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Research Article

Existence of Asymptotically Almost Automorphic Mild Solutions of Semilinear Fractional Differential Equations

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This paper is concerned with the existence of asymptotically almost automorphic mild solutions to a class of abstract semilinear fractional differential equations $D_t^\alpha x(t) = Ax(t) + D_t^{\alpha-1}F(t, x(t), Bx(t))$, $t \in \mathbb{R}$, where $1 < \alpha < 2$, A is a linear densely defined operator of sectorial type on a complex Banach space X and B is a bounded linear operator defined on X, F is an appropriate function defined on phase space, and the fractional derivative is understood in the Riemann-Liouville sense. Combining the fixed point theorem due to Krasnoselskii and a decomposition technique, we prove the existence of asymptotically almost automorphic mild solutions to such problems. Our results generalize and improve some previous results since the (locally) Lipschitz continuity on the nonlinearity F is not required. The results obtained are utilized to study the existence of asymptotically almost automorphic mild solutions to a fractional relaxation-oscillation equation.

1. Introduction

The almost periodic function introduced seminally by Bohr in 1925 plays an important role in describing the phenomena that are similar to the periodic oscillations which can be observed frequently in many fields, such as celestial mechanics, nonlinear vibration, electromagnetic theory, plasma physics, engineering, and ecosphere. The concept of almost automorphy, which is an important generalization of the classical almost periodicity, was first introduced in the literature [1–4] by Bochner in relation to some aspects of differential geometry. Since then, this pioneer work has attracted more and more attention and has been substantially extended in several different directions. Many authors have made important contributions to this theory (see, for instance, [5–17] and the references therein). Especially, in [5, 6], the authors gave an important overview about the theory of almost automorphic functions and their applications to differential equations.

As a natural extension of almost automorphy, the concept of asymptotic almost automorphy, which is the central issue to be discussed in this paper, was introduced in the literature [18] by N'Guérékata in the early eighties. Since then, this notion has found several developments and has been generalized into different directions. Until now, the asymptotically almost automorphic functions as well as the asymptotically almost automorphic solutions for differential systems have been investigated by many mathematicians; see [19] by Bugajewski and N'Guérékata, [20] by Diagana, Hernández, and dos Santos, and [21] by Ding, Xiao, and Liang for the asymptotically almost automorphic solutions to integrodifferential equations, see [22] by Zhao, Chang, and N'Guérékata for the asymptotically almost automorphic solutions to the nonlinear delay integral equations, and see [23] by Chang and Tang and [24] by Zhao, Chang, and Nieto for the asymptotically almost automorphic solutions to stochastic differential equations, and the existence of asymptotically almost automorphic solutions has become one of the most attractive topics in the qualitative theory of differential equations due to its significance and applications in physics, mathematical biology, control theory, and so on. We refer the reader to the monographs of N'Guérékata [25] for the recently theory and applications of asymptotically almost automorphic functions.

With motivation coming from a wide range of engineering and physical applications, fractional differential equations have recently attracted great attention of mathematicians and scientists. This kind of equations is a generalization of ordinary differential equations to arbitrary noninteger orders. Fractional differential equations find numerous applications in the field of viscoelasticity, feedback amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry, and biological sciences [26–32]. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; we only enumerate here the monographs of Kilbas et al. [26, 27], Diethelm [28], Hilfer [29], Podlubny [30], Miller [31], and Zhou [32] and the papers of Agarwal et al. [33, 34], Benchohra et al. [35, 36], El-Borai [37], Lakshmikantham et al. [38–41], Mophou et al. [42–45], N'Guérékata [46], and Zhou et al. [47–50] and the reference therein.

The study of almost periodic and almost automorphic type solutions to fractional differential equations was initiated by Araya and Lizama [11]. In their work, the authors investigated the existence and uniqueness of an almost automorphic mild solution of the semilinear fractional differential equation

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R},\ 1 < \alpha < 2, \tag{1}$$

when A is a generator of an α-resolvent family and D_t^{α} is the Riemann-Liouville fractional derivative. In [51], Cuevas and Lizama considered the fractional differential equation:

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t, x(t)), \quad t \in \mathbb{R},\ 1 < \alpha < 2, \tag{2}$$

where A is a linear operator of sectorial negative type on a complex Banach space X and the fractional derivative is understood in the Riemann-Liouville sense. Under suitable conditions on $F(t, x)$, the authors proved the existence and uniqueness of an almost automorphic mild solution to (2). Cuevas et al. [52, 53] studied, respectively, the pseudo almost periodic and pseudo almost periodic class infinity mild solutions to (2) assuming that $F : \mathbb{R} \times X \longrightarrow X$ and $(t, x) \longrightarrow F(t, x)$ is a pseudo almost periodic and pseudo almost periodic of class infinity function satisfying suitable conditions in $x \in X$. Agarwal et al. [54] studied the existence and uniqueness of a weighted pseudo almost periodic mild solution to equation (2). Ding et al. [55] investigated the existence and uniqueness of almost automorphic solution to (2) assuming that $F : \mathbb{R} \times X \longrightarrow X$ and $(t, x) \longrightarrow F(t, x)$ is Stepanov-like almost automorphic in $t \in \mathbb{R}$ satisfying some kind of Lipschitz conditions. Cuevas et al. [56] studied the existence of almost periodic (resp., pseudo almost periodic) mild solutions to equation (2) assuming that $F : \mathbb{R} \times X \longrightarrow X$ and $(t, x) \longrightarrow F(t, x)$ is Stepanov almost (resp., Stepanov-like pseudo almost) periodic in $t \in \mathbb{R}$ uniformly for $x \in X$. Chang et al. [57] studied the existence and uniqueness of weighted pseudo almost automorphic solution to equation (2) with Stepanov-like weighted pseudo almost automorphic coefficient. He et al. [58] studied also the existence and uniqueness of weighted Stepanov-like pseudo almost automorphic mild solution to (2). Cao et al. [59] studied the existence and uniqueness of antiperiodic mild solution to (2). In [60], Cuevas et al. showed sufficient conditions to ensure the existence and uniqueness of mild solution for (2) in the following classes of vector-valued function spaces: periodic functions, asymptotically periodic functions, pseudo periodic functions, almost periodic functions, asymptotically almost periodic functions, pseudo almost periodic functions, almost automorphic functions, asymptotically almost automorphic functions, pseudo almost automorphic functions, compact almost automorphic functions, asymptotically compact almost automorphic functions, pseudo compact almost automorphic functions, S-asymptotically ω-periodic functions, decay functions, and mean decay functions.

Recently, Xia et al. [61] established some sufficient criteria for the existence and uniqueness of (μ, ν)-pseudo almost automorphic solution to the semilinear fractional differential equation

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t, Bx(t)), \quad t \in \mathbb{R}, \tag{3}$$

where $1 < \alpha < 2$, A is a sectorial operator of type $\omega < 0$ on a complex Banach space X and B is a bounded linear operator. The fractional derivative is understood in the Riemann-Liouville sense. Their discussion is divided into two cases, i.e., $F : \mathbb{R} \times X \longrightarrow X$, $(t, x) \longrightarrow F(t, x)$ is (μ, ν)-pseudo almost automorphic and $F : \mathbb{R} \times X \longrightarrow X$, and $(t, x) \longrightarrow F(t, x)$ is Stepanov-like (μ, ν)-pseudo almost automorphic. Kavitha et al. [62] studied weighted pseudo almost automorphic solutions of the fractional integrodifferential equation

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t, x(t), Kx(t)), \quad t \in \mathbb{R}, \tag{4}$$

where $1 < \alpha < 2$ and

$$Kx(t) = \int_{-\infty}^{t} k(t-s)\, h(s, x(s))\, \mathrm{d}s, \tag{5}$$

A is a linear densely defined sectorial operator on a complex Banach space X, $F : \mathbb{R} \times X \times X \longrightarrow X$, and $(t, x, y) \longrightarrow F(t, x, y)$ is a weighted pseudo almost automorphic function in $t \in \mathbb{R}$ for each $x, y \in X$ satisfying suitable conditions. The fractional derivative is understood in the Riemann-Liouville sense. Mophou [63] investigated the existence and uniqueness of weighted pseudo almost automorphic mild solution to the fractional differential equation:

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t, x(t), Bx(t)), \quad t \in \mathbb{R},\ 1 < \alpha < 2, \tag{6}$$

where $A : D(A) \subset X \longrightarrow X$ is a linear densely operator of sectorial type on a complex Banach space X, $B : X \longrightarrow X$ is a bounded linear operator and $F : \mathbb{R} \times X \times X \longrightarrow X$, and $(t, x, y) \longrightarrow F(t, x, y)$ is a weighted pseudo almost automorphic function in $t \in \mathbb{R}$ for each $x, y \in X$ satisfying suitable conditions. The fractional derivative D_t^{α} is to be understood in Riemann-Liouville sense. Chang et al.

[64] investigated some existence results of μ-pseudo almost automorphic mild solutions to (6) assuming that $F : \mathbb{R} \times X \times X \longrightarrow X$ and $(t, x, y) \longrightarrow F(t, x, y)$ is a μ-pseudo almost automorphic function in $t \in \mathbb{R}$ for each $x, y \in X$ satisfying suitable conditions. For more on the almost periodicity and almost automorphy for fractional differential equations and related issues, we refer the reader to [65–67] and others.

Equation (6) is motivated by physical problems. Indeed, due to their applications in fields of science where characteristics of anomalous diffusion are presented, type (6) equations are attracting increasing interest (cf. [68–70] and references therein). For example, anomalous diffusion in fractals [69] or in macroeconomics [71] has been recently well studied in the setting of fractional Cauchy problems like (6). For this reason, (6) has gotten a considerable attention in recent years (cf. [51–64, 68–71] and the references therein).

To the best of our knowledge, much less is known about the existence of asymptotically almost automorphic mild solutions to (6) when the nonlinearity $F(t, x, y)$ as a whole loses the Lipschitz continuity with respect to x and y. Motivated by the abovementioned works, the purpose of this paper is to establish some new existence results of asymptotically almost automorphic mild solutions to (6). In our results, the nonlinearity $F : \mathbb{R} \times X \times X \longrightarrow X$, $(t, x, y) \longrightarrow F(t, x, y)$ does not have to satisfy a (locally) Lipschitz condition (see Remark 22). However, in many papers (for instance, [11, 51–64]) on almost periodic type and almost automorphic type solutions to fractional differential equations, to be able to apply the well-known Banach contraction principle, a (locally) Lipschitz condition for the nonlinearity of corresponding fractional differential equations is needed. As can be seen, our results generalize those as well as related research and have more broad applications. In particular, as application and to illustrate our main results, we will examine some sufficient conditions for the existence of asymptotically almost automorphic mild solutions to the fractional relaxation-oscillation equation given by

$$\begin{aligned} \partial_t^{\alpha} u(t,x) &= \partial_x^2 u(t,x) - p u(t,x) + \partial_t^{\alpha-1}\Big[\mu a(t) \\ &\quad \cdot \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)[\sin u(t,x) + u(t,x)] \\ &\quad + \nu e^{-|t|}[u(t,x) + \sin u(t,x)]\Big], \quad t \in \mathbb{R},\ x \in [0,\pi] \end{aligned} \tag{7}$$

with boundary conditions $u(t, 0) = u(t, \pi) = 0$, $t \in \mathbb{R}$, where $a(t) \in BC(\mathbb{R}, \mathbb{R}^+)$ is a function and p, μ, and ν are positive constants.

The rest of this paper is organized as follows. In Section 2, some concepts, the related notations, and some useful lemmas are introduced and established. In Section 3, we prove the existence of asymptotically almost automorphic mild solutions to such problems. The results obtained are utilized to study the existence of asymptotically almost automorphic mild solutions to a fractional relaxation-oscillation equation given in Section 4.

2. Preliminaries

This section is concerned with some notations, definitions, lemmas, and preliminary facts which are used in what follows.

From now on, let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces and $BC(\mathbb{R}, X)$ (resp., $BC(\mathbb{R} \times Y \times Y, X)$) is the space of all X-valued bounded continuous functions (resp., jointly bounded continuous functions $F : \mathbb{R} \times Y \times Y \longrightarrow X$). Furthermore, $C_0(\mathbb{R}, X)$ (resp., $C_0(\mathbb{R} \times Y \times Y, X)$) is the closed subspace of $BC(\mathbb{R}, X)$ (resp., $BC(\mathbb{R} \times Y \times Y, X)$) consisting of functions vanishing at infinity (vanishing at infinity uniformly in any compact subset of $Y \times Y$, in other words,

$$\lim_{|t| \longrightarrow +\infty} \|g(t, x, y)\| = 0 \quad \text{uniformly for } (x, y) \in \mathbb{K}, \tag{8}$$

where $\mathbb{K}$ is an any compact subset of $Y \times Y$). Let also $\mathbb{L}(X)$ be the Banach space of all bounded linear operators from X into itself endowed with the norm:

$$\|T\|_{\mathbb{L}(X)} = \sup\{\|Tx\| : x \in X,\ \|x\| = 1\}. \tag{9}$$

For a bounded linear operator $A \in \mathbb{L}(X)$, let $\rho(A)$ and $D(A)$ stand for the resolvent and domain of A, respectively.

