Research Article

Affine Discontinuous Galerkin Method Approximation of Second-Order Linear Elliptic Equations in Divergence Form with Right-Hand Side in L^1

Abdeluaab Lidouh and Rachid Messaoudi

Department of Mathematics and Computer Science, Laboratory LACSA, Faculty of Sciences, Mohammed Ist University, BV Mohammed VI, P.O. Box 717, 60000 Oujda, Morocco

Correspondence should be addressed to Rachid Messaoudi; m_rachid_ens@yahoo.fr

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We consider the standard **affine** discontinuous Galerkin method approximation of the second-order linear elliptic equation in divergence form with coefficients in $L^\infty(\Omega)$ and the right-hand side belongs to $L^1(\Omega)$; we extend the results where the case of **linear** finite elements approximation is considered. We prove that the unique solution of the discrete problem converges in $W_0^{1,q}(\Omega)$ for every q with $1 \leq q < d/(d-1)$ ($d = 2$ or $d = 3$) to the unique renormalized solution of the problem. Statements and proofs remain valid in our case, which permits obtaining a weaker result when the right-hand side is a bounded Radon measure and, when the coefficients are smooth, an error estimate in $W_0^{1,q}(\Omega)$ when the right-hand side f belongs to $L^r(\Omega)$ verifying $T_k(f) \in H^1(\Omega)$ for every $k > 0$, for some $r > 1$.

1. Introduction

In this work we consider, in dimension $d = 2$ or 3 , the $\mathbb{P}_1$ discontinuous Galerkin (dG) method approximation of the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where Ω is an open bounded set of $\mathbb{R}^d$, A is a coercive matrix with coefficients in $L^\infty(\Omega)$, and f belongs to $L^1(\Omega)$.

The solution of (1) does not belong to $H_0^1(\Omega)$ for a general right-hand side in $L^1(\Omega)$. Actually, in order to correctly define the solution of (1), one has to consider a specific framework, the concept of renormalized (or equivalently entropy) solution (see for example [1, 2]). These definitions allow one to prove that in this new sense problem (1) is well-posed in the terminology of Hadamard.

For this problem the standard $\mathbb{P}_1$ -nonconforming finite elements approximation, related to a triangulation $\mathcal{T}_h$ of Ω , namely,

$$\begin{aligned} \text{Find } u_h &\in V_h, \\ \forall v_h &\in V_h, \end{aligned} \quad (2)$$

$$a_h^{swip}(u_h, v_h) = \int_{\Omega} f v_h dx,$$

where

$$\begin{aligned} V_h &= \left\{ v_h \in L^2(\Omega) : \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_1[T], \forall F \right. \\ &\quad \left. \in \mathcal{F}_h, \int_F [v_h] = 0 \right\} \end{aligned} \quad (3)$$

with the discrete bilinear form a_h^{swip} yet to be designed, has a unique solution, since the right-hand side (2) $\int_{\Omega} f v_h dx$ is

correctly defined for $f \in L^1(\Omega)$ and the bilinear form a_h^{swip} is consistent.

Using the ideas which are at the root of the SWIP (Symmetric Weighted Interior Penalty) method, in the case $f \in L^2(\Omega)$, D. A. Di Pietro and A. Ern have proved, in [3], that the unique solution u_h of (2) converges to the unique solution u of (1) in the following sense:

$$\begin{aligned} u_h &\longrightarrow u \quad \text{strongly in } L^2(\Omega), \\ \nabla_h u_h &\longrightarrow \nabla u \quad \text{strongly in } [L^2(\Omega)]^d, \\ |u_h|_J &\longrightarrow 0, \end{aligned} \quad (4)$$

when $h \longrightarrow 0$,

with the broken gradient ∇_h and the jump seminorm $|\cdot|_J$ yet to be designed.

To do that, the authors in [3] assume that the family of triangulations $\mathcal{T}_h$ belong to an admissible mesh sequence in the sense of [17] and is compatible with the partition $\mathcal{P}_\Omega$ (see Assumption 3).

The framework in this paper is the same as in [3]. The unique difference here is that $f \in L^1(\Omega)$ is considered instead of $f \in L^2(\Omega)$; and we ourselves focus on the two cases $d = 2$ and $d = 3$. The same convergence results are proved.

Notations. In the present work, Ω denotes an open bounded subset of $\mathbb{R}^d$ with $d = 2$ or $d = 3$. A particular case is the case where Ω is an open bounded polyhedron. We use the notation $A \cdot w$ for the scalar product of the vector A by the vector w (which is often denoted by ${}^t w \cdot A$). For a measurable set $S \subset \Omega$, we denote by $|S|$ the measure of S and by S^c the complement $\Omega \setminus S$ of S .

For $1 < p < +\infty$ and $m \geq 0$, we have

$$\begin{aligned} H^m(\mathcal{T}_h) &:= \{v \in L^2(\Omega) / \forall T \in \mathcal{T}_h, v|_T \in H^m(T)\}, \\ W^{m,p}(\mathcal{T}_h) &:= \{v \in L^p(\Omega) / \forall T \in \mathcal{T}_h, v|_T \in W^{m,p}(T)\}. \end{aligned} \quad (5)$$

We define also the following function spaces:

$$\begin{aligned} H(\text{div}, \Omega) &:= \left\{ \tau \in [L^2(\Omega)]^d / \nabla \cdot \tau \in L^2(\Omega) \right\}, \\ H(\text{div}, \mathcal{T}_h) &:= \left\{ \tau \in [L^2(\Omega)]^d / \forall T \in \mathcal{T}_h, \tau|_T \in H(\text{div}, T) \right\}. \end{aligned} \quad (6)$$

For $k \geq 0$, we define the broken polynomial space

$$\mathbb{P}_d^k(\mathcal{T}_h) := \{v \in L^2(\Omega) / \forall T \in \mathcal{T}_h, v|_T \in \mathbb{P}_d^k(T)\}, \quad (7)$$

with polynomial degree $k \geq 1$.

In that case, $\mathbb{P}_d^1(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$, which leads us to define the broken gradient $\nabla_h : H^1(\mathcal{T}_h) \rightarrow [L^2(\Omega)]^d$ such that, $\forall v \in H^1(\mathcal{T}_h)$,

$$\forall T \in \mathcal{T}_h: \quad (\nabla_h v)|_T := \nabla(v|_T), \quad (8)$$

and the broken divergence operator $\nabla_h : H(\text{div}, \mathcal{T}_h) \rightarrow [L^2(\Omega)]^d$ such that, $\forall v \in H(\text{div}, \mathcal{T}_h)$,

$$\forall T \in \mathcal{T}_h: \quad (\nabla_h \cdot v)|_T := \nabla \cdot (v|_T). \quad (9)$$

Moreover, for any mesh element $T \in \mathcal{T}_h$, we denote

$$\begin{aligned} \forall T \in \mathcal{T}_h, \quad \mathcal{F}_T &:= \{F \in \mathcal{F}_h / F \in \partial T\}, \\ \forall F \in \mathcal{F}_h, \quad \mathcal{T}_F &:= \{T \in \mathcal{T}_h / F \in \partial T\}. \end{aligned} \quad (10)$$

And for a scalar-valued function v defined on Ω (which can admit two possible traces) the average of v is defined as

$$\begin{aligned} \{v\}_F(x) &:= \frac{1}{2} \left(v|_{T_1}(x) + v|_{T_2}(x) \right); \\ &\text{if } F \in \mathcal{F}_h^i \text{ such that } F \subset \partial T_1 \cap \partial T_2, \\ \{v\}_F(x) &:= v|_T(x), \\ &\text{if } F \in \mathcal{F}_h^b \text{ such that } F \subset \partial T \cap \partial \Omega, \end{aligned} \quad (11)$$

and the jump of v as

$$\begin{aligned} [v]_F(x) &:= v|_{T_1}(x) - v|_{T_2}(x), \\ &\text{if } F \in \mathcal{F}_h^i \text{ such that } F \subset \partial T_1 \cap \partial T_2, \\ [v]_F(x) &:= v|_T(x), \\ &\text{if } F \in \mathcal{F}_h^b \text{ such that } F \subset \partial T \cap \partial \Omega. \end{aligned} \quad (12)$$

For any face $F \in \mathcal{F}_h$ and for any integer $l \geq 0$, we define the (local) lifting operator $r_{F,A}^l : L^2(F) \rightarrow \mathbb{P}_d^l(\mathcal{T}_h)$ as follows. For all $\varphi \in L^2(F)$,

$$\begin{aligned} \int_\Omega A r_{F,A}^l(\varphi) \cdot \tau_h &= \int_F \{A \tau_h\} \cdot n_F \varphi \\ \forall \tau_h &\in [\mathbb{P}_d^l(\mathcal{T}_h)]^d, \end{aligned} \quad (13)$$

and for any function $v \in H^1(\mathcal{T}_h)$, we define the (global) lifting of its interface and boundary jumps as

$$R_{h,A}^l([v]) = \sum_{F \in \mathcal{F}_h} r_{F,A}^l([v]) \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d. \quad (14)$$

We also introduce the normal diffusion coefficient to one face F as

$$\begin{aligned} \forall F \in \mathcal{F}_h, \quad F &= \partial T_1 \cap \partial T_2: \\ A_i &:= (A|_{T_i} n_F) \cdot n_F, \quad i \in \{1, 2\}, \end{aligned} \quad (15)$$

the diffusion-dependent penalty parameter (harmonic average of normal diffusion) as

$$\gamma_{A,F} := \begin{cases} \frac{2A_1 A_2}{A_1 + A_2} & \text{if } F \subset \partial T_1 \cap \partial T_2, \\ (A|_T n) \cdot n & \text{if } F \subset \partial T \cap \partial \Omega, \end{cases} \quad (16)$$

the weighted average operator for all $F \in \mathcal{F}_h^i$ such that $F \subset \partial T_1 \cap \partial T_2$ as

$$\begin{aligned}\omega_{T_1,F} &:= \frac{A_2}{A_1 + A_2}, \\ \omega_{T_2,F} &:= \frac{A_1}{A_1 + A_2},\end{aligned}\quad (17)$$

the weighted average operator for all $F \in \mathcal{F}_h^i$ and for a.e. $x \in F$ as

$$\{v\}_{\omega,F}(x) := \omega_{T_1,F} v|_{T_1}(x) + \omega_{T_2,F} v|_{T_2}(x), \quad (18)$$

on boundary faces $F \in \mathcal{F}_h^b$ such that $F \subset \partial T \cap \partial \Omega$, we set

$$\{v\}_{\omega,F}(x) := v|_T(x), \quad (19)$$

and the skew-weighted average operator for all $F \in \mathcal{F}_h^i$ and for a.e. $x \in F$, as

$$\{v\}_{\bar{\omega},F}(x) := \omega_{T_2,F} v|_{T_1}(x) + \omega_{T_1,F} v|_{T_2}(x). \quad (20)$$

The SWIP bilinear form is defined by (see Lemma 4.47 in [3])

$$\begin{aligned}a_h^{swip}(v, w) &:= \underbrace{\int_{\Omega} A \nabla_h v \nabla_h w \, dx}_{\text{consistency term}} + \underbrace{\sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_{A,F}}{h_F} \int_F [v, w]}_{\text{penalty term}} \\ &\quad - \underbrace{\sum_{F \in \mathcal{F}_h} \int_F (\{A \nabla_h v\}_{\bar{\omega}} \cdot n_F [w] + \{A \nabla_h w\}_{\omega} \cdot n_F [v])}_{\text{symmetry term}},\end{aligned}\quad (21)$$

where the quantity $\eta > 0$ denotes a user-dependent penalty parameter which is independent of the diffusion coefficient.

