3 S_8 \\
&+ 190854144S_5S_7 - 2697315S_6S_8^2 \\
&- 214158080S_5S_7 + 5466300S_5S_7S_4^2 \\
&- 385 \left(776160 \left(S_6S_8 - 192S_7 \right) + 280S_5S_6 \right) S_2 \\
&- \left(134502525440S_3S_5^2 S_5 \right) + 3259872000S_3S_5^2 S_6 \\
&+ 3520661760S_3S_5^2 S_8 \\
&- 4004448000S_3^3 - 183456000S_3^2 S_6 \\
&- 371804160S_3^2 S_8 - 431200S_3S_6^2 \\
&- 274176S_5S_6S_8 + 19726336S_3S_7 \\
&- 3265920S_5S_8^2 - 364S_5^2 S_8 \\
&+ 182784S_5S_7 - 441S_6S_8^2 \\
&+ 103936S_6S_8 - 11220S_6^2 \right) S_1 \\
&+ 1536796800 \left(77 \left(192 \left(10C_9 - 3S_2 \right) S_4^2 \right) + \left(112S_5 + S_8 \right) S_4 \right) \\
&+ \left(120S_6 + 17S_6 + 60272S_5 \right) S_4^2 B_9S_4 \\
&- 91476000 \left(3696 \left(5404 \left(4B_9 + S_3 \right) S_4^2 \right) \right) \left(112S_5 + S_8 \right) S_2 \\
&- \left(603904S_5^2 + 1792S_5S_6 \right) + 5392S_5S_6 - 26S_6S_8^2 \\
&- 2688S_7 + 75S_5^2 \right) A_0 S_4^2 \\
&+ 11 \left(11176704000 \left(24S_6 \right) - 11S_8 + 6856S_5 \\
&- 554S_5 \right) B_0 S_4^2 \\
&+ 21451820544000S_5S_3S_6S_8^2 \\
&+ 71624044800S_5S_6S_8 \\
&- 2067692400S_5S_6S_8 \\
&- 400444800S_5^2 \\
&+ 133593600S_5S_6 \\
&- 107251200S_5S_8 \\
&- 39200S_5S_6 + 49007280S_3S_6S_8 \\
&+ 3140157440S_3S_7 \\
&- 81867600S_5S_8^2 + 188188S_5^2 S_8 \\
&+ 12348672S_5S_7 - 476955S_6S_8 \\
&- 10622080S_5S_8 \\
&+ 249900S_5^3 - 55440 \left(847S_6S_8 ight) \\
&+ 5913S_7 - 1125S_8^2 + 7840S_5S_8 \\
&+ 27941760S_5^2 S_7 S_4 \left(A_1 S_4 \right) \left(553246848000S_4^2 \right)^{-1}, \\
&\left(A.23 \right)
\end{align*}
\[-1106493696000B_0S_2S_4^2 + 161423592000B_0S_2^2S_5 + 4790016000B_0S_2^4S_6 + 1106493696000D_0S_3^4 + 44259748000S_3S_5^3S_6^2 + 62092800S_5S_6^3 - 27720S_5S_6S_8 - 74511360S_2S_7 + 82601164800S_3^2S_7 - 2567980800S_3S_5^2S_6 - 119750400S_5^2S_6^3 + 572006400S_5^3S_7 - 19084800S_5^2S_6 - 15321600S_5^2S_6^2 + 30605920S_5S_6^3S_8 + 38760S_5S_6S_7 + 36305920S_5S_7S_8 + 512S_5^2S_6 + 49728S_6S_7 - 1935S_6^2 - 34880S_6S_8 + 2550S_6^3 \]

\[\cdot (1106493696000S_4^3)^{-1}, \quad (A.24)\]

\[S_{6xx} = \left(7 \left(33 \left(240 (108662400H_4 - 778447S_2S_4^2 + 2199120D_2S_4 - 1811040D_2x - 1164240B_2S_4 + 136171 (112S_5 + S_8) S_6^2 - 53760 (4851S_1 - 2188S_5^2) C_4S_4^2 \right) + 2 \left(4303031040B_2^2S_2 - 2283635200B_2^2S_5 - 18627840B_2^2S_6\right) + 20913200B_2^2S_8 - 1434343680C_0S_4^2 - 14343436800D_2S_4 + 1290909321S_4S_8 - 1052792048S_8 + 5588352S_1S_5 + 6852307S_5S_8 + 2189453112S_4S_5^2 - 30514805S_5S_6^2 + 2469168S_6S_7 - 4097730S_6^2S_7 \right) A_1S_4 \cdot (162993600S_2^4)^{-1}. \quad (A.25)\]

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References