First, let us recall some basic definitions and results on almost automorphic and asymptotically almost automorphic functions.

Definition 1 ((Bochner) [1] (N'Guérékata) [6]). A continuous function $F : \mathbb{R} \longrightarrow X$ is said to be almost automorphic if for every sequence of real numbers $\{s_n'\}$, there exists a subsequence $\{s_n\}$ such that

$$\Theta(t) = \lim_{n \longrightarrow \infty} F(t + s_n) \tag{10}$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n \longrightarrow \infty} \Theta(t - s_n) = F(t) \quad \text{for each } t \in \mathbb{R}. \tag{11}$$

Denote by $AA(\mathbb{R}, X)$ the set of all such functions.

Remark 2 (see [6]). By the point-wise convergence, the function $\Theta(t)$ in Definition 1 is measurable but not necessarily continuous. Moreover, if $\Theta(t)$ is continuous, then $F(t)$ is uniformly continuous (cf., e.g., [17], Theorem 2.6), and if the convergence in Definition 1 is uniform on $\mathbb{R}$, one gets almost periodicity (in the sense of Bochner and von Neumann). Almost automorphy is thus a more general concept than almost periodicity. There exists an almost automorphic function which is not almost periodic. The function $F : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$F(t) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) \tag{12}$$

is an example of such functions [72].

Lemma 3 (see [5]). *$AA(\mathbb{R}, X)$ is a Banach space with the norm $\|F\|_\infty = \sup_{t \in \mathbb{R}} \|F(t)\|$.*

Definition 4 (see [6]). A continuous function $F : \mathbb{R}\times Y\times Y \longrightarrow X$ is said to be almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where K is any bounded subset of $Y\times Y$, if for every sequence of real numbers $\{s_n'\}$, there exists a subsequence $\{s_n\}$ such that

$$\lim_{n\longrightarrow\infty} F(t + s_n, x, y) = \Theta(t, x, y) \text{ exists} \quad \text{for each } t \in \mathbb{R} \text{ and each } (x, y) \in K \tag{13}$$

and

$$\lim_{n\longrightarrow\infty} \Theta(t - s_n, x, y) = F(t, x, y) \text{ exists} \quad \text{for each } t \in \mathbb{R} \text{ and each } (x, y) \in K. \tag{14}$$

The collection of those functions is denoted by $AA(\mathbb{R} \times Y \times Y, X)$.

Remark 5. The function $F : \mathbb{R} \times X \times X \longrightarrow X$ given by

$$F(t, x, y) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)[\sin(x) + y] \tag{15}$$

is almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where K is any bounded subset of $X \times X$, $X = L^2[0, \pi]$.

Similar to Lemma 2.2 of [73] and Proposition 3.2 of [63], we have the following result on almost automorphic functions.

Lemma 6. *Let $F : \mathbb{R}\times X\times X \longrightarrow X$ be almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where K is any bounded subset of $X \times X$, and assume that $F(t, x, y)$ is uniformly continuous on K uniformly for $t \in \mathbb{R}$, that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_1, x_2, y_1, y_2 \in K$ and $\|x_1 - y_1\| + \|x_2 - y_2\| < \delta$ imply that*

$$\|F(t, x_1, x_2) - F(t, y_1, y_2)\| < \varepsilon \quad \forall t \in \mathbb{R}. \tag{16}$$

Let $x, y : \mathbb{R} \longrightarrow X$ be almost automorphic. Then the function $\Upsilon : \mathbb{R} \longrightarrow X$ defined by $\Upsilon(t) = F(t, x(t), y(t))$ is almost automorphic.

Proof. Suppose that $\{s_n\}$ is a sequence of real numbers. Then by the definition of almost automorphic functions, we can extract a subsequence $\{\tau_n\}$ of $\{s_n\}$ such that

$$\begin{aligned}
&(P_1)\ \lim_{n\longrightarrow\infty} x(t + \tau_n) = \tilde{x}(t) \quad \text{for each } t \in \mathbb{R},\\
&(P_2)\ \lim_{n\longrightarrow\infty} \tilde{x}(t - \tau_n) = x(t) \quad \text{for each } t \in \mathbb{R},\\
&(P_3)\ \lim_{n\longrightarrow\infty} y(t + \tau_n) = \tilde{y}(t) \quad \text{for each } t \in \mathbb{R},\\
&(P_4)\ \lim_{n\longrightarrow\infty} \tilde{y}(t - \tau_n) = y(t) \quad \text{for each } t \in \mathbb{R},\\
&(P_5)\ \lim_{n\longrightarrow\infty} F(t + \tau_n, x, y) = \tilde{F}(t, x, y) \quad \text{for each } t \in \mathbb{R},\ x, y \in X,\\
&(P_6)\ \lim_{n\longrightarrow\infty} \tilde{F}(t - \tau_n, x, y) = F(t, x, y) \quad \text{for each } t \in \mathbb{R},\ x, y \in X.
\end{aligned} \tag{17}$$

Write

$$\tilde{\Upsilon}(t) := \tilde{F}(t, \tilde{x}(t), \tilde{y}(t)), \quad t \in \mathbb{R}. \tag{18}$$

Then

$$\begin{aligned}
&\|\Upsilon(t + \tau_n) - \tilde{\Upsilon}(t)\|\\
&\quad = \|F(t + \tau_n, x(t + \tau_n), y(t + \tau_n))\\
&\quad - \tilde{F}(t, \tilde{x}(t), \tilde{y}(t))\|\\
&\quad \le \|F(t + \tau_n, x(t + \tau_n), y(t + \tau_n))\\
&\quad - F(t + \tau_n, \tilde{x}(t), \tilde{y}(t))\| + \|F(t + \tau_n, \tilde{x}(t), \tilde{y}(t))\\
&\quad - \tilde{F}(t, \tilde{x}(t), \tilde{y}(t))\|.
\end{aligned} \tag{19}$$

Since $x(t)$ and $y(t)$ are almost automorphic, then $x(t)$, $y(t)$ and $\tilde{x}(t)$, and $\tilde{y}(t)$ are bounded. Therefore we can choose a bounded subset $K \subset X \times X$, such that

$$\begin{aligned}
&(x(t), y(t)) \in K,\\
&(\tilde{x}(t), \tilde{y}(t)) \in K\\
&\qquad \forall t \in \mathbb{R}.
\end{aligned} \tag{20}$$

By (P_1), (P_3), and the uniform continuity of $F(t, x, y)$ in $(x(t), y(t)) \in K$, we have

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \|F(t + \tau_n, x(t + \tau_n), y(t + \tau_n))\\
&\quad - F(t + \tau_n, \tilde{x}(t), \tilde{y}(t))\| = 0.
\end{aligned} \tag{21}$$

Moreover, by (P_5),

$$\lim_{n\longrightarrow\infty} \|F(t + \tau_n, \tilde{x}(t), \tilde{y}(t)) - \tilde{F}(t, \tilde{x}(t), \tilde{y}(t))\| = 0, \tag{22}$$

so remembering the above triangle inequality, we deduce that

$$\lim_{n\longrightarrow\infty} \|\Upsilon(t + \tau_n) - \tilde{\Upsilon}(t)\| = 0 \quad \text{for each } t \in \mathbb{R}. \tag{23}$$

Using the same argument we can prove that

$$\lim_{n\longrightarrow\infty} \|\tilde{\Upsilon}(t - \tau_n) - \Upsilon(t)\| = 0 \quad \text{for each } t \in \mathbb{R}. \tag{24}$$

This proves that $\Upsilon(t)$ is almost automorphic by the definition. □

Remark 7. If $F(t, x, y)$ satisfies a Lipschitz condition with respect to x and y uniformly in $t \in \mathbb{R}$, i.e., for each pair $x_1, x_2, y_1, y_2 \in X$,

$$\begin{aligned}
&\|F(t, x_1, x_2) - F(t, y_1, y_2)\|\\
&\quad \le L(\|x_1 - y_1\| + \|x_2 - y_2\|)
\end{aligned} \tag{25}$$

uniformly in $t \in \mathbb{R}$, where $L > 0$ is called the Lipschitz constant for the function $F(t, x, y)$, then $F(t, x, y)$ is uniformly continuous on K uniformly for $t \in \mathbb{R}$, where K is any bounded subset of $X \times X$.

Remark 8. If $F(t,x,y)$ satisfies a local Lipschitz condition with respect to x and y uniformly in $t \in \mathbb{R}$, i.e., for each pair $x_1, x_2, y_1, y_2 \in X$, $t \in \mathbb{R}$,

$$\begin{aligned} &\|F(t,x_1,x_2) - F(t,y_1,y_2)\| \\ &\quad \le L(t)\left(\|x_1 - y_1\| + \|x_2 - y_2\|\right), \end{aligned} \tag{26}$$

where $L(t) \in BC(\mathbb{R},\mathbb{R}^+)$, then $F(t,x,y)$ is uniformly continuous on K uniformly for $t \in \mathbb{R}$, where K is any bounded subset of $X \times X$.

Definition 9 (see [6]). A continuous function $F : \mathbb{R} \longrightarrow X$ is said to be asymptotically almost automorphic if it can be decomposed as $F(t) = G(t) + \Phi(t)$, where

$$\begin{aligned} &G(t) \in AA(\mathbb{R},X), \\ &\Phi(t) \in C_0(\mathbb{R},X). \end{aligned} \tag{27}$$

Denote by $AAA(\mathbb{R},X)$ the set of all such functions.

Remark 10. The function $F : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} F(t) &= G(t) + \Phi(t) \\ &= \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) + e^{-|t|} \end{aligned} \tag{28}$$

is an asymptotically almost automorphic function with

$$\begin{aligned} &G(t) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) \in AA(\mathbb{R},\mathbb{R}), \\ &\Phi(t) = e^{-|t|} \in C_0(\mathbb{R},\mathbb{R}). \end{aligned} \tag{29}$$

Lemma 11 (see [6]). *$AAA(\mathbb{R},X)$ is also a Banach space with the supremum norm $\|\cdot\|_\infty$.*

Definition 12 (see [6]). A continuous function $F : \mathbb{R} \times Y \times Y \longrightarrow X$ is said to be asymptotically almost automorphic if it can be decomposed as $F(t,x,y) = G(t,x,y) + \Phi(t,x,y)$, where

$$\begin{aligned} &G(t,x,y) \in AA(\mathbb{R}\times Y\times Y, X), \\ &\Phi(t,x,y) \in C_0(\mathbb{R}\times Y\times Y, X). \end{aligned} \tag{30}$$

Denote by $AAA(\mathbb{R}\times Y\times Y, X)$ the set of all such functions.

Remark 13. The function $F : \mathbb{R}\times X\times X \longrightarrow X$ given by

$$\begin{aligned} F(t,x,y) &= G(t,x,y) + \Phi(t,x,y) \\ &= \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)[\sin(x)+y] \\ &\quad + e^{-|t|}[x+\sin(y)] \end{aligned} \tag{31}$$

is asymptotically almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x,y) \in K$, where K is any bounded subset of $X \times X$, $X = L^2[0,\pi]$ and

$$\begin{aligned} G(t,x,y) &= \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)[\sin(x)+y] \\ &\in AA(\mathbb{R}\times X\times X, X), \\ \Phi(t,x,y) &= e^{-|t|}[x+\sin(y)] \in C_0(\mathbb{R}\times X\times X, X). \end{aligned} \tag{32}$$

Next we give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 14 (see [26]). The fractional integral of order $\alpha > 0$ with the lower limit t_0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t > t_0,\ \alpha > 0 \tag{33}$$

provided that the right-hand side is point-wise defined on $[t_0,\infty)$, where Γ is the Gamma function.

Definition 15 (see [26]). Riemann-Liouville derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : [t_0,\infty) \longrightarrow \mathbb{R}$ can be written as

$$\begin{aligned} D_t^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}(t-s)^{-\alpha} f(s)\,\mathrm{d}s, \\ &\qquad t > t_0,\ n-1 < \alpha < n. \end{aligned} \tag{34}$$

The first and maybe the most important property of Riemann-Liouville fractional derivative is that, for $t > t_0$ and $\alpha > 0$, one has $D_t^\alpha(I^\alpha f(t)) = f(t)$, which means that Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order α.

It is important to define sectorial operator for the definition of mild solution of any fractional abstract equations. So, let us now give the definitions of sectorial linear operators and their associated solution operators.