And the SWIP norms are defined by

$$\begin{aligned}\|v\|_{swip} &:= \left(\|A^{1/2} \nabla_h v\|_{[L^2(\Omega)]^d}^2 + |v|_{A,J}^2 \right)^{1/2}, \\ \forall q \geq 1:\end{aligned}\quad (22)$$

$$\|v\|_{swip,q} := \left(\|A^{1/2} \nabla_h v\|_{[L^q(\Omega)]^d}^q + |v|_{J,A,q}^q \right)^{1/q},$$

with the diffusion-dependent jump seminorms

$$\begin{aligned}|v|_{J,A}^2 &:= \sum_{F \in \mathcal{F}_h} \frac{\gamma_{A,F}}{h_F} \| [v] \|_{L^2(F)}^2, \\ \forall q \geq 1: \quad |v|_{J,A,q}^q &:= \sum_{F \in \mathcal{F}_h} \frac{\gamma_{A,F}}{h_F^{q-1}} \| [v] \|_{L^q(F)}^q.\end{aligned}\quad (23)$$

The discrete Galerkin norm is defined by

$$\|v\|_{dG} := \left(\|\nabla_h v\|_{[L^2(\Omega)]^d}^2 + |v|_J^2 \right)^{1/2}, \quad (24)$$

with the jump seminorm

$$|v|_J^2 := \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \| [v] \|_{L^2(F)}^2. \quad (25)$$

For every r with $1 < r < +\infty$, we denote by $L^{r,\infty}(\Omega)$ the Marcinkiewicz space whose norm is defined by

$$\|v\|_{L^{r,\infty}(\Omega)} = \sup_{\lambda > 0} \lambda |\{x \in \Omega : |v(x)| \geq \lambda\}|^{1/r}. \quad (26)$$

For every real number $k > 0$ we define the truncation $T_k : \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_k(s) = \max(-k, \min(k, s)) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| \geq k. \end{cases} \quad (27)$$

For every d -simplex T in $\mathbb{R}^d$, we adopt the following notations:

- (i) $a_{i,T}$, $i = 0, \dots, d$, denote the vertices of T .
- (ii) $m_{i,T}$, $i = 0, \dots, d$, denote the centers of the faces $F_i \in T$.
- (iii) $\lambda_{i,T}$, $i = 0, \dots, d$, designate the barycentric coordinates with respect to the $a_{i,T}$'s.
- (iv) for every $x \in \mathbb{R}^d$ we put

$$\varphi_{i,T}(x) := 1 - d\lambda_{i,T}(x) \quad \text{for } i = 0, \dots, d, \quad (28)$$

where $(\varphi_{i,T})_{0 \leq i \leq d}$ are the $\mathbb{P}_1$ shape functions related to T ; it is known that

$$\begin{aligned}\varphi_i &\in L^2(\Omega_h), \\ \varphi_i|_T &\in \mathbb{P}_1(T), \\ \forall i &\in \{0, 1, \dots, d\}, \quad \forall T \in \mathcal{T}_h,\end{aligned}$$

$$\begin{aligned}\varphi_i|_{F_i} &= 1, \\ \varphi_i(a_i) &= 1 - d,\end{aligned}\quad (29)$$

$$\varphi_i(a_j) = 1 \text{ such that } j \neq i,$$

$$\begin{aligned}\frac{1}{|F_j|} \int_{F_j} \varphi_i &= \varphi_i(m_j) = \delta_{i,j}, \\ \sum_{i=0}^d \varphi_{i,T}(x) &= 1, \quad \text{for every } x \in \mathbb{R}^d,\end{aligned}$$

$$\text{with } \Omega_h := \cup\{T, T \in \mathcal{T}_h\} \subset \bar{\Omega}.$$

- (v) If N designate the number of all interior centers m_i of faces F in $\mathcal{T}_h$ we define the interpolation operator Π_h and the truncated interpolation operator I_h^k by

$$\begin{aligned} \forall v \in L^2(\overline{\Omega}) \text{ avec } \int_F [v_h] &= 0, \\ \Pi_h(v) &\in V_h, \\ \Pi_h(v) &:= \sum_{1 \leq i \leq N} \alpha_i^v \varphi_i, \\ I_h^k(v) &:= \sum_{1 \leq i \leq N} T_k(\alpha_i^v) \varphi_i, \end{aligned} \quad (30)$$

with $\alpha_i^v = \Pi_h(v)(m_i) = (1/|F_i|) \int_{F_i} \Pi_h(v) = (1/|F_i|) \int_{F_i} v$.

- (vi) Finally, we define the $N \times N$ stiffness matrix $Q = (Q_{ij})$; namely,

$$Q_{i,j} = a_h^{swip}(\varphi_i, \varphi_j) \quad \text{for } i, j \text{ in } \{1, 2, \dots, N\}. \quad (31)$$

As in [4], the main assumption of the present paper is that Q is a diagonally dominant matrix; namely,

$$\forall i \in \{1, 2, \dots, N\}: \quad Q_{ii} - \sum_{\substack{1 \leq j \leq N \\ j \neq i}} |Q_{ij}| \geq 0. \quad (32)$$

2. Statement of the Main Result

We consider a matrix A such that

$$A \in L^\infty(\Omega)^{d \times d}, \quad (33)$$

$$\text{a.e } x \in \Omega: \forall \xi \in \mathbb{R}^d: A(x) \xi \xi \geq \alpha |\xi|^2, \quad (34)$$

for some $\alpha > 0$, and a right-hand side f such that

$$f \in L^1(\Omega). \quad (35)$$

A function u is the renormalized solution of the problem

- (1) if u satisfies

$$u \in L^1(\Omega), \quad (36)$$

$$\forall k > 0, \quad T_k(u) \in H_0^1(\Omega), \quad (37)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\Omega} |\nabla T_k(u)|^2 dx = 0, \quad (38)$$

$$\forall k > 0, \quad \forall S \in \mathcal{C}_c^1(\mathbb{R}) \text{ with } \text{supp}(S) \subset [-k, k],$$

$$\forall v \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

$$\begin{aligned} &\int_{\Omega} A \nabla T_k(u) \nabla v S(u) dx \\ &+ \int_{\Omega} A \nabla T_k(u) \nabla T_k(u) S'(u) v dx \\ &= \int_{\Omega} f S(u) v dx. \end{aligned} \quad (39)$$

It is known (see [1, 5]) that when f belongs to $L^1(\Omega) \cap H^{-1}(\Omega)$, the usual weak solution of (1), namely,

$$\begin{aligned} u &\in H_0^1(\Omega), \\ \forall v &\in H_0^1(\Omega), \end{aligned} \quad (40)$$

$$\int_{\Omega} A \nabla v \nabla u dx = \int_{\Omega} f v dx,$$

is a renormalized solution of (1) and conversely the main interest of definition of renormalized solution is the following existence, uniqueness, and continuity theorem (see [1, 4]).

Theorem 1. Assume that A and f satisfy (33), (34), and (35). Then there exists a renormalized solution of (1). This solution is unique. Moreover this unique solution belongs to $W_0^{1,q}(\Omega)$ for every q with $1 \leq q < d/(d-1)$. It depends continuously on the right-hand side f in the following sense: if f^ε is a sequence which satisfies

$$f^\varepsilon \longrightarrow f \quad \text{strongly in } L^1(\Omega), \quad (41)$$

when ε tends to zero, then the sequence u^ε of the renormalized solutions of (1) for the right-hand sides f^ε satisfies for every $k > 0$ and for every q with $1 \leq q < d/(d-1)$

$$T_k(u^\varepsilon) \longrightarrow T_k(u) \quad \text{strongly in } H_0^1(\Omega), \quad (42)$$

$$u^\varepsilon \longrightarrow u \quad \text{strongly in } W_0^{1,q}(\Omega),$$

when ε tends to zero, where u is the renormalized solution of (1) for the right-hand side f . Finally, if f_1 and f_2 belong to $L^1(\Omega)$, and if u_1 and u_2 are the renormalized solutions of (1) for the right-hand sides f_1 and f_2 , then, for every $k > 0$, the function $T_k(u_1 - u_2)$ belongs to $H_0^1(\Omega)$ and for every q with $1 \leq q < d/(d-1)$ one has

$$\begin{aligned} \alpha \|T_k(u_1 - u_2)\|_{H_0^1(\Omega)}^2 &\leq k \|f_1 - f_2\|_{L^1(\Omega)}, \\ \alpha \|u_1 - u_2\|_{W_0^{1,q}(\Omega)} &\leq C(d, |\Omega|, q) \|f_1 - f_2\|_{L^1(\Omega)}, \end{aligned} \quad (43)$$

Where the constant $C(d, |\Omega|, q)$ only depends on d , $|\Omega|$, and q .

Remark 2. Throughout all this paper, we denote by $C(p_1, p_2, p_3, \dots)$ any real constant which only depends on the parameters p_1 , p_2 , and $p_3 \dots$. We can use the same notation for different constants.