Definition 16 ([74] sectorial operator). A closed and linear operator A is said to be sectorial of type ω and angle θ if there exist $0 < \theta < \pi/2$, $M > 0$, and $\omega \in \mathbb{R}$ such that its resolvent $\rho(A)$ exists outside the sector $\omega + S_\theta := \{\omega+\lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$ and

$$\left\|(\lambda - A)^{-1}\right\| \le \frac{M}{|\lambda-\omega|}, \quad \lambda \notin \omega + S_\theta. \tag{35}$$

Sectorial operators are well studied in the literature, usually for the case $\omega = 0$. For a recent reference including several examples and properties we refer the reader to [74]. Note that an operator A is sectorial of type ω if and only if $\omega I - A$ is sectorial of type 0.

Definition 17 (see [75]). Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A

the generator of a solution operator if there are $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \longrightarrow \mathbb{L}(X)$ such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subseteq \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x \, dt, \quad \operatorname{Re}\lambda > \omega, \ x \in X. \tag{36}$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Note that if A is sectorial of type ω with $0 \le \theta \le \pi(1-\alpha/2)$, then A is the generator of a solution operator given by

$$S_\alpha(t) := \frac{1}{2\pi i} \int_\gamma e^{-\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} \, d\lambda, \tag{37}$$

where γ is a suitable path lying outside the sector $\omega + \Sigma_\theta$ (cf. [74]).

Very recently, Cuesta in [74](Theorem 1) has proved that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $0 \le \theta < \pi(1-\alpha/2)$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\|_{\mathbb{L}(X)} \le \frac{CM}{1 + |\omega| t^\alpha} \quad \text{for } t \ge 0. \tag{38}$$

In the border case $\alpha = 1$, this is analogous to saying that A is the generator of a exponentially stable C_0-semigroup. The main difference is that in the case $\alpha > 1$ the solution family $S_\alpha(t)$ decays like $t^{-\alpha}$. Cuesta's result proves that $S_\alpha(t)$ is, in fact, integrable.

In the following, we present the following compactness criterion, which is a special case of the general compactness result of Theorem 2.1 in [76].

Lemma 18 (see [76]). *A set $D \subset C_0(\mathbb{R}, X)$ is relatively compact if*

(1) *D is equicontinuous;*

(2) $\lim_{|t|\longrightarrow\infty} x(t) = 0$ *uniformly for $x \in D$;*

(3) *the set $D(t) := \{x(t) : x \in D\}$ is relatively compact in X for every $t \in \mathbb{R}$.*

The following Krasnoselskii's fixed point theorem plays a key role in the proofs of our main results, which can be found in many books.

Lemma 19 (see [77]). *Let U be a bounded closed and convex subset of X and J_1, J_2 be maps of U into X such that $J_1 x + J_2 y \in U$ for every pair $x, y \in U$. If J_1 is a contraction and J_2 is completely continuous, then $J_1 x + J_2 x = x$ has a solution on U.*

3. Asymptotically Almost Automorphic Mild Solutions

In this section, we study the existence of asymptotically almost automorphic mild solutions for the semilinear fractional differential equations of the form

$$\mathrm{D}_t^\alpha x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t, x(t), Bx(t)), \quad t \in \mathbb{R}, \ 1 < \alpha < 2, \tag{39}$$

where $A : D(A) \subset X \longrightarrow X$ is a linear densely defined operator of sectorial type of $\omega < 0$ on a complex Banach space X, $B : X \longrightarrow X$ is a bounded linear operator and $F : \mathbb{R} \times X \times X \longrightarrow X$, and $(t, x, y) \longrightarrow F(t, x, y)$ is a given function to be specified later. The fractional derivative D_t^α is to be understood in Riemann-Liouville sense.

We recall the following definition that will be essential for us.

Definition 20 (see [63]). Assume that A generates an integrable solution operator $S_\alpha(t)$. A continuous function $x : \mathbb{R} \longrightarrow X$ satisfying the integral equation

$$x(t) = \int_{-\infty}^t S_\alpha(t-\sigma) F(\sigma, x(\sigma), Bx(\sigma)) \, d\sigma, \quad t \in \mathbb{R} \tag{40}$$

is called a mild solution on $\mathbb{R}$ to (39).

In the proofs of our results, we need the following auxiliary result.

Lemma 21. *Given $Y(t) \in AA(\mathbb{R}, X)$ and $Z(t) \in C_0(\mathbb{R}, X)$, let*

$$\Phi_1(t) := \int_{-\infty}^t S_\alpha(t-s) Y(s) \, ds, \quad \Phi_2(t) := \int_{-\infty}^t S_\alpha(t-s) Z(s) \, ds, \quad t \in \mathbb{R}. \tag{41}$$

Then $\Phi_1(t) \in AA(\mathbb{R}, X)$, $\Phi_2(t) \in C_0(\mathbb{R}, X)$.

Proof. Firstly, note that

$$\int_0^\infty \frac{1}{1 + |\omega| s^\alpha} ds = \frac{|\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \quad \text{for } 1 < \alpha < 2. \tag{42}$$

Then

$$\begin{aligned} \|\Phi_1(t)\| &= \left\| \int_{-\infty}^t S_\alpha(t-s) Y(s) \, ds \right\| \\ &= \left\| \int_0^{+\infty} S_\alpha(\tau) Y(t-\tau) \, d\tau \right\| \\ &\le CM \|Y\|_\infty \int_0^\infty \frac{1}{1 + |\omega| \tau^\alpha} d\tau \\ &= \frac{CM |\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \|Y\|_\infty, \end{aligned} \tag{43}$$

which implies that $\Phi_1(t)$ is well defined and continuous on $\mathbb{R}$. Since $Y(t) \in AA(\mathbb{R}, X)$, then for any $\varepsilon > 0$ and every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$, a function $\widetilde{Y}(t)$, and $N \in \mathbb{N}$ such that

$$\left\| Y(s + s_n) - \widetilde{Y}(s) \right\| < \varepsilon \quad \text{for each } n > N \text{ and every } s \in \mathbb{R}. \tag{44}$$

Define

$$\widetilde{\Phi_1}(t) := \int_{-\infty}^{t} T(t-s)\,\widetilde{Y}(s)\,\mathrm{d}s. \tag{45}$$

Then

$$\begin{aligned}
\left\|\Phi_1(t+s_n) - \widetilde{\Phi_1}(t)\right\| &= \left\|\int_{-\infty}^{t+s_n} S_\alpha(t+s_n-s)\,Y(s)\,\mathrm{d}s \right.\\
&\quad \left. - \int_{-\infty}^{t} S_\alpha(t-s)\,Y(s)\,\mathrm{d}s\right\| \\
&= \left\|\int_{0}^{+\infty} S_\alpha(s)\,Y(t+s_n-s)\,\mathrm{d}s \right.\\
&\quad \left. - \int_{0}^{+\infty} S_\alpha(s)\,Y(t-s)\,\mathrm{d}s\right\| \\
&\le CM\int_0^\infty \frac{1}{1+|\omega|\,s^\alpha}\left\|Y(s+s_n) - \widetilde{Y}(s)\right\|\mathrm{d}s \\
&\le \frac{CM\,|\omega|^{-1/\alpha}\,\pi\varepsilon}{\alpha\sin(\pi/\alpha)}
\end{aligned} \tag{46}$$

for each $n > N$ and every $t \in \mathbb{R}$. This implies that

$$\widetilde{\Phi_1}(t) = \lim_{n\longrightarrow\infty}\Phi_1(t+s_n) \tag{47}$$

is well defined for each $t \in \mathbb{R}$.

By a similar argument one can obtain

$$\lim_{n\longrightarrow\infty}\widetilde{\Phi_1}(t-s_n) = \Phi_1(t) \quad \text{for each } t \in \mathbb{R}. \tag{48}$$

Thus $\Phi_1(t) \in AA(\mathbb{R}, X)$.

Since $Z(t) \in C_0(\mathbb{R}, X)$, one can choose an $N_1 > 0$ such that $\|Z(t)\| < \varepsilon$ for all $t > N_1$. This enables us to conclude that, for all $t > N_1$,

$$\begin{aligned}
\|\Phi_2(t)\| &\le \left\|\int_{-\infty}^{N_1} S_\alpha(t-s)\,Z(s)\,\mathrm{d}s\right\| \\
&\quad + \left\|\int_{N_1}^{t} S_\alpha(t-s)\,Z(s)\,\mathrm{d}s\right\| \\
&\le CM\,\|Z\|_\infty \int_{-\infty}^{N_1}\frac{1}{1+|\omega|\,(t-s)^\alpha}\mathrm{d}s \\
&\quad + \varepsilon CM\int_{N_1}^{t}\frac{1}{1+|\omega|\,(t-s)^\alpha}\mathrm{d}s \\
&\le \frac{CM\,\|Z\|_\infty}{|\omega|}\int_{-\infty}^{N_1}\frac{1}{(t-s)^\alpha}\mathrm{d}s \\
&\quad + \frac{CM\,|\omega|^{-1/\alpha}\,\pi\varepsilon}{\alpha\sin(\pi/\alpha)} \\
&\le \frac{CM\,\|Z\|_\infty}{|\omega|}\frac{1}{(\alpha-1)\,(t-N_1)^{\alpha-1}} \\
&\quad + \frac{CM\,|\omega|^{-1/\alpha}\,\pi\varepsilon}{\alpha\sin(\pi/\alpha)},
\end{aligned} \tag{49}$$

which implies

$$\lim_{t\longrightarrow+\infty}\|\Phi_2(t)\| = 0. \tag{50}$$

On the other hand, from $Z(t) \in C_0(\mathbb{R}, X)$ it follows that there exists an $N_2 > 0$ such that $\|Z(t)\| < \varepsilon$ for all $t < -N_2$. This enables us to conclude that, for all $t < -N_2$,

$$\begin{aligned}
\|\Phi_2(t)\| &= \left\|\int_{-\infty}^{t} S_\alpha(t-s)\,Z(s)\,\mathrm{d}s\right\| \\
&\le \int_{-\infty}^{t}\|S_\alpha(t-s)\|\,\|Z(s)\|\,\mathrm{d}s \\
&\le CM\varepsilon\int_{-\infty}^{t}\frac{1}{1+|\omega|\,(t-s)^\alpha}\mathrm{d}s \\
&= \frac{CM\,|\omega|^{-1/\alpha}\,\pi\varepsilon}{\alpha\sin(\pi/\alpha)},
\end{aligned} \tag{51}$$

which implies

$$\lim_{t\longrightarrow-\infty}\|\Phi_2(t)\| = 0. \tag{52}$$

□

Now we are in position to state and prove our first main result. To prove our main result, let us introduce the following assumptions:

(H_1) $F(t,x,y) = F_1(t,x,y) + F_2(t,x,y) \in AAA(\mathbb{R}\times X\times X, X)$ with

$$\begin{aligned}
&F_1(t,x,y) \in AA(\mathbb{R}\times X\times X, X), \\
&F_2(t,x,y) \in C_0(\mathbb{R}\times X\times X, X)
\end{aligned} \tag{53}$$

and there exists a constant $L > 0$ such that, for all $t \in \mathbb{R}$ and $x_1, x_2, y_1, y_2 \in X$,

$$\begin{aligned}
&\|F_1(t,x_1,x_2) - F_1(t,y_1,y_2)\| \\
&\quad \le L\left(\|x_1-y_1\| + \|x_2-y_2\|\right).
\end{aligned} \tag{54}$$

(H_2) There exist a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that, for all $t \in \mathbb{R}$ and $x, y \in X$ with $\|x\| + \|y\| \le r$,

$$\begin{aligned}
&\|F_2(t,x,y)\| \le \beta(t)\,\Phi(r) \\
&\text{and } \liminf_{r\longrightarrow+\infty}\frac{\Phi(r)}{r} = \rho_1.
\end{aligned} \tag{55}$$

Remark 22. Assuming that $F(t,x,y)$ satisfies the assumption (H_1), it is noted that $F(t,x,y)$ does not have to meet the

Lipschitz continuity with respect to x and y. Such class of asymptotically almost automorphic functions $F(t,x,y)$ are more complicated than those with Lipschitz continuity with respect to x and y and little is known about them.

Let $\beta(t)$ be the function involved in assumption (H_2). Define

$$\sigma(t) := \int_{-\infty}^{t} \frac{\beta(s)}{1+|\omega|(t-s)^{\alpha}} ds, \quad t \in \mathbb{R}. \tag{56}$$

Lemma 23. $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$.

Proof. Since $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$, one can choose a $T_1 > 0$ such that $\|\beta(t)\| < \varepsilon$ for all $t > T_1$. This enables us to conclude that, for all $t > T_1$,

$$\begin{aligned}
\|\sigma(t)\| &\le \left\| \int_{-\infty}^{T_1} \frac{\beta(s)}{1+|\omega|(t-s)^{\alpha}} ds \right\| \\
&\quad + \left\| \int_{T_1}^{t} \frac{\beta(s)}{1+|\omega|(t-s)^{\alpha}} ds \right\| \\
&\le \|\beta\|_{\infty} \int_{-\infty}^{T_1} \frac{1}{1+|\omega|(t-s)^{\alpha}} ds \\
&\quad + \varepsilon \int_{T_1}^{t} \frac{1}{1+|\omega|(t-s)^{\alpha}} ds \\
&\le \frac{\|\beta\|_{\infty}}{|\omega|} \int_{-\infty}^{T_1} \frac{1}{(t-s)^{\alpha}} ds + \frac{|\omega|^{-1/\alpha} \pi \varepsilon}{\alpha \sin(\pi/\alpha)} \\
&\le \frac{\|\beta\|_{\infty}}{|\omega|} \frac{1}{(\alpha-1)(t-T_1)^{\alpha-1}} + \frac{|\omega|^{-1/\alpha} \pi \varepsilon}{\alpha \sin(\pi/\alpha)},
\end{aligned} \tag{57}$$

which implies

$$\lim_{t \longrightarrow +\infty} \|\sigma(t)\| = 0. \tag{58}$$

On the other hand, from $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ it follows that there exists a $T_2 > 0$ such that $\|\beta(t)\| < \varepsilon$ for all $t < -T_2$. This enables us to conclude that, for all $t < -T_2$,

$$\begin{aligned}
\|\sigma(t)\| &= \left\| \int_{-\infty}^{t} \frac{\beta(s)}{1+|\omega|(t-s)^{\alpha}} ds \right\| \\
&\le \varepsilon \int_{-\infty}^{t} \frac{1}{1+|\omega|(t-s)^{\alpha}} ds = \frac{|\omega|^{-1/\alpha} \pi \varepsilon}{\alpha \sin(\pi/\alpha)},
\end{aligned} \tag{59}$$

which implies

$$\lim_{t \longrightarrow -\infty} \|\sigma(t)\| = 0. \tag{60}$$

□

Theorem 24. *Assume that A is sectorial of type $\omega < 0$. Let $F : \mathbb{R} \times X \times X \longrightarrow X$ satisfy the hypotheses (H_1) and (H_2). Put $\rho_2 := \sup_{t \in \mathbb{R}} \sigma(t)$. Then (39) has at least one asymptotically almost automorphic mild solution provided that*

$$\begin{aligned}
&\frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \\
&\quad + CM\left(1+\|B\|_{\mathbb{L}(X)}\right)\rho_1 \rho_2 < 1.
\end{aligned} \tag{61}$$

Proof. The proof is divided into the following five steps.

Step 1. Define a mapping Λ on $AA(\mathbb{R}, X)$ by

$$(\Lambda v)(t) = \int_{-\infty}^{t} S_{\alpha}(t-s) F_1(s, v(s), Bv(s)) ds, \quad t \in \mathbb{R} \tag{62}$$

and prove Λ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

Firstly, since the function $s \longrightarrow F_1(s, v(s), Bv(s))$ is bounded in $\mathbb{R}$ and

$$\begin{aligned}
\|[\Lambda v](t)\| &\le \int_{-\infty}^{t} \|S_{\alpha}(t-s)\| \|F_1(s, v(s), Bv(s))\| ds \\
&\le CM \int_{-\infty}^{t} \frac{1}{1+|\omega|(t-s)^{\alpha}} \|F_1(s, v(s), Bv(s))\| ds \\
&\le CM \|F_1\|_{\infty} \int_{-\infty}^{t} \frac{1}{1+|\omega|(t-s)^{\alpha}} ds \\
&= \frac{CML|\omega|^{-1/\alpha} \pi \|F_1\|_{\infty}}{\alpha \sin(\pi/\alpha)},
\end{aligned} \tag{63}$$

this implies that $(\Lambda v)(t)$ exists. Moreover from $F_1(t,x,y) \in AA(\mathbb{R} \times X \times X, X)$ satisfying (54), together with Lemma 6 and Remark 7, it follows that

$$F_1(\cdot, v(\cdot), Bv(\cdot)) \in AA(\mathbb{R}, X) \quad \text{for every } v(\cdot) \in AA(\mathbb{R}, X). \tag{64}$$

This, together with Lemma 21, implies that Λ is well defined and maps $AA(\mathbb{R}, X)$ into itself.

In the sequel, we verify that Λ is continuous.

Let $v_n(t), v(t)$ be in $AA(\mathbb{R}, X)$ with $v_n(t) \longrightarrow v(t)$ as $n \longrightarrow \infty$; then one has

$$\begin{aligned}
\|[\Lambda v_n](t) - [\Lambda v](t)\| &= \left\| \int_{-\infty}^{t} S_{\alpha}(t-s) \right. \\
&\quad \cdot [F_1(s, v_n(s), Bv_n(s)) \\
&\quad \left. - F_1(s, v(s), Bv(s))] ds \right\| \le L \int_{-\infty}^{t} \|S_{\alpha}(t-s)\| \\
&\quad \cdot [\|v_n(s) - v(s)\| + \|Bv_n(s) - Bv(s)\|] ds \\
&\le CML \int_{-\infty}^{t} \frac{1}{1+|\omega|(t-s)^{\alpha}} \left(1+\|B\|_{\mathbb{L}(X)}\right) \|v_n(s) \\
&\quad - v(s)\| ds \le CML\left(1+\|B\|_{\mathbb{L}(X)}\right) \|v_n - v\|_{\infty}
\end{aligned}$$

$$\cdot \int_{-\infty}^{t} \frac{1}{1+|\omega|(t-s)^{\alpha}} \mathrm{d}s = \frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \|v_n - v\|_{\infty}. \tag{65}$$

Therefore, as $n \longrightarrow \infty$ and $\Lambda v_n \longrightarrow \Lambda v$, hence Λ is continuous.

Next, we prove that Λ is a contraction on $AA(\mathbb{R}, X)$ and has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

In fact, let $v_1(t), v_2(t)$ be in $AA(\mathbb{R}, X)$, and similar to the above proof of the continuity of Λ, one has

$$\|[\Lambda v_1](t) - [\Lambda v_2](t)\| \le \frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \|v_1 - v_2\|_{\infty}, \tag{66}$$

which implies

$$\|[\Lambda v_1](t) - [\Lambda v_2](t)\|_{\infty} \le \frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \|v_1 - v_2\|_{\infty}. \tag{67}$$

Together with (61), this proves that Λ is a contraction on $AA(\mathbb{R}, X)$. Thus, Banach's fixed point theorem implies that Λ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

Step 2. Set

$$\Omega_r := \{\omega(t) \in C_0(\mathbb{R}, X) : \|\omega(t)\| \le r\}. \tag{68}$$

For the above $v(t)$, define $\Gamma := \Gamma^1 + \Gamma^2$ on $C_0(\mathbb{R}, X)$ as

$$\begin{aligned}\left(\Gamma^1\omega\right)(t) &= \int_{-\infty}^{t} S_{\alpha}(t-s) \cdot [F_1(s, v(s)+\omega(s), B(v(s)+\omega(s))) - F_1(s, v(s), Bv(s))]\,\mathrm{d}s, \\ \left(\Gamma^2\omega\right)(t) &= \int_{-\infty}^{t} S_{\alpha}(t-s) F_2(s, v(s) + \omega(s), B(v(s)+\omega(s)))\,\mathrm{d}s\end{aligned} \tag{69}$$

and prove that Γ maps Ω_{k_0} into itself, where k_0 is a given constant.

Firstly, from (54) it follows that, for all $s \in \mathbb{R}$ and $\omega(s) \in X$,

$$\begin{aligned}&\|F_1(s, v(s)+\omega(s), B(v(s)+\omega(s))) - F_1(s, v(s), Bv(s))\| \le L[\|\omega(s)\| + \|B\omega(s)\|] \\ &\le L\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega(s)\|,\end{aligned} \tag{70}$$

which implies that

$$F_1(\cdot, v(\cdot)+\omega(\cdot), B(v(\cdot)+\omega(\cdot))) - F_1(\cdot, v(\cdot), Bv(\cdot)) \in C_0(\mathbb{R}, X) \quad \text{for every } \omega(\cdot) \in C_0(\mathbb{R}, X). \tag{71}$$

According to (55), one has

$$\begin{aligned}&\|F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\| \le \beta(s) \cdot \Phi\left(\|\omega(s) + B\omega(s)\| + \sup_{s\in\mathbb{R}}\|v(s)+Bv(s)\|\right) \\ &\le \beta(s)\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega(s)\| + \left(1+\|B\|_{\mathbb{L}(X)}\right)\sup_{s\in\mathbb{R}}\|v(s)\|\right) = \beta(s) \cdot \Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left[\|\omega(s)\| + \sup_{s\in\mathbb{R}}\|v(s)\|\right]\right)\end{aligned} \tag{72}$$

for all $s \in \mathbb{R}$ and $\omega(s) \in X$ with $\|\omega(s)\| \le r$; then

$$F_2(\cdot, v(\cdot)+\omega(\cdot), B(v(\cdot)+\omega(\cdot))) \in C_0(\mathbb{R}, X) \quad \text{as } \beta(\cdot) \in C_0\left(\mathbb{R}, \mathbb{R}^+\right). \tag{73}$$

Those, together with Lemma 21, yield that Γ is well defined and maps $C_0(\mathbb{R}, X)$ into itself.

On the other hand, in view of (55) and (61) it is not difficult to see that there exists a constant $k_0 > 0$ such that

$$\frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} k_0 + CM\rho_2\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0 + \sup_{s\in\mathbb{R}}\|v(s)\|\right)\right) \le k_0. \tag{74}$$

This enables us to conclude that, for any $t \in \mathbb{R}$ and $\omega_1(t)$, $\omega_2(t) \in \Omega_{k_0}$,

$$\begin{aligned}&\left\|\left(\Gamma^1\omega_1\right)(t) + \left(\Gamma^2\omega_2\right)(t)\right\| \le \left\|\int_{-\infty}^{t} S_{\alpha}(t-s) \cdot [F_1(s, v(s)+\omega_1(s), B(v(s)+\omega_1(s))) - F_1(s, v(s), Bv(s))]\,\mathrm{d}s\right\| \\ &+ \left\|\int_{-\infty}^{t} S_{\alpha}(t-s) \cdot F_2(s, v(s)+\omega_2(s), B(v(s)+\omega_2(s)))\,\mathrm{d}s\right\| \\ &\le \int_{-\infty}^{t} \|S_{\alpha}(t-s)\| \|F_1(s, v(s) + \omega_1(s), B(v(s)+\omega_1(s))) - F_1(s, v(s), Bv(s))\|\,\mathrm{d}s + \int_{-\infty}^{t} \|S_{\alpha}(t-s)\| \cdot \|F_2(s, v(s)+\omega_2(s), B(v(s)+\omega_2(s)))\|\,\mathrm{d}s \\ &\le CM\int_{-\infty}^{t} \frac{1}{1+|\omega|(t-s)^{\alpha}} [\|\omega_1(s)\|\end{aligned}$$

$$
\begin{aligned}
&+ \|B\omega_1(s)\|]\,\mathrm{d}s \\
&+ CM\int_{-\infty}^{t}\frac{\beta(s)}{1+|\omega|(t-s)^{\alpha}}\Phi\left(\|\omega_2(s)\|+\|B\omega_2(s)\|\right. \\
&\left.+\|v(s)\|+\|Bv(s)\|\right)\mathrm{d}s \le CML\left(1+\|B\|_{\mathbb{L}(X)}\right) \\
&\cdot\|\omega_1\|_{\infty}\int_{-\infty}^{t}\frac{1}{1+|\omega|(t-s)^{\alpha}}\mathrm{d}s + CM\sigma(t)\Phi\left(\left(1\right.\right. \\
&\left.\left.+\|B\|_{\mathbb{L}(X)}\right)\left(\|\omega_2\|_{\infty}+\|v(s)\|_{\infty}\right)\right) \\
&= \frac{CML|\omega|^{-1/\alpha}\pi\left(1+\|B\|_{\mathbb{L}(X)}\right)}{\alpha\sin(\pi/\alpha)}\|\omega\|_{\infty} \\
&+ CM\rho_2\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(\|\omega_2\|_{\infty}+\|v(s)\|_{\infty}\right)\right) \\
&\le \frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}k_0 + CM\rho_2\Phi\left(\left(1\right.\right. \\
&\left.\left.+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+\|v(s)\|_{\infty}\right)\right) \le k_0,
\end{aligned}
\tag{75}
$$

which implies that $(\Gamma^1\omega_1)(t)+(\Gamma^2\omega_2)(t)\in\Omega_{k_0}$. Thus Γ maps Ω_{k_0} into itself.

Step 3. Show that Γ^1 is a contraction on Ω_{k_0}.