Now we consider a family of triangulations $\mathcal{T}_h$ satisfying for each $h > 0$ the following assumption:

the triangulation $\mathcal{T}_h$ is made of a finite number of closed d -simplices T (namely triangles when $d = 2$ and tetrahedra

when $d = 3$) such that :

- (i) $\Omega_h = \bigcup \{T : T \in \mathcal{T}_h\} \subset \bar{\Omega}$,
- (ii) for every compact set K with $K \subset \Omega$, there exists $h_0(K) > 0$ such that $K \subset \Omega_h$ for every h with $h < h_0(K)$,
- (iii) for $(T_1, T_2) \in \mathcal{T}_h^2$ with $T_1 \neq T_2$, one has $|T_1 \cap T_2| = 0$,
- (iv) every face of every T of $\mathcal{T}_h$ is either a subset of $\partial\Omega_h$, or a face of another T' of $\mathcal{T}_h$.

Note that because of (iv) the triangulations are conforming. A particular case is the case where Ω is a polyhedron of $\mathbb{R}^d$, and where Ω_h coincides with Ω for every h .

In practice, the diffusion coefficient (i.e., matrix A) has more regularity than just belonging to $L^\infty(\Omega)$. Henceforth, we make the following assumption (assumption 4.43 [3]):

there is a partition P_Ω

$:= \{\Omega_i\}_{1 \leq i \leq N_\Omega}$ of Ω such that :

- (i) Each Ω_i , $1 \leq i \leq N_\Omega$, is apolyhedron;
- (ii) The restriction of A to each Ω_i , $1 \leq i \leq N_\Omega$, is constant.

An important assumption on the mesh sequence $\mathcal{T}_h := (\mathcal{T}_h)_{h \in \mathcal{H}}$ is its compatibility with the partition P_Ω in the following sense (assumption 4.45 [3]).

Assumption 3 (mesh compatibility). We suppose that the admissible mesh sequence $\mathcal{T}_h$ is such that, for each $h \in \mathcal{H}$, each $T \in \mathcal{T}_h$ is a subset of only one set Ω_i of the partition P_Ω . In this situation, the meshes are said to be compatible with the partition P_Ω .

For every $T \in \mathcal{T}_h$, we denote by h_T the diameter of T and by ρ_T the diameter of the ball inscribed in T . We set

$$h = \sup_{T \in \mathcal{T}_h} h_T \quad (46)$$

and we assume that h tends to zero.

We also assume that the family of triangulations $\mathcal{T}_h$ is regular in the sense of P. G. Ciarlet [6]; namely, there exists a constant σ such that

$$\forall h, \forall T \in \mathcal{T}_h, \frac{h_T}{\rho_T} \leq \sigma. \quad (47)$$

For every triangulation $\mathcal{T}_h$, we consider the discrete problem:

$$\begin{aligned} \text{Find } u_h &\in V_h, \\ \forall v_h &\in V_h, \end{aligned} \quad (48)$$

$$a_h^{swip}(u_h, v_h) = \int_{\Omega} f v_h dx.$$

Note that the right-hand side of (48) makes sense since $f \in L^1(\Omega)$ and $V_h \subset L^\infty(\Omega)$. The discrete bilinear forme a_h^{swip} is consistent and coercive (see (128)) on V_h , so a straightforward consequence of the Lax-Milgram Lemma is that the discrete problem (48) is well-posed. The solution u_h of (48) exists and is unique.

As in [4], the main result of this paper is the following.

Theorem 4. Assume that A , f , and $\mathcal{T}_h$ satisfy (33), (34), (35), (44), (46), (47), and (32). Then the unique solution u_h of (48) satisfies for every $k > 0$ and for every q with $1 \leq q < d/(d-1)$

$$\begin{aligned} u_h &\longrightarrow u \quad \text{strongly in } L^q(\Omega), \\ \nabla_h u_h &\longrightarrow \nabla u \quad \text{strongly in } [L^q(\Omega)]^d, \\ |u_h|_{J,A,q} &\longrightarrow 0, \\ I_h^k(u_h) &\longrightarrow T_k(u) \quad \text{strongly in } L^2(\Omega), \\ \nabla_h I_h^k(u_h) &\longrightarrow \nabla T_k(u) \quad \text{strongly in } [L^2(\Omega)]^d, \\ |I_h^k(u_h)|_{J,A} &\longrightarrow 0, \end{aligned} \quad (49)$$

when h tends to zero, where u is the unique renormalized solution of (1).

This theorem will be proved in Section 4, using the tools that we will prepare in Section 3. In Section 5, we will explain why the results of [4] when f is a bounded Radon measure remain valid in our case. In Section 6 we also show that if we assume in addition that $T_k(f) \in H^1(\Omega)$ for every $k > 0$, we obtain for smooth solutions an $O(h^{4(1-1/r)})$ error estimate in $\|\cdot\|_{swip,q}$ -norm (Section 6.1), and for Low-Regularity solutions an $O(h^{4\alpha_p(1-1/r)})$ error estimate in $\|\cdot\|_{swip,q}$ -norm (Section 6.2). Finally, in Section 7 we show that in the case where A is the identity matrix, condition (32) remains satisfied when every inner angle of every d -simplex of $\mathcal{T}_h$ is acute.

3. Tools

We are going to prove Theorem 4 in several steps. We begin by proving the following result which is a piecewise $\mathbb{P}_1$ variant of a result of L. Boccardo & T. Gallouët [2, 5].

Theorem 5. Assume that $v_h \in V_h$ satisfies

$$\forall k > 0, \int_{\Omega} |\nabla_h I_h^k(v_h)|^2 dx \leq kM \quad (50)$$

for some $M > 0$. Then, for every q with $1 \leq q < d/(d-1)$,

$$\|v_h\|_{\text{swip},q} \leq C_2(d, |\Omega|, q, \sigma, \|A\|_{L^\infty(\Omega)^{d \times d}}) M \quad (51)$$

where the constant $C_2(d, |\Omega|, q)$ only depends on d , $|\Omega|$, and q .

As in [4], to prove Theorem 5, we use the following lemmas.

Lemma 6. Under assumption (47), for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, one has

$$h_F |F| \leq 2\sigma^d |T| \quad (52)$$

Proof. Indeed, let $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, so

$$h_F |F| \leq h_T^d \quad (53)$$

and by (47) one has

$$h_F |F| \leq \sigma^d \rho_T^d \quad (54)$$

which combined with the fact that

$$|T| \geq c\rho_T^d, \quad (55)$$

where $c = \Pi/4$ in 2D, and $c = \Pi/6$ in 3D, implies (52). $\square$

Lemma 7. Under assumption (47), for every q such that $1 \leq q$, the following bound holds for any $v_h \in V_h$:

$$|v|_{J,A,q}^q \leq C(q, \sigma, d, \|A\|_{L^\infty(\Omega)^{d \times d}}) \|\nabla_h v\|_{L^q(\Omega)}^q. \quad (56)$$

Proof. For every $T \in \mathcal{T}_h$, we denote by $(\nabla_h v)_T$ the (constant) gradient of the restriction of v to T . With this notation, using the continuity of v across any F in $\mathcal{F}_T$ at the mass center x_F of any internal F , the fact that v vanishes at the mass center x_F of any external F , and the known inequality

$$\forall q \geq 1, \quad (|a| + |b|)^q \leq 2^{q-1} (|a|^q + |b|^q) \quad (57)$$

and using (52) we get

$$\begin{aligned} |v|_{J,A,q}^q &= \sum_{F \in \mathcal{F}_h^i} \frac{\gamma_{A,F}}{h_F^{q-1}} \int_F [v]^q dx + \sum_{F \in \mathcal{F}_h^b} \frac{\gamma_{A,F}}{h_F^{q-1}} \int_F [v]^q dx \\ &\leq \sum_{\substack{F \in \mathcal{F}_h^i \\ F=T_1 \cap T_2}} \frac{\gamma_{A,F}}{h_F^{q-1}} \\ &\quad \cdot \int_F \left(((\nabla_h v)_{T_1} - (\nabla_h v)_{T_2}) \cdot (x - x_F) \right)^q dx \\ &\quad + \sum_{\substack{F \in \mathcal{F}_h^b \\ F \in \mathcal{F}_T}} \int_F \frac{\gamma_{A,F}}{h_F^{q-1}} ((\nabla_h v)_T \cdot (x - x_F))^q dx \\ &\leq \sum_{\substack{F \in \mathcal{F}_h^i \\ F=T_1 \cap T_2}} 2^q \min(\lambda_{T_1,F}, \lambda_{T_2,F}) h_F |F| \\ &\quad \cdot (|(\nabla_h v)_{T_1}|^q + |(\nabla_h v)_{T_2}|^q) + \sum_{\substack{F \in \mathcal{F}_h^b \\ F \in \mathcal{F}_T}} \lambda_{T,F} h_F |F| \\ &\quad \cdot |(\nabla_h v)_T|^q \leq 2\sigma^d \|A\|_{L^\infty(\Omega)^{d \times d}} \sum_{\substack{F \in \mathcal{F}_h^i \\ F=T_1 \cap T_2}} 2^q \\ &\quad \cdot (|T_1| |(\nabla_h v)_{T_1}|^q + |T_2| |(\nabla_h v)_{T_2}|^q) \\ &\quad + 2\sigma^d \|A\|_{L^\infty(\Omega)^{d \times d}} \sum_{\substack{F \in \mathcal{F}_h^b \\ F \in \mathcal{F}_T}} |T| |(\nabla_h v)_T|^q \\ &\leq 2^{q+3} \sigma^d \|A\|_{L^\infty(\Omega)} \sum_{T \in \mathcal{T}_h} |T| |(\nabla_h v)_T|^q, \end{aligned} \quad (58)$$

which is (56) with $C(q, \sigma, d, \|A\|_{L^\infty(\Omega)^{d \times d}}) = 2^{q+3} \sigma^d \|A\|_{L^\infty(\Omega)^{d \times d}}$. $\square$

Lemma 8. Let $v_h \in V_h$ and let $k > 0$. If for some $T \in \mathcal{T}_h$ there exists $y \in T$ with $|v_h(y)| \geq k$, then there exists a d -simplex $S \subset T$ with $|S| = c(d)|T|$ such that

$$\forall x \in S: \quad \left| I_h^k(v_h(x)) \right| \geq \frac{k}{2}, \quad (59)$$

and the strictly positive constant $c(d)$ only depends on d .