In fact, for any $\omega_1(t),\omega_2(t)\in\Omega_{k_0}$ and $t\in\mathbb{R}$, from (54) it follows that

$$
\begin{aligned}
&\|[F_1(s,v(s)+\omega_1(s),B(v(s)+\omega_1(s))) \\
&\quad - F_1(s,v(s),Bv(s))] \\
&\quad - [F_1(s,v(s)+\omega_2(s),B(v(s)+\omega_2(s))) \\
&\quad - F_1(s,v(s),Bv(s))]\| \le L[\|\omega_1(s)-\omega_2(s)\| \\
&\quad + \|B\omega_1(s)-B\omega_2(s)\|] \le L\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega_1(s) \\
&\quad - \omega_2(s)\|.
\end{aligned}
\tag{76}
$$

Thus

$$
\begin{aligned}
&\left\|\left(\Gamma^1\omega_1\right)(t)-\left(\Gamma^1\omega_2\right)(t)\right\| = \left\|\int_{-\infty}^{t}S_{\alpha}(t-s)\left[\left(F_1(s,v(s)+\omega_1(s),B(v(s)+\omega_1(s)))-F_1(s,v(s),Bv(s))\right)\right.\right. \\
&\quad\left.\left.-\left(F_1(s,v(s)+\omega_2(s),B(v(s)+\omega_2(s)))-F_1(s,v(s),Bv(s))\right)\right]\mathrm{d}s\right\| \le L\int_{-\infty}^{t}\|S_{\alpha}(t-s)\|\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega_1(s) \\
&\quad-\omega_2(s)\|\,\mathrm{d}s \le CML\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega_1-\omega_2\|_{\infty}\int_{-\infty}^{t}\frac{1}{1+|\omega|(t-s)^{\alpha}}\mathrm{d}s = \frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\|\omega_1 \\
&\quad-\omega_2\|_{\infty},
\end{aligned}
\tag{77}
$$

which implies that

$$
\begin{aligned}
&\left\|\left(\Gamma^1\omega_1\right)(t)-\left(\Gamma^1\omega_2\right)(t)\right\|_{\infty} \\
&\quad\le \frac{CML\left(1+\|B\|_{\mathbb{L}(X)}\right)|\omega|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\|\omega_1-\omega_2\|_{\infty}.
\end{aligned}
\tag{78}
$$

Thus, in view of (61), one obtains the conclusion.

Step 4. Show that Γ^2 is completely continuous on Ω_{k_0}.

Given $\varepsilon>0$. Let $\{\omega_k\}_{k=1}^{+\infty}\subset\Omega_{k_0}$ with $\omega_k\longrightarrow\omega_0$ in $C_0(\mathbb{R},X)$ as $k\longrightarrow+\infty$. Since $\sigma(t)\in C_0(\mathbb{R},\mathbb{R}^+)$, one may choose a $t_1>0$ big enough such that, for all $t\ge t_1$,

$$
\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+\|v\|_{\infty}\right)\right)\sigma(t) < \frac{\varepsilon}{3CM}.
\tag{79}
$$

Also, in view of (H_1), we have

$$
\begin{aligned}
&F_2(s,v(s)+\omega_k(s),B(v(s)+\omega_k(s))) \\
&\quad\longrightarrow F_2(s,v(s)+\omega_0(s),B(v(s)+\omega_0(s)))
\end{aligned}
\tag{80}
$$

for all $s\in(-\infty,t_1]$ as $k\longrightarrow+\infty$ and

$$
\begin{aligned}
&\|F_2(\cdot,v(\cdot)+\omega_k(\cdot),B(v(\cdot)+\omega_k(\cdot))) \\
&\quad - F_2(\cdot,v(\cdot)+\omega_0(\cdot),B(v(\cdot)+\omega_0(\cdot)))\| \\
&\quad\le 2\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+\|v\|_{\infty}\right)\right)\beta(\cdot) \\
&\quad\in L^1(-\infty,t_1].
\end{aligned}
\tag{81}
$$

Hence, by the Lebesgue dominated convergence theorem we deduce that there exists an $N>0$ such that

$$
\begin{aligned}
&CM\int_{-\infty}^{t_1}\frac{1}{1+|\omega|(t-s)^{\alpha}}\|F_2(s,v(s) \\
&\quad+\omega_k(s),B(v(s)+\omega_k(s)))-F_2(s,v(s) \\
&\quad+\omega_0(s),B(v(s)+\omega_0(s)))\|\,\mathrm{d}s \le \frac{\varepsilon}{3}
\end{aligned}
\tag{82}
$$

whenever $k \geq N$. Thus

$$\begin{aligned}
&\left\|\left(\Gamma^2\omega_k\right)(t)-\left(\Gamma^2\omega_0\right)(t)\right\| = \left\|\int_{-\infty}^{t} S_\alpha(t-s) F_2(s, v(s) \right. \\
&+ \omega_k(s), B(v(s)+\omega_k(s))) \mathrm{d}s - \int_{-\infty}^{t} S_\alpha(t \\
&\left. - s) F_2(s, v(s)+\omega_0(s), B(v(s)+\omega_0(s))) \mathrm{d}s\right\| \\
&\leq CM \int_{-\infty}^{t_1} \frac{1}{1+|\omega|(t-s)^\alpha} \|F_2(s, v(s) \\
&+ \omega_k(s), B(v(s)+\omega_k(s))) - F_2(s, v(s) \\
&+ \omega_0(s), B(v(s)+\omega_0(s)))\| \mathrm{d}s + 2CM\Phi((1 \\
&+ \|B\|_{\mathbb{L}(X)})(k_0+\|v\|_\infty)) \int_{t_1}^{\max\{t,t_1\}} \frac{\beta(s)}{1+|\omega|(t-s)^\alpha} \mathrm{d}s \\
&\leq CM \int_{-\infty}^{t_1} \frac{1}{1+|\omega|(t-s)^\alpha} \|F_2(s, v(s) \\
&+ \omega_k(s), B(v(s)+\omega_k(s))) - F_2(s, v(s) \\
&+ \omega_0(s), B(v(s)+\omega_0(s)))\| \mathrm{d}s + 2CM\Phi((1 \\
&+ \|B\|_{\mathbb{L}(X)})(k_0+\|v\|_\infty)) \sigma(t) \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon
\end{aligned} \tag{83}$$

whenever $k \geq N$. Accordingly, Γ^2 is continuous on Ω_{k_0}.

In the sequel, we consider the compactness of Γ^2.

Set $B_r(X)$ for the closed ball with center at 0 and radius r in X, $V = \Gamma^2(\Omega_{k_0})$, and $z(t) = \Gamma^2(u(t))$ for $u(t) \in \Omega_{k_0}$. First, for all $\omega(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$,

$$\begin{aligned}
&\left\|\left(\Gamma^2\omega\right)(t)\right\| = \left\|\int_{-\infty}^{t} S_\alpha(t-s) \right. \\
&\cdot F_2(s, v(s)+\omega(s), B(v(s)+\omega(s))) \mathrm{d}s\Big\| \\
&\leq CM\sigma(t)\Phi((1+\|B\|_{\mathbb{L}(X)})(k_0+\|v\|_\infty)),
\end{aligned} \tag{84}$$

and in view of $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$, which follows from Lemma 23, one concludes that

$$\lim_{|t|\longrightarrow+\infty} \left(\Gamma^2\omega\right)(t) = 0 \quad \text{uniformly for } \omega(t) \in \Omega_{k_0}. \tag{85}$$

As

$$\begin{aligned}
&\left\|\left(\Gamma^2\omega\right)(t)\right\| = \left\|\int_{-\infty}^{t} S_\alpha(t-s) F_2(s, v(s) \right. \\
&\left. + \omega(s), B(v(s)+\omega(s))) \mathrm{d}s\right\| = \left\|\int_{0}^{+\infty} S_\alpha(\tau) \right. \\
&\cdot F_2(t-\tau, v(t-\tau) \\
&\left. + \omega(t-\tau), B(v(t-\tau)+\omega(t-\tau))) \mathrm{d}\tau\right\|.
\end{aligned} \tag{86}$$

Hence, given $\varepsilon_0 > 0$, one can choose a $\xi > 0$ such that

$$\begin{aligned}
&\left\|\int_{\xi}^{+\infty} S_\alpha(\tau) F_2(t-\tau, v(t-\tau) \right. \\
&\left. + \omega(t-\tau), B(v(t-\tau)+\omega(t-\tau))) \mathrm{d}\tau\right\| \\
&< \varepsilon_0.
\end{aligned} \tag{87}$$

Thus we get

$$z(t) \in \xi \overline{c\left(\left\{S_\alpha(\tau) F_2(\lambda, v(\lambda)+\omega(\lambda), B(v(\lambda)+\omega(\lambda))) : 0 \leq \tau \leq \xi,\ t-\xi \leq \lambda \leq \xi,\ \|\omega\|_\infty \leq r\right\}\right)} + B_{\varepsilon_0}(X), \tag{88}$$

where $c(K)$ denotes the convex hull of K. Using that $S_\alpha(\cdot)$ is strongly continuous, we infer that

$$\begin{aligned}
K = \{&S_\alpha(\tau) F_2(\lambda, v(\lambda)+\omega(\lambda), B(v(\lambda)+\omega(\lambda))) : 0 \\
&\leq \tau \leq \xi,\ t-\xi \leq \lambda \leq \xi,\ \|\omega\|_\infty \leq r\}
\end{aligned} \tag{89}$$

is a relatively compact set and $V \subset \xi\overline{c(K)} + B_{\varepsilon_0}(X)$, which implies that V is a relatively compact subset of X.

Next, we verify the equicontinuity of the set $\{(\Gamma^2\omega)(t) : \omega(t) \in \Omega_{k_0}\}$.

Let $k > 0$ be small enough and $t_1, t_2 \in \mathbb{R}$ and $\omega(t) \in \Omega_{k_0}$. Then by (55) we have

$$\begin{aligned}
&\left\|\left(\Gamma^2\omega\right)(t_2) - \left(\Gamma^2\omega\right)(t_1)\right\| \leq \int_{t_1}^{t_2} \|S_\alpha(t_2-s) \\
&\cdot F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\| \mathrm{d}s \\
&+ \int_{-\infty}^{t_1-k} \|[S_\alpha(t_2-s) - S_\alpha(t_1-s)] \\
&\cdot F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\| \mathrm{d}s \\
&+ \int_{t_1-k}^{t_1} \|[S_\alpha(t_2-s) - S_\alpha(t_1-s)] \\
&\cdot F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\| \mathrm{d}s \\
&\leq CM\Phi((1+\|B\|_{\mathbb{L}(X)})(k_0+\|v\|_\infty)) \\
&\cdot \int_{t_1}^{t_2} \frac{\beta(s)}{1+|\omega|(t_2-s)^\alpha} \mathrm{d}s + \Phi((1+\|B\|_{\mathbb{L}(X)})(k_0 \\
&+ \|v\|_\infty)) \sup_{s\in[-\infty,t_1-k]} \|S_\alpha(t_2-s) - S_\alpha(t_1-s)\| \\
&\cdot \int_{-\infty}^{t_1-k} \beta(s) \mathrm{d}s + CM\Phi((1+\|B\|_{\mathbb{L}(X)})(k_0
\end{aligned}$$

$$+ \|v\|_\infty)) \int_{t_1-k}^{t_1} \left(\frac{\beta(s)}{1+|\omega|(t_2-s)^\alpha} + \frac{\beta(s)}{1+|\omega|(t_1-s)^\alpha} \right) ds \longrightarrow 0 \quad \text{as } t_2 - t_1 \longrightarrow 0,\ k \longrightarrow 0, \tag{90}$$

which implies the equicontinuity of the set $\{(\Gamma^2\omega)(t) : \omega(t) \in \Omega_{k_0}\}$.

Now an application of Lemma 18 justifies the compactness of Γ^2.

Step 5. Show that (39) has at least one asymptotically almost automorphic mild solution.

Firstly, the complete continuity of Γ^2, together with the results of Steps 2 and 3 as well as Lemma 19, yields that Γ has at least one fixed point $\omega(t) \in \Omega_{k_0}$; furthermore $\omega(t) \in C_0(\mathbb{R}, X)$.

Then, consider the following coupled system of integral equations:

$$\begin{aligned} v(t) &= \int_{-\infty}^{t} S_\alpha(t-s) F_1(s, v(s), Bv(s))\, ds, \quad t \in \mathbb{R}, \\ \omega(t) &= \int_{-\infty}^{t} S_\alpha(t-s) \\ &\quad \cdot [F_1(s, v(s)+\omega(s), B(v(s)+\omega(s))) \\ &\quad - F_1(s, v(s), Bv(s))]\, ds + \int_{-\infty}^{t} S_\alpha(t-s) \\ &\quad \cdot F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\, ds, \\ &\qquad t \in \mathbb{R}. \end{aligned} \tag{91}$$

From the result of Step 1, together with the above fixed point $\omega(t) \in C_0(\mathbb{R}, X)$, it follows that

$$(v(t), \omega(t)) \in AA(\mathbb{R}, X) \times C_0(\mathbb{R}, X) \tag{92}$$

is a solution to system (91). Thus

$$x(t) := v(t) + \omega(t) \in AAA(\mathbb{R}, X) \tag{93}$$

and it is a solution to the integral equation

$$x(t) = \int_{-\infty}^{t} S_\alpha(t-s) F(s, x(s), Bx(s))\, ds, \quad t \in \mathbb{R}; \tag{94}$$

that is, $x(t)$ is an asymptotically almost automorphic mild solution to (39). □

Taking $A = -\rho^\alpha I$ with $\rho > 0$ in (39), the above theorem gives the following corollary.

Corollary 25. *Let $F : \mathbb{R} \times X \times X \longrightarrow X$ satisfy (H_1) and (H_2). Put $\rho_2 := \sup_{t\in\mathbb{R}} \sigma(t)$. Then (39) admits at least one asymptotically almost automorphic mild solution whenever*

$$\frac{CL(1+\|B\|_{\mathbb{L}(X)})\rho\pi}{\alpha \sin(\pi/\alpha)} + C(1+\|B\|_{\mathbb{L}(X)})\rho_1\rho_2 < 1. \tag{95}$$

Remark 26. It is interesting to note that the function $\alpha \longrightarrow \alpha\sin(\pi/\alpha)/\rho\pi$ is increasing from 0 to $2/\rho\pi$ in the interval $1 < \alpha < 2$. Therefore, with respect to condition (61), the class of admissible terms $F_1(t, x(t), Bx(t))$ is the best in the case $\alpha = 2$ and the worst in the case $\alpha = 1$.

Theorem 24 can be extended to the case of $F_1(t, x, y)$ being locally Lipschitz continuous with respect to x and y, where we have the following result.

(H_1') $F(t, x, y) = F_1(t, x, y) + F_2(t, x, y) \in AAA(\mathbb{R} \times X \times X, X)$ with

$$\begin{aligned} &F_1(t, x, y) \in AA(\mathbb{R} \times X \times X, X), \\ &F_2(t, x, y) \in C_0(\mathbb{R} \times X \times X, X) \end{aligned} \tag{96}$$

and for all $x_1, x_2, y_1, y_2 \in X$, $t \in \mathbb{R}$,

$$\begin{aligned} &\|F_1(t, x_1, x_2) - F_1(t, y_1, y_2)\| \\ &\quad \le L(t)(\|x_1 - y_1\| + \|x_2 - y_2\|), \end{aligned} \tag{97}$$

where $L(t)$ is a function on $\mathbb{R}$.

Theorem 27. *Assume that A is sectorial of type $\omega < 0$. Let $F : \mathbb{R} \times X \times X \longrightarrow X$ satisfy the hypotheses (H_1') and (H_2) with $L(t) \in BC(\mathbb{R}, \mathbb{R}^+)$. Put $\rho_2 := \sup_{t\in\mathbb{R}} \sigma(t)$. Let $\|L\| = \sup_{t\in\mathbb{R}} \int_t^{t+1} L(s)ds$. Then (39) has at least one asymptotically almost automorphic mild solution provided that*

$$\begin{aligned} &\frac{CML\|L\||\omega|^{-1/\alpha}\pi(1+\|B\|_{\mathbb{L}(X)})}{\alpha\sin(\pi/\alpha)} \\ &\quad + CM\rho_1\rho_2(1+\|B\|_{\mathbb{L}(X)}) < 1. \end{aligned} \tag{98}$$

Proof. The proof is divided into the following five steps.

Step 1. Define a mapping Λ on $AA(\mathbb{R}, X)$ by (62) and prove that Λ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

Firstly, similar to the proof in Step 1 of Theorem 24, we can prove that $(\Lambda v)(t)$ exists. Moreover from $F_1(t, x, y) \in AA(\mathbb{R} \times X \times X, X)$ satisfying (97), together with Lemma 6 and Remark 8, it follows that

$$F_1(\cdot, v(\cdot), Bv(\cdot)) \in AA(\mathbb{R}, X) \quad \text{for every } v(\cdot) \in AA(\mathbb{R}, X). \tag{99}$$

This, together with Lemma 21, implies that Λ is well defined and maps $AP(\mathbb{R}, X)$ into itself.

In the sequel, we verify that Λ is continuous.

Let $v_n(t), v(t)$ be in $AA(\mathbb{R}, X)$ with $v_n(t) \longrightarrow v(t)$ as $n \longrightarrow \infty$; then one has

$$\begin{aligned} \|[\Lambda v_n](t) - [\Lambda v](t)\| &= \Big\| \int_{-\infty}^{t} S_\alpha(t-s) \\ &\quad \cdot [F_1(s, v_n(s), Bv_n(s)) \\ &\quad - F_1(s, v(s), Bv(s))]\, ds \Big\| \le \int_{-\infty}^{t} L(s) \|S_\alpha(t \\ &\quad - s)\| [\|v_n(s) - v(s)\| + \|Bv_n(s) - Bv(s)\|]\, ds \end{aligned}$$

$$\begin{aligned}
&\le CM\int_{-\infty}^{t}\frac{L(s)}{1+|\omega|(t-s)^{\alpha}}\left(1+\|B\|_{\mathbb{L}(X)}\right)\|v_n(s)\\
&-v(s)\|\,ds\le CM\left(1+\|B\|_{\mathbb{L}(X)}\right)\\
&\cdot\left(\sum_{m=0}^{+\infty}\int_{t-(m+1)}^{t-m}\frac{L(s)}{1+|\omega|(t-s)^{\alpha}}ds\right)\|v_n-v\|_{\infty}\\
&\le CM\left(1+\|B\|_{\mathbb{L}(X)}\right)\\
&\cdot\left(\sum_{m=0}^{+\infty}\frac{1}{1+|\omega|m^{\alpha}}\int_{t-(m+1)}^{t-m}L(s)\,ds\right)\|v_n-v\|_{\infty}\\
&\le\frac{CM\|L\||\omega|^{-1/\alpha}\pi\left(1+\|B\|_{\mathbb{L}(X)}\right)}{\alpha\sin(\pi/\alpha)}\|v_n-v\|_{\infty}.
\end{aligned}\tag{100}$$

Therefore, as $n\longrightarrow\infty$ and $\Lambda v_n\longrightarrow\Lambda v$, hence Λ is continuous.

Next, we prove that Λ is a contraction on $AA(\mathbb{R},X)$ and has a unique fixed point $v(t)\in AA(\mathbb{R},X)$.

In fact, for $v_1(t),v_2(t)$ in $AA(\mathbb{R},X)$, similar to the above proof of the continuity of Λ, one has

$$\begin{aligned}
&\|(\Lambda v_1)(t)-(\Lambda v_2)(t)\|\\
&\quad\le\frac{CM\|L\||\omega|^{-1/\alpha}\pi\left(1+\|B\|_{\mathbb{L}(X)}\right)}{\alpha\sin(\pi/\alpha)}\|v_1-v_2\|_{\infty},
\end{aligned}\tag{101}$$

which implies that

$$\begin{aligned}
&\|(\Lambda v_1)(t)-(\Lambda v_2)(t)\|_{\infty}\\
&\quad\le\frac{CM\|L\||\omega|^{-1/\alpha}\pi\left(1+\|B\|_{\mathbb{L}(X)}\right)}{\alpha\sin(\pi/\alpha)}\|v_1-v_2\|_{\infty}.
\end{aligned}\tag{102}$$

Hence, by (98), together with the contraction principle, Λ has a unique fixed point $v(t)\in AA(\mathbb{R},X)$.

Step 2. Set

$$\Omega_r:=\{\omega(t)\in C_0(\mathbb{R},X):\|\omega(t)\|\le r\}.\tag{103}$$

For the above $v(t)$, define $\Gamma:=\Gamma^1+\Gamma^2$ on $C_0(\mathbb{R},X)$ as (69) and prove that Γ maps Ω_{k_0} into itself, where k_0 is a given constant.

Firstly, from (97) it follows that, for all $s\in\mathbb{R}$, $\omega(s)\in X$,

$$\begin{aligned}
&\|F_1(s,v(s)+\omega(s),B(v(s)+\omega(s)))\\
&\quad-F_1(s,v(s),Bv(s))\|\le L(s)\left[\|\omega(s)\|+\|B\omega(s)\|\right]\\
&\quad\le L(s)\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega(s)\|,
\end{aligned}\tag{104}$$

which together with $L(s)\in BC(\mathbb{R},\mathbb{R}^+)$ implies that

$$\begin{aligned}
&F_1(\cdot,v(\cdot)+\omega(\cdot),B(v(\cdot)+\omega(\cdot)))\\
&\quad-F_1(\cdot,v(\cdot),Bv(\cdot))\in C_0(\mathbb{R},X)\\
&\qquad\text{for every }\omega(\cdot)\in C_0(\mathbb{R},X).
\end{aligned}\tag{105}$$

According to (55), one has

$$\begin{aligned}
&\|F_2(s,v(s)+\omega(s),B(v(s)+\omega(s)))\|\le\beta(s)\\
&\quad\cdot\Phi\left(\|\omega(s)+B\omega(s)\|+\sup_{s\in\mathbb{R}}\|v(s)+Bv(s)\|\right)\\
&\quad\le\beta(s)\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega(s)\|\right.\\
&\quad\left.+\left(1+\|B\|_{\mathbb{L}(X)}\right)\sup_{s\in\mathbb{R}}\|v(s)\|\right)\le\beta(s)\\
&\quad\cdot\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left[\|\omega(s)\|+\sup_{s\in\mathbb{R}}\|v(s)\|\right]\right)
\end{aligned}\tag{106}$$

for all $s\in\mathbb{R}$ and $\omega(s)\in X$ with $\|\omega(s)\|\le r$; then

$$\begin{aligned}
&F_2(\cdot,v(\cdot)+\omega(\cdot),B(v(\cdot)+\omega(\cdot)))\in C_0(\mathbb{R},X)\\
&\qquad\text{as }\beta(\cdot)\in C_0(\mathbb{R},\mathbb{R}^+).
\end{aligned}\tag{107}$$

Those, together with Lemma 21, yield that Γ is well defined and maps $C_0(\mathbb{R},X)$ into itself.

On the other hand, in view of (55) and (98) it is not difficult to see that there exists a constant $k_0>0$ such that

$$\begin{aligned}
&\frac{CM\|L\||\omega|^{-1/\alpha}\pi\left(1+\|B\|_{\mathbb{L}(X)}\right)}{\alpha\sin(\pi/\alpha)}k_0\\
&\quad+CM\rho_2\Phi\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+\sup_{s\in\mathbb{R}}\|v(s)\|\right)\right)\\
&\quad\le k_0.
\end{aligned}\tag{108}$$

This enables us to conclude that, for any $t\in\mathbb{R}$ and $\omega_1(t)$, $\omega_2(t)\in\Omega_{k_0}$,