Proof. Let T be a d -simplex from the triangulation $\mathcal{T}_h$, $v_h \in V_h$, and $k > 0$, such that $\sup_T |v_h| \geq k$. Consider $w_h = I_h^k(v_h) \in \mathbb{P}_1(T)$.

(i) If $\max_{m_i \in T} |v_h| < k$, thus

$$\forall i \in \{0, 1, \dots, d\}, \quad (60)$$

$$T_k(\alpha_i^{v_h}) = T_k(v_h(m_i)) = v_h(m_i) = \alpha_i^{v_h}$$

so $w_h = v_h$, and $\exists y \in T$ such that $|w_h(y)| \geq k$.

(ii) If $\max_{m_i \in T} |v_h| \geq k$, thus $\exists m_i \in T$, such that $|v_h(m_i)| \geq k$

so $|T_k(v_h(m_i))| = k$, and $|w_h(m_i)| = T_k(\alpha_i^{v_h}) = k$.

In both cases, there exists an element y in T such that $|w_h(y)| \geq k$.

But $w_h \in \mathbb{P}_1(T)$, so

$$w_h(y) - w_h(x) = \nabla w_h(y - x), \quad (61)$$

In other words

$$\forall i \neq j \neq k \text{ dans } \{0, 1, \dots, d\},$$

$$\nabla \lambda_i(a_i - a_k) = 1, \quad (62)$$

$$\nabla \lambda_i(a_j - a_k) = 0.$$

and since

$$y - x = \sum_{k=1}^d (\lambda_k(y) - \lambda_k(x)) (a_k - a_0), \quad (63)$$

one obtains

$$\nabla w_h = -d \sum_{i=0}^d T_k(\alpha_i^{v_h}) \nabla \lambda_i, \quad (64)$$

$$\nabla w_h(a_k - a_0) = -d (T_k(\alpha_k^{v_h}) - T_k(\alpha_0^{v_h})),$$

so

$$|\nabla w_h(a_k - a_0)| \leq 2kd, \quad (65)$$

and

$$|w_h(y) - w_h(x)| \leq 2d \sum_{k=1}^d |\lambda_k(y) - \lambda_k(x)| \leq \frac{k}{2}, \quad (66)$$

as soon as

$$\sum_{k=1}^d |\lambda_k(y) - \lambda_k(x)| \leq \frac{1}{4d}. \quad (67)$$

For this purpose we define the d -simplex $S = (y - a_0) + S_0$ such that

$$S_0 = \left\{ x, \lambda_0 \geq 1 - \frac{1}{4d} = \frac{4d-1}{4d} \right\}, \quad (68)$$

so

$$|w_h(x)| \geq |w_h(y)| - |w_h(y) - w_h(x)| \geq k - \frac{k}{2} = \frac{k}{2}, \quad (69)$$

and to estimate the measure of S , it is clear to verify first that $|S| = |S_0|$. Let $\widehat{T}$ be the reference unit d -simplex with vertices $\widehat{a}_0 = 0, \widehat{a}_1 = e_1, \dots, \widehat{a}_d = e_d$, where $\{e_i, i = 1, \dots, d\}$ is the canonical basis of $\mathbb{R}^d$. Let F_T be the invertible affine mapping that maps $F_T(\widehat{T}) = T$ onto T and set $\widehat{S} = F_T^{-1}(S)$.

Since F_T is affine, it is easy to check that $\widehat{\lambda}_i = F_T^{-1} \circ \lambda_{i,T}$ for $i = 0, 1, \dots, d$ are the barycentric coordinates with respect to the $\widehat{a}_i$'s and that

$$\widehat{S} = \left\{ \widehat{x} \in \widehat{T} : \widehat{\lambda}_0(\widehat{x}) \geq \frac{4d-1}{4d} \right\}, \quad (70)$$

$$|S| = \frac{|\widehat{S}|}{|\widehat{T}|} |T| = C(d) |T|,$$

where $c(d) = |\widehat{S}|/|\widehat{T}|$ is a constant that depends only on d . This proves the result. $\square$

Lemma 9. Assume that $v_h \in V_h$ satisfies (50), then

$$|B_k(v_h)| \leq C(d, |\Omega|, 2^*) \left(\frac{M}{k} \right)^{2^*/2}, \quad (71)$$

for every $k > 0$, where $2^* = 2d/(d-2) = 6$ if $d = 3$ and 2^* is any real number with $2^* \geq 1$ if $d = 2$; $B_k(v_h)$ is defined by

$$B_k(v_h) = \bigcup \left\{ T \in \mathcal{T}_h : \max_T |v_h| \geq k \right\}, \quad (72)$$

and $C(d, |\Omega|, 2^*)$ is a constant depending only on $d, 2^*, \theta_0$, and $|\Omega|$.

Proof. Discrete Sobolev's theorem (see theorem 5.3 in [3]) asserts that

$$\forall v \in V_h, \quad \|v\|_{L^{2^*}(\Omega)} \leq \sigma_{2,2^*}(d, |\Omega|) \|v\|_{dG}, \quad (73)$$

and we will also need (56) with $q = 2$

$$\forall v \in V_h, \quad |v|_T^2 \leq C(\sigma, d) \|\nabla_h v\|_{L^2(\Omega)}. \quad (74)$$

Here $d = 2$ or $d = 3$ so $2^* = 2d/(d-2) = 6$ if $d = 3$ and 2^* can be any real number with $1 \leq 2^*$.

Fix $k > 0$. If $T \subset B_k(v_h)$, from Lemma 8, we know that there exists $S \subset T$, with $|S| = c(d)|T|$ and

$$\forall x \in S, \quad |I_h^k(v_h(x))| \geq \frac{k}{2}. \quad (75)$$

Therefore

$$\begin{aligned} \int_T |I_h^k(v_h(x))|^{2^*} dx &\geq \int_S |I_h^k(v_h(x))|^{2^*} dx \\ &\geq \left(\frac{k}{2} \right)^{2^*} |S| = c(d) |T| \left(\frac{k}{2} \right)^{2^*}. \end{aligned} \quad (76)$$

Hence

$$\begin{aligned} |B_k(v_h)| &= \sum_{T \subset B_k(v_h)} |T| \\ &\leq \sum_{T \subset B_k(v_h)} \frac{1}{c(d)} \left(\frac{2}{k} \right)^{2^*} \int_T |I_h^k(v_h(x))|^{2^*} dx \\ &\leq \frac{1}{c(d)} \left(\frac{2}{k} \right)^{2^*} \int_\Omega |I_h^k(v_h(x))|^{2^*} dx. \end{aligned} \quad (77)$$

Combining with (73) and (74) one has

$$|B_k(v_h)| \leq \frac{(2\sigma_{2,2^*}(d, |\Omega|))^{2^*} (1 + C(\sigma, d))^{2^*/2}}{c(d) k^{2^*}} \left(\int_{\Omega} |\nabla_h I_h^k(v_h(x))|^2 dx \right)^{2^*/2}. \quad (78)$$

Finally using (50), we obtain

$$\begin{aligned} |B_k(v_h)| &\leq \frac{(2\sigma_{2,2^*}(d, |\Omega|))^{2^*} (1 + C(\sigma, d))^{2^*/2}}{c(d)} \left(\frac{M}{k} \right)^{2^*/2}, \end{aligned} \quad (79)$$

which is (71) with

$$\begin{aligned} C(d, |\Omega|, 2^*, \sigma, \|A\|_{L^\infty(\Omega)^{d \times d}}) &= \frac{(2\sigma_{2,2^*}(d, |\Omega|))^{2^*}}{C(d)} (1 + C(\sigma, d))^{2^*/2}. \end{aligned} \quad (80)$$

□

Proof (proof of Theorem 5; see [4]). Fix q with $1 \leq q < d/(d-1)$. Take $r = (2 \times 2^*)/(2 + 2^*) = 3/2$ in the case $d = 3$, and verify $(2 \times 2^*)/(2 + 2^*) > q$, in the case $d = 2$.

The embedding inequality $(L^q(\Omega) \hookrightarrow L^{r,\infty}(\Omega))$ writes

$$\|\nabla_h v_h\|_{L^q(\Omega)} \leq C(q, r, |\Omega|) \|\nabla_h v_h\|_{L^{r,\infty}(\Omega)}, \quad (81)$$

and by (56) one can have

$$\begin{aligned} \forall v \in W_0^{1,q}(\mathcal{T}_h), \\ |v|_{J,A,q}^q \leq C(q, \sigma, d, \|A\|_{L^\infty(\Omega)^{d \times d}}) \|\nabla_h v\|_{L^q(\Omega)}. \end{aligned} \quad (82)$$

So to prove Theorem 5, it suffices to estimate

$$\|\nabla_h v_h\|_{L^{r,\infty}(\Omega)} = \sup_{\lambda > 0} \left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \right|^{1/r}. \quad (83)$$

So let $\lambda > 0$. For every $k > 0$, we can write

$$\begin{aligned} &\left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \right| \\ &\leq \left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \cap B_k(v_h) \right| \\ &\quad + \left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \cap [B_k(v_h)]^c \right| \\ &\leq |B_k(v_h)| \\ &\quad + \left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \cap [B_k(v_h)]^c \right|. \end{aligned} \quad (84)$$

But $\{x \in \Omega : |\nabla_h v_h(x)| \geq \lambda\} \cap [B_k(v_h)]^c$ coincides, up to a set of measure zero, with

$$\begin{aligned} &\bigcup_{T \in \mathcal{T}_h} \left(\{x \in T : |\nabla_h v_h| \geq \lambda\} \right. \\ &\quad \left. \cap \left\{ T \in \mathcal{T}_h : \max_T |v_h| < k \right\} \right). \end{aligned} \quad (85)$$

On the other hand, if $\max_T |v_h| < k$, then $I_h^k(v_h)|_T = v_h|_T$, so

$$\begin{aligned} &\left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \cap [B_k(v_h)]^c \right| \\ &\leq \left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla \Pi_h T_k(v_h)| \geq \lambda\} \right| \\ &\leq \frac{1}{\lambda^2} \int_{\Omega} |\nabla_h \Pi_h(T_k(v_h))|^2 dx. \end{aligned} \quad (86)$$

By the use of (50), one has

$$\left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \cap [B_k(v_h)]^c \right| \leq \frac{kM}{\lambda^2}. \quad (87)$$

We fix $k = \lambda^{2-r} M^{r-1}$ to have $kM/\lambda^2 = (M/\lambda)^r$; using (71) we obtain

$$\begin{aligned} &\left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \right| \\ &\leq C(d, |\Omega|, 2^*) \left(\frac{M}{k} \right)^{2^*/2} + \left(\frac{M}{\lambda} \right)^r, \end{aligned} \quad (88)$$

so, for $r = (2 \times 2^*)/(2 + 2^*)$ to have $(M/k)^{2^*/2} = (M/\lambda)^r$, in both cases $d = 2$ or $d = 3$, one has

$$\begin{aligned} &\left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \right| \\ &\leq (C(d, |\Omega|, 2^*) + 1) \left(\frac{M}{\lambda} \right)^r, \end{aligned} \quad (89)$$

and then

$$\lambda \left| \bigcup_{T \in \mathcal{T}_h} \{x \in T : |\nabla_h v_h| \geq \lambda\} \right|^{1/r} \leq C(d, |\Omega|, q) M, \quad (90)$$

for every $\lambda > 0$, which finishes the proof. □

Lemma 10. Let $v \in V_h$. For every s, k with $0 \leq s < k$, the set $B_{k,s}(v)$, defined by

$$B_{k,s}(v) = \bigcup_{T \in \mathcal{T}_h} \left\{ T \in \mathcal{T}_h : \min_T |v| \leq s, \max_T |v| \geq k \right\}, \quad (91)$$

satisfies

$$|B_{k,s}(v)| \leq \frac{h^2}{(k-s)^2} \int_{\Omega} |\nabla_h(v)|^2 dx. \quad (92)$$

Proof. If $T \in B_{k,s}(v)$, $\max_T |v| = |v(a)| \geq k$, and $\min_T |v| = |v(b)| \leq s$, so

$$\begin{aligned} k - s &\leq |v(a)| - |v(b)| \leq |v(a) - v(b)| \\ &\leq |\nabla(v)| |a - b|. \end{aligned} \quad (93)$$

Therefore

$$1 \leq |\nabla(v)|^2 \frac{|a - b|^2}{(k - s)^2} \quad (94)$$

Hence

$$\begin{aligned} |B_{k,s}(v)| &= \sum_{T \in B_{k,s}(v)} |T| = \sum_{T \in B_{k,s}(v)} \int_T 1 dx \\ &\leq \sum_{T \in B_{k,s}(v)} \int_T |\nabla(v)|^2 \frac{|a - b|^2}{(k - s)^2} dx \\ &\leq \frac{h^2}{(k - s)^2} \int_{\Omega} |\nabla_h(v)|^2 dx. \end{aligned} \quad (95)$$

The estimate (92) follows. $\square$

Lemma 11. Let $v_h \in V_h$ and $0 \leq s < k$, then the set $\tilde{B}_{k,s}(v_h)$, defined by

$$\begin{aligned} \tilde{B}_{k,s}(v_h) \\ = \bigcup \left\{ T \in \mathcal{T}_h : \min_{m_i \in T} |v_h| \leq s, \max_T |v_h| \geq k \right\}, \end{aligned} \quad (96)$$

satisfies

$$|\tilde{B}_{k,s}(v_h)| \leq \frac{h^2}{(k-s)^2} \int_{\Omega} |\nabla_h I_h^k(v_h)|^2 dx. \quad (97)$$

Proof. Indeed, if $T \in \tilde{B}_{k,s}(v_h)$ then there are two possibilities:

- (i) If $\max_{m_i \in T} |v_h| \leq k$ so $I_h^k(v_h)|_T = v_h|_T$.
- (ii) If $\max_{m_i \in T} |v_h| \geq k$ so $\max_T |I_h^k(v_h)| \geq k$ and obviously $\min_T |I_h^k(v_h)| \leq s$.

In the two cases $T \subset B_{k,s}(I_h^k(v_h))$. The estimate (97) follows. $\square$

Remark 12. It is then clear that, under hypothesis (50),

$$|\tilde{B}_{k,s}(v_h)| \xrightarrow{h \rightarrow 0} 0, \quad (98)$$

$$|B_{k,s}(I_h^k(v_h))| \xrightarrow{h \rightarrow 0} 0.$$

In addition, one has the following.

Proposition 13. Let $v_h \in V_h$ and $0 \leq s < k$. If v_h satisfies (50), then

$$|B_{k,s}(v_h)| \xrightarrow{h \rightarrow 0} 0. \quad (99)$$

Proof. Fix $k > 0$ and $s > 0$, $s < k$. For $h > 0$ such that $1/h \geq k$, we can write

$$\begin{aligned} B_{k,s}(v_h) &= (B_{k,s}(v_h) \cap B_{1/h}(v_h)) \\ &\cup (B_{k,s}(v_h) \cap [B_{1/h}(v_h)]^c). \end{aligned} \quad (100)$$

On the one hand, with $2^* = 6$ in (71), one has

$$|B_{k,s}(v_h) \cap B_{1/h}(v_h)| \leq |B_{1/h}(v_h)| \leq C(|\Omega|) h^3. \quad (101)$$

On the other hand,

$$B_{k,s}(v_h) \cap [B_{1/h}(v_h)]^c \subset B_{k,s}(I_h^{1/h}(v_h)). \quad (102)$$

Indeed, if $x \in B_{k,s}(v_h) \cap [B_{1/h}(v_h)]^c$ and $T \in \mathcal{T}_h$ such that $x \in T$, then $\max_T |v_h| \geq k$, $\min_T |v_h| \leq s$, and, for every y in T , $|v_h(y)| \leq 1/h$, which means $I_h^{1/h}(v_h)|_T = v_h|_T$, and $T \in B_{k,s}(I_h^{1/h}(v_h))$.

Therefore, with Lemma 10, and (50), one has

$$\begin{aligned} |B_{k,s}(v_h) \cap B_{1/h}(v_h)| &\leq |B_{k,s}(I_h^{1/h}(v_h))| \\ &\leq \frac{h^2}{(k-s)^2} \int_{\Omega} |\nabla_h I_h^{1/h}(v_h)|^2 dx \\ &\leq \frac{h}{(k-s)^2} M. \end{aligned} \quad (103)$$

The convergence (99) is then a consequence of (101) and (103). $\square$

Lemma 14. Let $v_h \in V_h$. For every s and every k with $0 < s < k$, one has

$$\begin{aligned} \{x \in \Omega : T_s(I_h^k(v_h)) \neq T_s(v_h(x))\} \\ \subset B_{k,s}(v_h) \cup B_{k,s}(I_h^k(v_h)). \end{aligned} \quad (104)$$

Proof. Let $x \in \Omega$ such that $T_s(I_h^k(v_h(x))) \neq T_s(v_h(x))$, and $T \in \mathcal{T}_h$ with $x \in T$. It is easily checked that $I_h^k(v_h)|_T \neq v_h|_T$, $\max_T |v_h| \geq k$, and $\max_T |I_h^k(v_h)| \geq k$. So there are three possibilities.

- (i) $|I_h^k(v_h(x))| \geq s$ and $|v_h(x)| < s$, and then $T \subset B_{k,s}(v_h)$,
- (ii) $|I_h^k(v_h(x))| < s$ and $|v_h(x)| \geq s$, and then $T \subset B_{k,s}(I_h^k(v_h))$,
- (iii) $|I_h^k(v_h(x))| < s$ and $|v_h(x)| < s$, and then

$$T \subset B_{k,s}(I_h^k(v_h)) \cap B_{k,s}(v_h). \quad (105)$$

In all cases $x \in T \subset B_{k,s}(I_h^k(v_h)) \cup B_{k,s}(v_h)$, and (104) follows. $\square$

Proposition 15. Assume that $v_h \in V_h$ satisfies (50). Then, for every $k > 0$, one has

$$I_h^k(v_h) - T_k(v_h) \longrightarrow 0 \quad \text{in measure,} \quad (106)$$

when h tends to zero.

Proof. Fix $k > 0$ and $\varepsilon > 0$ such that $\varepsilon < k$ and consider

$$I_\varepsilon = \{x \in \Omega : |I_h^k(v_h(x)) - T_k(v_h(x))| \geq \varepsilon\}. \quad (107)$$

Let $x \in I_\varepsilon$ and $T \in \mathcal{T}_h$ with $x \in T$. It is easily checked that

$$I_h^k(v_h)|_T \neq T_k(v_h)|_T, \quad (108)$$

which implies that $\max_T |v_h| > k$. So there are four possibilities.

(i) v_h changes sign in T ; then, by continuity,

$$T \subset B_{k,s}(v_h) \quad \text{for every } s \in]0, k[. \quad (109)$$

(ii) $I_h^k(v_h)$ changes sign in T ; then

$$T \subset B_{k,s}(I_h^k(v_h)) \quad \text{for every } s \in]0, k[. \quad (110)$$

(iii) $\max_{m_i \in T} |v_h| \leq k$ and $v_h|_T \geq 0$ (or $v_h|_T \leq 0$) so $I_h^k(v_h)|_T = v_h|_T$.

(a) If $v_h|_T \geq 0$, then $v_h(x) \geq k + \varepsilon$ and $T \subset B_{k+\varepsilon,k}(I_h^k(v_h))$.

(b) If $v_h|_T \leq 0$, then $v_h(x) \leq -k - \varepsilon$ and $T \subset B_{k+\varepsilon,k}(I_h^k(v_h))$.

(iv) $\max_{m_i \in T} |v_h| > k$ and $v_h|_T \geq 0$ (or $v_h|_T \leq 0$). So

$$\begin{aligned} & |I_h^k(v_h(x)) - T_k(v_h(x))| \\ &= \left| |I_h^k(v_h(x))| - |T_k(v_h(x))| \right|. \end{aligned} \quad (111)$$

(a) If $|I_h^k(v_h(x))| - |T_k(v_h(x))| \geq \varepsilon$. There are three possibilities:

case 1: $|v_h(x)| \geq k$, so $|I_h^k(v_h(x))| \geq k + \varepsilon$ and

$$T \subset B_{k+\varepsilon,k}(I_h^k(v_h)); \quad (112)$$

case 2: $|v_h(x)| < k - \varepsilon/2$, so $T \subset B_{k,k-\varepsilon/2}(v_h)$;

case 3: $k - \varepsilon/2 \leq |v_h(x)| < k$, so $|I_h^k(v_h(x))| \geq k + \varepsilon/2$ and

$$T \subset B_{k+\varepsilon/2,k}(I_h^k(v_h)). \quad (113)$$

(b) If $|T_k(v_h(x))| - |I_h^k(v_h(x))| \geq \varepsilon$, then

$$|I_h^k(v_h(x))| \leq |T_k(v_h(x))| - \varepsilon, \quad (114)$$

so

$$T \subset B_{k,k-\varepsilon}(I_h^k(v_h)). \quad (115)$$

We can then conclude that

$$\begin{aligned} I_\varepsilon &\subset B_{k+\varepsilon/2,k}(\Pi_h T_k(v_h)) \cup B_{k,k-\varepsilon}(\Pi_h T_k(v_h)) \\ &\cup B_{k,k-\varepsilon/2}(v_h). \end{aligned} \quad (116)$$

Convergence (106) is then consequence of (98) and (99). $\square$

The result and the proof of the Proposition (2.7) in [4] can be conserved without changes.