$$\begin{aligned}
&\left\|\left(\Gamma^1\omega_1\right)(t)+\left(\Gamma^2\omega_2\right)(t)\right\|\le\left\|\int_{-\infty}^{t}S_{\alpha}(t-s)\right.\\
&\quad\cdot\left[F_1\left(s,v(s)+\omega_1(s),B\left(v(s)+\omega_1(s)\right)\right)\right.\\
&\quad\left.\left.-F_1(s,v(s),Bv(s))\right]ds\right\|+\left\|\int_{-\infty}^{t}S_{\alpha}(t-s)\right.\\
&\quad\left.\cdot F_2\left(s,v(s)+\omega_2(s),B\left(v(s)+\omega_2(s)\right)\right)ds\right\|\\
&\quad\le\int_{-\infty}^{t}L(s)\|S_{\alpha}(t-s)\|\left[\|\omega_1(s)\|\right.\\
&\quad\left.+\|B\omega_1(s)\|\right]ds+CM\int_{-\infty}^{t}\frac{\beta(s)}{1+|\omega|(t-s)^{\alpha}}\Phi\\
&\quad\cdot\left(\|\omega_2(s)\|+\|B\omega_2(s)\|+\|v(s)\|+\|Bv(s)\|\right)ds\\
&\quad\le CM\int_{-\infty}^{t}\frac{L(s)}{1+|\omega|(t-s)^{\alpha}}\left(1+\|B\|_{\mathbb{L}(X)}\right)\\
&\quad\cdot\|\omega_1(s)\|ds+CM\int_{-\infty}^{t}\frac{\beta(s)}{1+|\omega|(t-s)^{\alpha}}\Phi\left(\left(1\right.\right.\\
&\quad\left.\left.+\|B\|_{\mathbb{L}(X)}\right)\left(\|\omega_2(s)\|+\|v(s)\|\right)ds\le CM\left(1\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \|B\|_{\mathbb{L}(X)}) \|\omega_1\|_\infty \int_{-\infty}^{t} \frac{L(s)}{1+|\omega|(t-s)^\alpha} \mathrm{d}s \\
&+ CM\sigma(t)\Phi\left((1+\|B\|_{\mathbb{L}(X)})\left(\|\omega_2\|_\infty\right.\right. \\
&\left.\left.+ \sup_{s\in\mathbb{R}} \|v(s)\|\right)\right) \\
&\le CM\left(\sum_{m=0}^{+\infty}\int_{t-(m+1)}^{t-m} \frac{L(s)}{1+|\omega|(t-s)^\alpha}\mathrm{d}s\right)(1 \\
&+ \|B\|_{\mathbb{L}(X)})\|\omega_1\|_\infty + CM\rho_2\Phi\left((1+\|B\|_{\mathbb{L}(X)})\right. \\
&\left.\cdot\left(\|\omega_2\|_\infty + \sup_{s\in\mathbb{R}}\|v(s)\|\right)\right) \\
&\le CM\left(\sum_{m=0}^{+\infty}\frac{1}{1+|\omega|m^\alpha}\int_{t-(m+1)}^{t-m} L(s)\,\mathrm{d}s\right)(1 \\
&+ \|B\|_{\mathbb{L}(X)})\|\omega_1\|_\infty + CM\rho_2\Phi\left((1+\|B\|_{\mathbb{L}(X)})\left(k_0\right.\right. \\
&\left.\left.+ \sup_{s\in\mathbb{R}}\|v(s)\|\right)\right) \le CM\left(\sum_{m=0}^{+\infty}\frac{1}{1+|\omega|m^\alpha}\right)\|L\|(1 \\
&+ \|B\|_{\mathbb{L}(X)})k_0 + CM\rho_2\Phi\left((1+\|B\|_{\mathbb{L}(X)})\left(k_0\right.\right. \\
&\left.\left.+ \sup_{s\in\mathbb{R}}\|v(s)\|\right)\right) \\
&= \frac{CM\|L\||\omega|^{-1/\alpha}\pi(1+\|B\|_{\mathbb{L}(X)})}{\alpha\sin(\pi/\alpha)}k_0 \\
&+ CM\rho_2\Phi\left((1+\|B\|_{\mathbb{L}(X)})\left(k_0 + \sup_{s\in\mathbb{R}}\|v(s)\|\right)\right) \\
&\le k_0,
\end{aligned}
\tag{109}
$$

which implies that $(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t) \in \Omega_{k_0}$. Thus Γ maps Ω_{k_0} into itself.

Step 3. Show that Γ^1 is a contraction on Ω_{k_0}.

In fact, for any $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$, from (97) it follows that

$$
\begin{aligned}
&\|[F_1(s, v(s)+\omega_1(s), B(v(s)+\omega_1(s))) \\
&\quad - F_1(s, v(s), Bv(s))] \\
&\quad - [F_1(s, v(s)+\omega_2(s), B(v(s)+\omega_2(s))) \\
&\quad - F_1(s, v(s), Bv(s))]\| \le L(s)[\|\omega_1(s)-\omega_2(s)\| \\
&\quad + \|B\omega_1(s) - B\omega_2(s)\|] \le L(s)(1+\|B\|_{\mathbb{L}(X)})\|\omega_1(s) \\
&\quad - \omega_2(s)\|.
\end{aligned}
\tag{110}
$$

Thus

$$
\begin{aligned}
&\|(\Gamma^1\omega_1)(t) - (\Gamma^1\omega_2)(t)\| = \left\|\int_{-\infty}^{t} S_\alpha(t-s)[(F_1(s, v(s)+\omega_1(s), B(v(s)+\omega_1(s))) - F_1(s, v(s), Bv(s))) \right. \\
&\left. - (F_1(s, v(s)+\omega_2(s), B(v(s)+\omega_2(s))) - F_1(s, v(s), Bv(s)))]\,\mathrm{d}s\right\| \le \int_{-\infty}^{t} L(s)\|S_\alpha(t-s)\|(1+\|B\|_{\mathbb{L}(X)})\|\omega_1(s) \\
&- \omega_2(s)\|\,\mathrm{d}s \le CM\int_{-\infty}^{t}\frac{L(s)}{1+|\omega|(t-s)^\alpha}(1+\|B\|_{\mathbb{L}(X)})\|\omega_1(s)-\omega_2(s)\|\,\mathrm{d}s \\
&\le CM\left(\sum_{m=0}^{+\infty}\int_{t-(m+1)}^{t-m}\frac{L(s)}{1+|\omega|(t-s)^\alpha}\mathrm{d}s\right)(1+\|B\|_{\mathbb{L}(X)})\|\omega_1-\omega_2\|_\infty \le CM\left(\sum_{m=0}^{+\infty}\frac{1}{1+|\omega|m^\alpha}\int_{t-(m+1)}^{t-m}L(s)\,\mathrm{d}s\right)(1 \\
&+ \|B\|_{\mathbb{L}(X)})\|\omega_1-\omega_2\|_\infty \le CM\left(\sum_{m=0}^{+\infty}\frac{1}{1+|\omega|m^\alpha}\right)\|L\|(1+\|B\|_{\mathbb{L}(X)})\|\omega_1-\omega_2\|_\infty \\
&= \frac{CM\|L\||\omega|^{-1/\alpha}\pi(1+\|B\|_{\mathbb{L}(X)})}{\alpha\sin(\pi/\alpha)}\|\omega_1-\omega_2\|_\infty,
\end{aligned}
\tag{111}
$$

which implies that

$$
\begin{aligned}
&\|(\Gamma^1\omega_1)(t) - (\Gamma^1\omega_2)(t)\|_\infty \\
&\quad \le \frac{CM\|L\||\omega|^{-1/\alpha}\pi(1+\|B\|_{\mathbb{L}(X)})}{\alpha\sin(\pi/\alpha)}\|\omega_1-\omega_2\|_\infty.
\end{aligned}
\tag{112}
$$

Thus, in view of (98), one obtains the conclusion.

Step 4. Show that Γ^2 is completely continuous on Ω_{k_0}.

The proof is similar to the proof in Step 4 of Theorem 24.

Step 5. Show that (39) has at least one asymptotically almost automorphic mild solution.

The proof is similar to the proof in Step 5 of Theorem 24. □

Taking $A = -\rho^{\alpha} I$ with $\rho > 0$ in (39), Theorem 27 gives the following corollary.

Corollary 28. *Let $F : \mathbb{R} \times X \times X \longrightarrow X$ satisfy (H_1') and (H_2) with $L(t) \in BC(\mathbb{R}, \mathbb{R}^+)$. Put $\rho_2 := \sup_{t\in\mathbb{R}} \sigma(t)$. Let $\|L\| = \sup_{t\in\mathbb{R}} \int_t^{t+1} L(s)\mathrm{d}s$. Then (39) admits at least one asymptotically almost automorphic mild solution whenever*

$$\frac{C\|L\|\rho\pi\left(1+\|B\|_{\mathbb{L}(X)}\right)}{\alpha\sin(\pi/\alpha)} + C\rho_1\rho_2\left(1+\|B\|_{\mathbb{L}(X)}\right) < 1. \tag{113}$$

Now we consider a more general case of equations introducing a new class of functions $L(t)$. We have the following result.

(H_2') There exists a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ such that, for all $t \in \mathbb{R}$ and $x, y \in X$,

$$\|F_2(t, x, y)\| \le \beta(t)\left(\|x\| + \|y\|\right). \tag{114}$$

Theorem 29. *Assume that A is sectorial of type $\omega < 0$. Let $F : \mathbb{R} \times X \times X \longrightarrow X$ satisfy the hypotheses (H_1') and (H_2') with $L(t) \in BC(\mathbb{R}, \mathbb{R}^+)$. Moreover the integral $\int_{-\infty}^{t} \max\{L(s), \beta(s)\}\mathrm{d}s$ exists for all $t \in \mathbb{R}$. Then (39) has at least one asymptotically almost automorphic mild solution.*

Proof. The proof is divided into the following five steps.

Step 1. Define a mapping Λ on $AA(\mathbb{R}, X)$ by (62) and prove that Λ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

Firstly, similar to the proof in Step 1 of Theorem 27, we can prove that Λ is well defined and maps $AP(\mathbb{R}, X)$ into itself; moreover Λ is continuous.

Next, we prove that Λ is a contraction on $AA(\mathbb{R}, X)$ and has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

In fact, for $v_1(t), v_2(t)$ is in $AA(\mathbb{R}, X)$ and defines a new norm

$$|\|x\|| := \sup_{t\in\mathbb{R}} \{\mu(t)\|x(t)\|\}, \tag{115}$$

where $\mu(t) := e^{-k\int_{-\infty}^{t}\max\{L(s),\beta(s)\}\mathrm{d}s}$ and k is a fixed positive constant. Let $C_\alpha := \sup_{t\in\mathbb{R}}\|S_\alpha(t)\|$; then we have

$$\begin{aligned}
\mu(t)\|\Lambda v_1(t) - \Lambda v_2(t)\| &= \mu(t)\left\|\int_{-\infty}^{t} S_\alpha(t-\sigma)\right. \\
&\quad \cdot \left[F_1(\sigma, v_1(\sigma), Bv_1(\sigma))\right. \\
&\quad \left.\left. - F_1(\sigma, v_2(\sigma), Bv_2(\sigma))\right]\mathrm{d}\sigma\right\| \\
&\le C_\alpha \int_{-\infty}^{t} \mu(t) L(\sigma)\left[\|v_1(\sigma) - v_2(\sigma)\| + \|Bv_1(\sigma)\right. \\
&\quad \left. - Bv_2(\sigma)\|\right]\mathrm{d}\sigma = C_\alpha \int_{-\infty}^{t} \mu(t)\mu(\sigma)L(\sigma) \\
&\quad \cdot \mu(\sigma)^{-1}\left(1+\|B\|_{\mathbb{L}(X)}\right)\|v_1(\sigma) - v_2(\sigma)\|\mathrm{d}\sigma \\
&\le C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)|\|v_1 - v_2|\| \int_{-\infty}^{t} \mu(t)\mu(\sigma)^{-1} \\
&\quad \cdot L(\sigma)\mathrm{d}\sigma = \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|\|v_1 - v_2|\| \\
&\quad \cdot \int_{-\infty}^{t} ke^{-k\int_\sigma^t \max\{L(\tau),\beta(\tau)\}\mathrm{d}\tau} L(\sigma)\mathrm{d}\sigma \\
&\le \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|\|v_1 - v_2|\| \\
&\quad \cdot \int_{-\infty}^{t} ke^{-k\int_\sigma^t L(\tau)\mathrm{d}\tau} L(\sigma)\mathrm{d}\sigma \\
&= \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|\|v_1 - v_2|\| \\
&\quad \cdot \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(e^{k\int_t^\sigma L(\tau)\mathrm{d}\tau}\right)\mathrm{d}\sigma \\
&= \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}\left(1 - e^{-k\int_{-\infty}^{t} L(\tau)\mathrm{d}\tau}\right)|\|v_1 - v_2|\| \\
&\le \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|\|v_1 - v_2|\|,
\end{aligned} \tag{116}$$

which implies that

$$|\|\Lambda x(t) - \Lambda y(t)|\| \le \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|\|x - y|\|. \tag{117}$$

Hence Λ has a unique fixed point $x \in AA(\mathbb{R}, X)$ when k is greater than $C_\alpha(1 + \|B\|_{\mathbb{L}(X)})$.

Step 2. Set $\Theta_r := \{\omega(t) \in C_0(\mathbb{R}, X) : |\|\omega(t)\|| \le r\}$. For the above $v(t)$, define $\Gamma := \Gamma^1 + \Gamma^2$ on $C_0(\mathbb{R}, X)$ as (69) and prove that Γ maps Θ_{k_0} into itself, where k_0 is a given constant.

Firstly, from (97) it follows that, for all $s \in \mathbb{R}$, $\omega(s) \in X$,

$$\begin{aligned}
&\|F_1(s, v(s) + \omega(s), B(v(s) + \omega(s))) \\
&\quad - F_1(s, v(s), Bv(s))\| \le L(s)\left[\|\omega(s)\| + \|B\omega(s)\|\right] \\
&\quad \le L(s)\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega(s)\| + \|B\omega(s)\|,
\end{aligned} \tag{118}$$

which together with $L(s) \in BC(\mathbb{R}, \mathbb{R}^+)$ implies that

$$\begin{aligned}
&F_1(\cdot, v(\cdot) + \omega(\cdot), B(v(\cdot) + \omega(\cdot))) - F_1(\cdot, v(\cdot), Bv(\cdot)) \\
&\quad \in C_0(\mathbb{R}, X) \quad \text{for every } \omega(\cdot) \in C_0(\mathbb{R}, X).
\end{aligned} \tag{119}$$

According to (114), one has

$$\begin{aligned}
&\|F_2(s, v(s) + \omega(s), B(v(s) + \omega(s)))\| \le \beta(s) \\
&\quad \cdot (\|\omega(s) + B\omega(s)\| + \|v(s) + Bv(s)\|) \le \beta(s) \\
&\quad \cdot \left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega(s)\| + \left(1+\|B\|_{\mathbb{L}(X)}\right)\|v(s)\|\right) \\
&\quad \le \beta(s)\left(\left(1+\|B\|_{\mathbb{L}(X)}\right)\left[\|\omega(s)\| + \|v(s)\|\right]\right)
\end{aligned} \tag{120}$$

for all $s \in \mathbb{R}$ and $\omega(s) \in X$ with $\|\omega(s)\| \le r$; then

$$F_2(\cdot, v(\cdot) + \omega(\cdot), B(v(\cdot) + \omega(\cdot))) \in C_0(\mathbb{R}, X) \quad \text{as } \beta(\cdot) \in C_0(\mathbb{R}, \mathbb{R}^+). \tag{121}$$

Those, together with Lemma 21, yield that Γ is well defined and maps $C_0(\mathbb{R}, X)$ into itself.