Proposition 16. Under assumption (32), one has for every $v_h \in V_h$ and every $k > 0$

$$a_h^{swip}(v_h - I_h^k(v_h), I_h^k(v_h)) \geq 0. \quad (117)$$

Proof (proof of Proposition 2.7 in [4]). Since

$$v_h = \sum_{i=0}^d \alpha_i^{v_h} \varphi_i, \quad (118)$$

$$I_h^k(v_h) = \sum_{i=0}^d T_k(\alpha_i^{v_h}) \varphi_i,$$

one has

$$\begin{aligned} a_h^{swip}(v_h - I_h^k(v_h), I_h^k(v_h)) &= \int_{\Omega} A \nabla_h(v_h - I_h^k(v_h)) \\ &\quad \cdot \nabla_h I_h^k(v_h) dx \\ &= \sum_{F \in \mathcal{F}_h^i} \int_F \{A \nabla_h(v_h - I_h^k(v_h))\}_{\bar{\omega}} \cdot n_F [I_h^k(v_h)] dx \\ &\quad - \sum_{F \in \mathcal{F}_h^i} \int_F \{A \nabla_h I_h^k(v_h)\}_{\omega} \cdot n_F [v_h - I_h^k(v_h)] dx \\ &\quad + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\gamma_{A,F}}{h_F} \eta [v_h - I_h^k(v_h)] [I_h^k(v_h)] dx \\ &= \int_{\Omega} \left(\sum_{i=0}^d (\alpha_i^{v_h} + T_k(\alpha_i^{v_h})) A \nabla_h \varphi_i \right) \left(\sum_{j=0}^d T_k(\alpha_j^{v_h}) \right. \\ &\quad \left. \cdot A \nabla_h \varphi_j \right) dx \\ &\quad - \sum_{F \in \mathcal{F}_h^i} \int_F \left(\sum_{i=0}^d (\alpha_i^{v_h} + T_k(\alpha_i^{v_h})) \{A \nabla_h \varphi_i\}_{\bar{\omega}} \right) \\ &\quad \cdot n_F \left(\sum_{j=0}^d T_k(\alpha_j^{v_h}) [\varphi_j] \right) dx \end{aligned}$$

$$\begin{aligned}
 & - \sum_{F \in \mathcal{F}_h^i} \int_F \left(\sum_{i=0}^d T_k(\alpha_i^{v_h}) \{\varphi_i\}_\omega \right) \\
 & \cdot n_F \left(\sum_{j=0}^d (\alpha_j^{v_h} + T_k(\alpha_j^{v_h})) [A \nabla_h \varphi_j] \right) dx \\
 & + \sum_{F \in \mathcal{F}_h^i} \int_F \left(\sum_{i=0}^d (\alpha_i^{v_h} + T_k(\alpha_i^{v_h})) [A \nabla_h \varphi_i] \right) \\
 & \cdot \left(\sum_{j=0}^d T_k(\alpha_j^{v_h}) [\varphi_j] \right) dx = \sum_{i=0}^d (\alpha_i^{v_h} - T_k(\alpha_i^{v_h})) \\
 & \cdot \sum_{j=0}^d T_k(\alpha_j^{v_h}) a_h^{swip}(\varphi_i, \varphi_j) = \sum_{i=0}^d (\alpha_i^{v_h} + T_k(\alpha_i^{v_h})) \\
 & \cdot \sum_{j=0}^d T_k(\alpha_j^{v_h}) Q_{i,j} = \sum_{i=0}^d S_i,
 \end{aligned} \tag{119}$$

where

$$\begin{aligned}
 S_i &= (\alpha_i^{v_h} - T_k(\alpha_i^{v_h})) T_k(\alpha_i^{v_h}) Q_{i,i} \\
 &+ (\alpha_i^{v_h} - T_k(\alpha_i^{v_h})) \sum_{\substack{0 \leq j \leq d \\ j \neq i}} \alpha_j^{T_k(v_h)} Q_{i,j}.
 \end{aligned} \tag{120}$$

Fix $\epsilon \in \{0, 1, \dots, d\}$;

- (i) if $|T_k(v_h(m_i))| < k$, then $T_k(v_h(m_i)) = v_h(m_i)$ et $\alpha_i^{v_h} - T_k(\alpha_i^{v_h}) = 0$ and $S_i = 0$,
- (ii) if $|T_k(v_h(m_i))| = k$,

then

$$\begin{aligned}
 & (\alpha_i^{v_h} - T_k(\alpha_i^{v_h})) T_k(\alpha_i^{v_h}) \\
 &= (v_h(m_i) - T_k(v_h(m_i))) T_k(v_h(m_i)) \\
 &= k |v_h(m_i) - T_k(v_h(m_i))|,
 \end{aligned} \tag{121}$$

and therefore

$$S_i \geq k |v_h(m_i) - T_k(v_h(m_i))| \left(Q_{i,i} - \sum_{\substack{0 \leq j \leq d \\ j \neq i}} |Q_{i,j}| \right) \tag{122}$$

in the light of the assumption (32). This proves that $S_i \geq 0$ for every i in $\{1, 2, \dots, N\}$; and so (117). $\square$

4. Proof of the Main Theorem

We first show an a priori estimate (compared with (50)) on the solution u_h of (48).

Lemma 17. Let $v_h \in V_h$, $T \in \mathcal{T}_h$, and $k > 0$ such that $\max_{m_i \in T} |v_h| \leq k$. Then

$$\max_T |v_h| \leq k(2d^2 - 1). \tag{123}$$

Proof. Let $v_h \in V_h$, $T \in \mathcal{T}_h$, and $k > 0$ such that $\max_{m_i \in T} |v_h| \leq k$. For every $x \in T$,

$$\begin{aligned}
 |v_h(x)| &= \left| \sum_{i=0}^d \alpha_i^{v_h} \varphi_i(x) \right| \leq \sum_{i=0}^d |\alpha_i^{v_h} \varphi_i(x)| \\
 &\leq \sum_{i=0}^d |v_h(m_i)| |\varphi_i(x)| \\
 &\leq -k \left(\sum_{\substack{i=0 \\ \lambda_i < 1/d}}^d \varphi_i(x) \right) + k \underbrace{\left(\sum_{\substack{i=0 \\ \lambda_i \geq 1/d}}^d \varphi_i(x) \right)}_{=1 - \sum_{\substack{i=0 \\ \lambda_i < 1/d}}^d \varphi_i(x)} \\
 &\leq k \left(1 + 2 \sum_{i=0}^d (d-1) \right) \leq k(2d^2 - 1),
 \end{aligned} \tag{124}$$

the inequality (123) is then proved. $\square$

Proposition 18. Under the assumption of Theorem 4, the solution u_h of (48) satisfies for every $k > 0$ and every $h > 0$

$$a_h^{swip}(I_h^k(u_h), I_h^k(u_h)) \leq \int_{\Omega} f I_h^k(u_h) dx, \tag{125}$$

and, in particular, u_h satisfies

$$\int_{\Omega} |\nabla_h I_h^k(u_h)|^2 dx \leq k \frac{2d^2 - 1}{\alpha C_{\eta}} \|f\|_{L^1(\Omega)}, \tag{126}$$

where the constant C_{η} only depends on η, σ , and d .

The proof makes use of results appearing in [3] that we reproduce as follows.

Lemma 19 (Lemma 1.46 in [3], discrete trace inequality). Under assumption (47), one has for all $v_h \in \mathbb{P}_d^1(\mathcal{T}_h)$, all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$

$$h_T^{1/2} \|v_h\|_{L^2(F)} \leq C_{tr} \|v_h\|_{L^2(T)} \tag{127}$$

where C_{tr} only depends on σ and d .

Lemma 20 (Lemma 4.51 in [3], discrete coercivity). For all $\eta > \underline{\eta} := (d+1)C_{tr}^2$, the SWIP bilinear form a_h^{swip} is coercive on V_h with respect to the $\|\cdot\|_{swip}$ -norm; i.e.,

$$\forall v_h \in V_h, \quad a_h^{swip}(v_h, v_h) \geq C_{\eta} \|v_h\|_{swip}^2 \tag{128}$$

where $C_{\eta} = (\eta - (d+1)C_{tr}^2)/(1 + \eta)$.

Proof (of Proposition 18). Using $I_h^k(u_h)$ as a test function in (48) one has

$$a_h^{swip}(u_h, I_h^k(u_h)) = \int_{\Omega} f I_h^k(u_h) dx, \tag{129}$$

and from (117) we obtain (125).

On the other hand, by combining (125) with coercivities (34) and (128) one obtains successively

$$\begin{aligned}
 & \alpha C_\eta \int_\Omega \left| \nabla_h (I_h^k(u_h)) \right|^2 dx \\
 & \leq C_\eta \int_\Omega \left| A \nabla_h (I_h^k(u_h)) \nabla_h (I_h^k(u_h)) \right| dx \\
 & \leq C_\eta \left\| I_h^k(u_h) \right\|_{swip}^2 \leq a_h^{swip} (I_h^k(u_h), I_h^k(u_h)) \quad (130) \\
 & \leq \int_\Omega f I_h^k(u_h) dx \leq \max_\Omega |I_h^k(u_h)| \int_\Omega |f| dx \\
 & \leq k(2d^2 - 1) \|f\|_{L^1(\Omega)},
 \end{aligned}$$

and by (123) the estimation (126) is then proved. $\square$

Theorem 21. *Under the assumptions of Theorem 4, the solution u_h of (48) satisfies for every q with $1 \leq q < d/(d-1)$*

$$\begin{aligned}
 u_h & \longrightarrow u \quad \text{strongly in } L^q(\Omega), \\
 \nabla_h u_h & \longrightarrow \nabla u \quad \text{strongly in } [L^q(\Omega)]^d, \quad (131) \\
 |u_h|_{J,A,q} & \longrightarrow 0,
 \end{aligned}$$

when h tends to zero, where u is the unique renormalized solution of (1).

Proof (proof of Theorem 3.2 in [4]). Consider a sequence f^ε in $L^2(\Omega)$, converging strongly in $L^1(\Omega)$ to f (for example, $f^\varepsilon = T_{1/\varepsilon}(f)$). Let u_h^ε be the unique solution of (48) for the right-hand side f^ε . Then $u_h - u_h^\varepsilon$ satisfies

$$\begin{aligned}
 u_h - u_h^\varepsilon & \in V_h, \\
 \forall v_h & \in V_h, \\
 a_h^{swip}(u_h - u_h^\varepsilon, v_h) & = \int_\Omega (f - f^\varepsilon) v_h dx. \quad (132)
 \end{aligned}$$

Applying estimate (126) to this problem, we obtain for every $k > 0$, every $h > 0$, and every $\varepsilon > 0$

$$\begin{aligned}
 & \int_\Omega \left| \nabla_h (I_h^k(u_h - u_h^\varepsilon)) \right|^2 dx \\
 & \leq k \frac{2d^2 - 1}{\alpha C_\eta} \|f - f^\varepsilon\|_{L^1(\Omega)}, \quad (133)
 \end{aligned}$$

Which implies by Theorem 5 that for every q with $1 \leq q < d/(d-1)$, every $h > 0$, and every $\varepsilon > 0$

$$\begin{aligned}
 & \|u_h - u_h^\varepsilon\|_{swip,q} \\
 & \leq \frac{C_2(d, |\Omega|, q, \sigma, \|A\|_{L^\infty(\Omega)^{d \times d}})}{\alpha C_\eta} \|f - f^\varepsilon\|_{L^1(\Omega)}. \quad (134)
 \end{aligned}$$

On the other hand, since $f^\varepsilon \in L^2(\Omega)$ and $\mathcal{T}_h$ satisfies (44), (46), and (47), it is known (see [3]) that for every fixed ε

$$u_h^\varepsilon \longrightarrow u^\varepsilon \quad \text{strongly in } L^2(\Omega), \quad (135)$$

$$\nabla_h u_h^\varepsilon \longrightarrow \nabla u^\varepsilon \quad \text{strongly in } [L^2(\Omega)]^d, \quad (136)$$

$$|u_h^\varepsilon|_{J,A} \longrightarrow 0, \quad (137)$$

when h tends to zero, where u^ε is the unique solution of

$$\begin{aligned}
 u^\varepsilon & \in H_0^1(\Omega), \\
 -\operatorname{div}(A \nabla u^\varepsilon) & = f^\varepsilon \quad \text{in } \mathcal{D}'(\Omega). \quad (138)
 \end{aligned}$$

Finally, the function u^ε is also the unique renormalized solution of the problem

$$\begin{aligned}
 -\operatorname{div}(A \nabla u^\varepsilon) & = f^\varepsilon \quad \text{in } \Omega, \\
 u^\varepsilon & = 0 \quad \text{on } \partial\Omega. \quad (139)
 \end{aligned}$$

The estimate (43) combined with the inequality

$$\|u^\varepsilon - u\|_{swip,q} \leq \|A\|_{L^\infty(\Omega)^{d \times d}} \|u^\varepsilon - u\|_{W_0^{1,q}(\Omega)} \quad (140)$$

allows one to have

$$\begin{aligned}
 & \|u^\varepsilon - u\|_{swip,q} \\
 & \leq \frac{\|A\|_{L^\infty(\Omega)^{d \times d}} C_1(d, |\Omega|, q)}{\alpha} \|f - f^\varepsilon\|_{L^1(\Omega)}, \quad (141)
 \end{aligned}$$

for every q with $1 \leq q < d/(d-1)$, where u is the unique renormalized solution of (1).

Writing now

$$\begin{aligned}
 \|u_h - u\|_{swip,q} & \leq \|u_h - u_h^\varepsilon\|_{swip,q} + \|u^\varepsilon - u\|_{swip,q} \\
 & \quad + \|u_h^\varepsilon - u^\varepsilon\|_{swip,q}, \quad (142)
 \end{aligned}$$

Using (134), (136), (137), and (141), one has for every $\varepsilon > 0$ and every q with $1 \leq q < d/(d-1)$

$$\begin{aligned}
 & \limsup_{h \rightarrow 0} \|u_h - u\|_{swip,q} \\
 & \leq \frac{C_3(d, |\Omega|, q, \sigma, \|A\|_{L^\infty(\Omega)^{d \times d}})}{\alpha C_\eta} \|f - f^\varepsilon\|_{L^1(\Omega)}, \quad (143)
 \end{aligned}$$

and passing to the limit when ε tends to zero proves Theorem 21. $\square$

To complete the proof of Theorem 4, it remains to prove the following proposition.

Proposition 22. *Under the assumptions of Theorem 4, the solution u_h of (48) satisfies*

$$I_h^k(u_h) \xrightarrow{h \rightarrow 0} T_k(u) \quad \text{strongly in } L^2(\Omega), \quad (144)$$

$$\nabla_h (I_h^k(u_h)) \xrightarrow{h \rightarrow 0} \nabla T_k(u) \quad \text{strongly in } [L^2(\Omega)]^d, \quad (145)$$

$$|I_h^k(u_h)|_{J,A} \xrightarrow{h \rightarrow 0} 0, \quad (146)$$

for every $k > 0$.

Proof (proof of Proposition 3.3 in [4]). First by result of Proposition 15 and the estimate (126) one can have (see theorem 5.7 in [3]) the two following convergences:

$$I_h^k(u_h) \xrightarrow{h \rightarrow 0} T_k(u) \quad \text{strongly in } L^2(\Omega), \quad (147)$$

$$G_h^l(I_h^k(u_h)) \xrightarrow{h \rightarrow 0} \nabla T_k(u) \quad \text{weakly in } [L^2(\Omega)]^d, \quad (148)$$

for every $k > 0$.

On the other hand, using (123) one has

$$|f I_h^k(u_h)| \leq k(2d^2 - 1)|f| \in L^1(\Omega), \quad (149)$$

so, by Lebesgue's dominated convergence theorem combined with discrete Rellich-Kondrachov's compactness theorem (see theorem 5.6 in [3]) one has

$$\int_{\Omega} f I_h^k(u_h) dx \xrightarrow{h \rightarrow 0} \int_{\Omega} f T_k(u) dx. \quad (150)$$

Therefore passing to the limit with respect to h in (125) yields

$$\limsup_{h \rightarrow 0} a_h^{swip}(I_h^k(u_h), I_h^k(u_h)) \leq \int_{\Omega} f T_k(u) dx. \quad (151)$$

Consequently

$$\begin{aligned} & \limsup_{h \rightarrow 0} \int_{\Omega} A G_h^l(I_h^k(u_h)) G_h^l(I_h^k(u_h)) \\ & \leq \int_{\Omega} f T_k(u) dx. \end{aligned} \quad (152)$$

By the fact that

$$\begin{aligned} & \int_{\Omega} A G_h^l(I_h^k(u_h)) G_h^l(I_h^k(u_h)) \\ & \leq a_h^{swip}(I_h^k(u_h), I_h^k(u_h)), \end{aligned} \quad (153)$$

and since u is the renormalized solution of (1), it is known that (see [4])

$$\int_{\Omega} A \nabla T_k(u) \nabla T_k(u) = \int_{\Omega} f T_k(u) dx. \quad (154)$$

Finally, from (152) and (154), we deduce that

$$\begin{aligned} & \limsup_{h \rightarrow 0} \int_{\Omega} A G_h^l(I_h^k(u_h)) G_h^l(I_h^k(u_h)) \\ & \leq \int_{\Omega} A \nabla T_k(u) \nabla T_k(u), \end{aligned} \quad (155)$$

which combined with the weak convergence (148) implies

$$G_h^l(I_h^k(u_h)) \xrightarrow{h \rightarrow 0} \nabla T_k(u) \quad \text{strongly in } [L^2(\Omega)]^d. \quad (156)$$

Owing to Proposition 4.36 in [3], for all $v_h \in V_h$ and all $\eta > (d+1)C_{tr}^2$, one can have

$$\begin{aligned} & |I_h^k(u_h)|_{J,A} \\ & \leq \frac{a_h^{swip}(I_h^k(u_h), I_h^k(u_h)) - \|A^{1/2} G_h^l(I_h^k(u_h))\|_{[L^2(\Omega)]^d}}{\eta - (d+1)C_{tr}^2} \end{aligned} \quad (157)$$

and, since the right-hand side tends to zero, the result (146) holds.

Finally, using the result of Proposition 4.34 in [3]

$$\|R_h^l([I_h^k(u_h)])\|_{[L^2(\Omega)]^d} \leq \sqrt{d+1} C_{tr} |I_h^k(u_h)|_{J,A} \quad (158)$$

with the triangle inequality that yields

$$\begin{aligned} & \|\nabla_h I_h^k(u_h) - \nabla T_k(u)\|_{[L^2(\Omega)]^d} \\ & \leq \|G_h^l(I_h^k(u_h)) - \nabla T_k(u)\|_{[L^2(\Omega)]^d} \\ & \quad + \|R_h^l([I_h^k(u_h)])\|_{[L^2(\Omega)]^d} \end{aligned} \quad (159)$$

concluding the proof of (145). $\square$

5. The Case Where f Is a Bounded Radon Measure

The materials used in [4], to handle the case where f belongs to $\mathcal{M}_b(\Omega)$, are not specific to the case of $\mathbb{P}_1$ finite elements approximation; only the weak convergence (148) requires clarification; in our approach it is based on the result of Proposition 15 whose proof involves only properties of v_h not f . So we can also state the following convergence result.

Theorem 23 (Theorem 4.1 in [4]). *Assume that f belongs to $\mathcal{M}_b(\Omega)$ and A and $\mathcal{T}_h$ satisfy (46), (33), (35), (44), and (47) and (32). Then there exist a subsequence, still denoted by h , and a function u such that for every $k > 0$ and for every q with $1 \leq q < d/(d-1)$ one has*

$$G_h^l(I_h^k(u_h)) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } [L^2(\Omega)]^d, \quad (160)$$

$$\nabla_h u_h \rightharpoonup \nabla u \quad \text{weakly in } [L^q(\Omega)]^d,$$

when h tends to zero along this subsequence, where u satisfies

$$\forall k > 0, \quad T_k(u) \in H_0^1(\Omega),$$

$$\forall q \text{ with } 1 \leq q < \frac{d}{d-1}, \quad u \in W_0^{1,q}(\Omega), \quad (161)$$

$$\forall v \in \mathcal{C}_c^\infty(\Omega), \quad \int_{\Omega} A \nabla u \nabla v dx = \int_{\Omega} v df.$$

6. Convergence Rate Estimation

6.1. Error Estimates for Smooth Solutions

Assumption 24 (regularity of exact solution and space V_*). As in [3], we assume that $\mathcal{T}_h$ is compatible with the partition P_Ω

in the sense of Assumption 3, and the unique solution v is such that

$$v \in V_* := H_0^1(\Omega) \cap H^2(\mathcal{T}_h). \quad (162)$$

And we set

$$V_{*h} := V_* + V_h. \quad (163)$$

The convergence analysis is performed in the spirit of (Theorem 1.35 [3]) by establishing discrete coercivity, consistency, and boundedness for a_h^{swip} . The discrete bilinear form a_h^{swip} is extended to $V_{*h} \times V_h$.

Without further knowledge on the exact solution v apart from the domain Ω and the datum $g \in L^2(\Omega)$, Assumption 24 can be asserted for instance if the domain Ω is convex; see Grisvard [7].

A straightforward consequence of the Lax-Milgram Lemma is that the discrete problem (48) is well-posed.

Theorem 25. *Under the assumptions of Theorem 4 ($d = 2$ or $d = 3$), if f belongs to $L^{r,\infty}(\Omega)$ for some r such that $1 < r < 2$, Ω a convex polyhedron ($\Omega_h = \Omega$), and $A \in [W^{1,\infty}(\Omega)]^{d \times d}$, then $\forall q$ with $1 \leq q < d/(d-1)$*

$$\begin{aligned} \|u_h - u\|_{swip,q} &\leq C(d, |\Omega|, q, r, \alpha, \|A\|_{W^{1,\infty}(\Omega)^{d \times d}}, \sigma) \\ &\cdot h^{2(1-1/r)} \|f\|_{L^{r,\infty}(\Omega)}. \end{aligned} \quad (164)$$

Proof (proof of Theorem 5.1 in [4]). From [3] the unique solution u^ϵ of (48) with right-hand side $f^\epsilon = T_{1/\epsilon}(f) \in L^\infty(\Omega) \subset L^2(\Omega)$ verifies

$$\|u_h^\epsilon - u^\epsilon\|_{swip} \leq C(\|A\|_{W^{1,\infty}(\Omega)^{d \times d}}) h \|f^\epsilon\|_{L^2(\Omega)} \quad (165)$$

so

$$\begin{aligned} \|u_h^\epsilon - u^\epsilon\|_{swip,q} \\ \leq C(q, d, \eta, |\Omega|, \|A\|_{W^{1,\infty}(\Omega)^{d \times d}}) h \|f^\epsilon\|_{L^2(\Omega)} \end{aligned} \quad (166)$$

Combined with (142), (134), and (141) allows one to have

$$\begin{aligned} \|u_h - u\|_{swip,q} &\leq C(q, d, \eta, |\Omega|, \|A\|_{W^{1,\infty}(\Omega)^{d \times d}}) \\ &\cdot (\|f - f^\epsilon\|_{L^1(\Omega)} + h \|f^\epsilon\|_{L^2(\Omega)}), \end{aligned} \quad (167)$$

and by proceeding as in [4], we obtain (173). $\square$

Remark 26. If $f \in L^{r,\infty}(\Omega)$ for some r with $1 < r < 2$ and

$$\forall k > 0: T_k(f) \in H^1(\Omega), \quad (168)$$

then a small adaptation of the proof given in [4] provides an $O(h^{4(1-1/r)})$ error estimate in $\|\cdot\|_{swip,q}$ -norm, with $1 \leq q < d/(d-1)$, since, with (168), it is known that

$$\begin{aligned} \|u_h^\epsilon - u^\epsilon\|_{swip,q} \\ \leq C(d, |\Omega|, q, r, \alpha, \|A\|_{W^{1,\infty}(\Omega)^{d \times d}}, \sigma) h^2 \|f^\epsilon\|_{L^2(\Omega)}. \end{aligned} \quad (169)$$

6.2. Error Estimates for Low-Regularity Solutions

Assumption 27 (regularity of exact solution and space V_*). As in [3], we assume that the mesh $\mathcal{T}_h$ is compatible with the partition P_Ω in the sense of Assumption 3, $d \geq 2$, and that there is p such that $2d/(d+2) < p \leq 2$; the unique solution v is such that

$$v \in V_* := H_0^1(\Omega) \cap W^{2,p}(\mathcal{T}_h), \quad (170)$$

where $W^{2,p}(P_\Omega) = W^{2,p}(\mathcal{T})$ designate that the mesh $\mathcal{T}$ is compatible with the partition P_Ω , and we set

$$V_{*h} := V_* + V_h. \quad (171)$$

We also assume $p < 2$ since, in the case $p = 2$, Assumption 27 amounts to Assumption 24.

Assumption 27 requires $p > 1$ for $d = 2$ and $p > 6/5$ for $d = 3$. In particular, we observe that, in two space dimensions, $v \in W^{2,p}(P_\Omega)$ with $p > 1$ holds true in polygonal domains; see, e.g., Dauge [8]. Moreover, using Sobolev embeddings (see [Evans [9], Sect. 5.6] or [Brézis [10], Sect. IX.3]), Assumption 27 implies

$$v \in H^{1+\alpha_p}(\Omega), \quad \text{with } \alpha_p = \frac{d+2}{2} - \frac{d}{p} > 0. \quad (172)$$

Theorem 28. *Under the assumptions of Theorem 4 ($d = 2$ or $d = 3$), if f belongs to $L^{r,\infty}(\Omega)$ for some r such that $1 < r < 2$, Ω a convex polyhedron ($\Omega_h = \Omega$), and $A \in [W^{1,\infty}(\Omega)]^{d \times d}$, then $\forall q$ with $1 \leq q < p$*

$$\begin{aligned} \|u_h - u\|_{swip,q} &\leq C(d, |\Omega|, q, r, \alpha, \|A\|_{W^{1,\infty}(\Omega)^{d \times d}}, \sigma) \\ &\cdot h^{2\alpha_p(1-1/r)} \|f\|_{L^{r,\infty}(\Omega)}. \end{aligned} \quad (173)$$

Remark 29. Moreover, under the assumption in Remark 26 one can have an $O(h^{4\alpha_p(1-1/r)})$ error estimate in $\|\cdot\|_{swip,q}$ -norm, with $1 \leq q < p$.

7. The Case Where A Is the Identity Matrix

Returning to the definition (31) of mass matrix Q where $A = I$, one has

for i, j in $\{1, 2, \dots, N\}$

$$Q_{i,j} = a_h^{swip}(\varphi_i, \varphi_j) = \underbrace{\int_{\Omega} \nabla_h \varphi_i \nabla_h \varphi_j dx}_{=I_1} + \underbrace{\sum_{F \in \mathcal{F}_h^i} \frac{1}{h_F} \eta[\varphi_i][\varphi_j] dx}_{=I_2} - \underbrace{\left(\sum_{F \in \mathcal{F}_h^i} \int_F \{\nabla_h \varphi_i\}_{\omega} \cdot n_F [\varphi_j] dx + \sum_{F \in \mathcal{F}_h^i} \int_F \{\nabla_h \varphi_j\}_{\omega} \cdot n_F [\varphi_i] dx \right)}_{=I_3}. \quad (174)$$

If $i = j$, so

$$\begin{aligned} I_1 &= \int_{\Omega} \nabla_h \varphi_i^2 dx \geq 0, \\ I_2 &= \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{h_F} [\varphi_i]^2 dx \geq 0, \\ I_3 &= 2 \left(\int_{F_i} \{\nabla_h \varphi_i\}_{\omega} \cdot n_F [\varphi_i] dx \right. \\ &\quad \left. + \sum_{\substack{F \in \mathcal{F}_h^i \\ F \neq F_i}} \{\nabla_h \varphi_i\}_{\omega} \cdot n_F [\varphi_i] dx \right) = 0, \end{aligned} \quad (175)$$

since $[\varphi_i]_{F_i} = 0$, $\{\nabla_h \varphi_i\}_{\omega} \cdot n_F = -(d/h_{F_i})n_i \cdot n_F$, and $\int_{F \neq F_i} \varphi_i dx = 0$.

If $i \neq j$, so

$$I_1 = \int_{\Omega} \nabla_h \varphi_i \nabla_h \varphi_j dx = \frac{1}{d^2} |F_i| |F_j| n_i \cdot n_j \leq 0 \quad (176)$$

(see Proposition 6.1 in [4]),

$$\begin{aligned} I_2 &= \int_{F \in F_i \cup F_j} \frac{\eta}{h_F} [\varphi_i][\varphi_j] dx + \sum_{\substack{F \in \mathcal{F}_h^i \\ F \neq F_i \neq F_j}} \frac{\eta}{h_F} [\varphi_i][\varphi_j] dx \\ &= -\frac{\eta}{d+1} \sum_{\substack{F \in \mathcal{F}_h^i \\ F \neq F_i \neq F_j}} \frac{|F|}{h_F} \leq 0, \\ I_3 &= -d \sum_{\substack{F \in \mathcal{F}_h^i \\ F \neq F_i \neq F_j}} \left(\frac{1}{h_{F_i}} n_i \cdot n_F \varphi_j + \frac{1}{h_{F_j}} n_j \cdot n_F \varphi_i \right) dx = 0. \end{aligned} \quad (177)$$

It is therefore concluded that, under the condition $n_l \cdot n_k \leq 0$ ($l \neq k$), one can have $Q_{lk} \leq 0$. Thus, the matrix Q verifies (32).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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