On the other hand, it is not difficult to see that there exists a constant $k_0 > 0$ such that

$$\frac{2C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} k_0 + \frac{C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} |||v(s)||| \le k_0, \tag{122}$$

when k is large enough. This enables us to conclude that, for any $t \in \mathbb{R}$ and $\omega_1(t), \omega_2(t) \in \Theta_{k_0}$,

$$\begin{aligned}
&\mu(t) \left\|\left(\Gamma^1\omega_1\right)(t) + \left(\Gamma^2\omega_2\right)(t)\right\| \le \mu(t) \left\| \int_{-\infty}^t S_\alpha(t-s) \right. \\
&\cdot [F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) \\
&- F_1(s, v(s), Bv(s))] \, ds \Big\| + \mu(t) \left\| \int_{-\infty}^t S_\alpha(t \right. \\
&- s) F_2(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s))) \, ds \Big\| \\
&\le C_\alpha \int_{-\infty}^t \mu(t) L(s) (\|\omega_1(s)\| + \|B\omega_1(s)\|) \, ds \\
&+ C_\alpha \int_{-\infty}^t \mu(t) \beta(s) (\|\omega_2\| + \|v(s)\| + \|B\omega_2\| \\
&+ \|Bv(s)\|) \, ds = C_\alpha \int_{-\infty}^t \mu(t) \mu(s) L(s) \mu(s)^{-1} \\
&\cdot (1 + \|B\|_{\mathbb{L}(X)}) \|\omega_1(s)\| \, ds + C_\alpha \int_{-\infty}^t \mu(t) \mu(s) \\
&\cdot \beta(s) \mu(s)^{-1} (1 + \|B\|_{\mathbb{L}(X)}) (\|\omega_2\| + \|v(s)\|) \, ds \\
&\le C_\alpha (1 + \|B\|_{\mathbb{L}(X)}) |||\omega_1||| \int_{-\infty}^t \mu(t) \mu(s)^{-1} L(s) \, ds \\
&+ C_\alpha (1 + \|B\|_{\mathbb{L}(X)}) (|||\omega_2||| + |||v(s)|||) \int_{-\infty}^t \mu(t) \\
&\cdot \mu(s)^{-1} \beta(s) \, ds = \frac{C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} |||\omega_1||| \\
&\cdot \int_{-\infty}^t k e^{-k \int_t^s \max\{L(\tau), \beta(\tau)\} d\tau} L(s) \, ds + C_\alpha (1 \\
&+ \|B\|_{\mathbb{L}(X)}) (|||\omega_2||| + |||v(s)|||) \\
&\cdot \int_{-\infty}^t k e^{-k \int_t^s \max\{L(\tau), \beta(\tau)\} d\tau} \beta(s) \, ds \\
&\le \frac{C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} |||\omega_1||| \\
&\cdot \int_{-\infty}^t k e^{-k \int_t^s L(\tau) d\tau} L(s) \, ds + C_\alpha (1 + \|B\|_{\mathbb{L}(X)}) \\
&\cdot (|||\omega_2||| + |||v(s)|||) \int_{-\infty}^t k e^{-k \int_t^s \beta(\tau) d\tau} \beta(s) \, ds \\
&= \frac{C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} |||\omega_1||| \int_{-\infty}^t \frac{d}{ds} \left(e^{k \int_t^s L(\tau) d\tau}\right) ds \\
&+ C_\alpha (1 + \|B\|_{\mathbb{L}(X)}) (|||\omega_2||| + |||v(s)|||) \\
&\cdot \int_{-\infty}^t \frac{d}{ds} \left(e^{k \int_t^s \beta(\tau) d\tau}\right) ds \\
&= \frac{C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} \left(1 - e^{-k \int_{-\infty}^t L(\tau) d\tau}\right) |||\omega_1||| \\
&+ \frac{C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} \left(1 - e^{-k \int_{-\infty}^t \beta(\tau) d\tau}\right) (|||\omega_2||| \\
&+ |||v(s)|||) \le \frac{C_\alpha(1 + \|B\|_{\mathbb{L}(X)})}{k} |||\omega_1||| + \frac{C_\alpha}{k} (1 \\
&+ \|B\|_{\mathbb{L}(X)}) (|||\omega_2||| + |||v(s)|||) \le k_0,
\end{aligned} \tag{123}$$

which implies that $(\Gamma^1\omega_1)(t) + (\Gamma^2\omega_2)(t) \in \Theta_{k_0}$. Thus Γ maps Θ_{k_0} into itself.

Step 3. Show that Γ^1 is a contraction on Θ_{k_0}.

In fact, for any $\omega_1(t), \omega_2(t) \in \Theta_{k_0}$ and $t \in \mathbb{R}$, from (97) it follows that

$$\begin{aligned}
&\|[F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) \\
&- F_1(s, v(s), Bv(s))] \\
&- [F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s))) \\
&- F_1(s, v(s), Bv(s))]\| \le L(s) [\|\omega_1(s) - \omega_2(s)\| \\
&+ \|B\omega_1(s) - B\omega_2(s)\|] \le L(s)(1 + \|B\|_{\mathbb{L}(X)}) \\
&\cdot \|\omega_1(s) - \omega_2(s)\|.
\end{aligned} \tag{124}$$

Thus

$$\begin{aligned}
&\mu(t) \left\|\left(\Gamma^1\omega_1\right)(t) - \left(\Gamma^1\omega_2\right)(t)\right\| = \mu(t) \left\| \int_{-\infty}^t S_\alpha(t-s) [(F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) - F_1(s, v(s), Bv(s))) \right. \\
&\left. - (F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s))) - F_1(s, v(s), Bv(s)))] \, ds \right\| \le C_\alpha \int_{-\infty}^t \mu(t) L(\sigma) (1 + \|B\|_{\mathbb{L}(X)}) \|\omega_1(\sigma)
\end{aligned}$$

$$
\begin{aligned}
&- \omega_2(\sigma)\| \mathrm{d}\sigma = C_\alpha \int_{-\infty}^{t} \mu(t)\mu(\sigma)L(\sigma)\mu(\sigma)^{-1}\left(1+\|B\|_{\mathbb{L}(X)}\right)\|\omega_1(\sigma)-\omega_2(\sigma)\| \mathrm{d}\sigma \le C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)|||\omega_1 \\
&- \omega_2||| \int_{-\infty}^{t} \mu(t)\mu(\sigma)^{-1}L(\sigma)\mathrm{d}\sigma = \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|||\omega_1-\omega_2||| \int_{-\infty}^{t} ke^{-k\int_\sigma^t \max\{L(\tau),\beta(\tau)\}\mathrm{d}\tau}L(\sigma)\mathrm{d}\sigma \\
&\le \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|||\omega_1-\omega_2||| \int_{-\infty}^{t} ke^{-k\int_\sigma^t L(\tau)\mathrm{d}\tau}L(\sigma)\mathrm{d}\sigma = \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|||\omega_1-\omega_2||| \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(e^{k\int_t^\sigma L(\tau)\mathrm{d}\tau}\right)\mathrm{d}\sigma \\
&= \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}\left(1-e^{-k\int_{-\infty}^t L(\tau)\mathrm{d}\tau}\right)|||\omega_1-\omega_2||| \le \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|||\omega_1-\omega_2|||,
\end{aligned}
\tag{125}
$$

which implies

$$
\begin{aligned}
&\left|\left|\left|\left(\Gamma^1\omega_1\right)(t)-\left(\Gamma^1\omega_2\right)(t)\right|\right|\right| \\
&\quad \le \frac{C_\alpha\left(1+\|B\|_{\mathbb{L}(X)}\right)}{k}|||\omega_1-\omega_2|||.
\end{aligned}
\tag{126}
$$

Thus, when k is greater than $C_\alpha(1+\|B\|_{\mathbb{L}(X)})$, one obtains the conclusion.

Step 4. Show that Γ^2 is completely continuous on Θ_{k_0}.

Given $\varepsilon > 0$. Let $\{\omega_n\}_{n=1}^{+\infty} \subset \Theta_{k_0}$ with $\omega_n \longrightarrow \omega_0$ in Θ_{k_0} as $n \longrightarrow +\infty$. Since $\sigma(t) \in C_0(\mathbb{R},\mathbb{R}^+)$, one may choose a $t_1 > 0$ big enough such that, for all $t \ge t_1$,

$$
\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+|||v|||\right)\sigma(t) < \frac{\varepsilon}{3CM}. \tag{127}
$$

Also, in view of (H_1'), we have

$$
\begin{aligned}
&F_2\left(s, v(s)+\omega_k(s), B\left(v(s)+\omega_k(s)\right)\right) \\
&\quad \longrightarrow F_2\left(s, v(s)+\omega_0(s), B\left(v(s)+\omega_0(s)\right)\right)
\end{aligned}
\tag{128}
$$

for all $s \in (-\infty, t_1]$ as $k \longrightarrow +\infty$ and

$$
\begin{aligned}
&\mu(\cdot)\|F_2\left(\cdot, v(\cdot)+\omega_n(\cdot), B\left(v(\cdot)+\omega_n(\cdot)\right)\right) \\
&\quad - F_2\left(\cdot, v(\cdot)+\omega_0(\cdot), Bv(\cdot)+\omega_0(\cdot)\right)\ )\| \le \mu(\cdot) \\
&\quad \cdot \beta(\cdot)\left(\|\omega_n(\cdot)\|+\|v(\cdot)\|+\|B\omega_n(\cdot)\|+\|Bv(\cdot)\|\right. \\
&\quad \left.+\|\omega_0(\cdot)\|+\|v(\cdot)\|+\|B\omega_0(\cdot)\|+\|Bv(\cdot)\|\right) \le \beta(\cdot) \\
&\quad \cdot\left(|||\omega_n|||+|||v|||+|||B\omega_n|||+|||Bv|||+|||\omega_0|||\right. \\
&\quad \left.+|||v|||+|||B\omega_0|||+|||Bv|||\right) \le \beta(\cdot) \\
&\quad \cdot\left(2\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+|||v|||\right)\right) \in L^1(-\infty, t_1].
\end{aligned}
\tag{129}
$$

Hence, by the Lebesgue dominated convergence theorem we deduce that there exists an $N > 0$ such that

$$
\begin{aligned}
&CM\int_{-\infty}^{t_1} \frac{1}{1+|\omega|(t-s)^\alpha}\mu(t) \\
&\quad \cdot\|F_2\left(s, v(s)+\omega_k(s), B\left(v(s)+\omega_k(s)\right)\right) \\
&\quad - F_2\left(s, v(s)+\omega_0(s), B\left(v(s)+\omega_0(s)\right)\right)\| \mathrm{d}s \\
&\quad \le \frac{\varepsilon}{3}
\end{aligned}
\tag{130}
$$

whenever $k \ge N$. Thus

$$
\begin{aligned}
&\mu(t)\left\|\left(\Gamma^2\omega_k\right)(t)-\left(\Gamma^2\omega_0\right)(t)\right\| = \mu(t)\left\|\int_{-\infty}^{t} S_\alpha(t-s)\right. \\
&\quad \cdot F_2\left(s, v(s)+\omega_k(s), B\left(v(s)+\omega_k(s)\right)\right)\mathrm{d}s \\
&\quad - \int_{-\infty}^{t} \mathrm{S}_\alpha(t-s) \\
&\quad \left.\cdot F_2\left(s, v(s)+\omega_0(s), B\left(v(s)+\omega_0(s)\right)\right)\mathrm{d}s\right\| \\
&\quad \le CM\int_{-\infty}^{t_1} \frac{1}{1+|\omega|(t-s)^\alpha}\mu(t) \\
&\quad \cdot\|F_2\left(s, v(s)+\omega_k(s), B\left(v(s)+\omega_k(s)\right)\right) \\
&\quad - F_2\left(s, v(s)+\omega_0(s), B\left(v(s)+\omega_0(s)\right)\right)\| \mathrm{d}s \\
&\quad + CM\left(2\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+|||v|||\right)\right) \\
&\quad \cdot\int_{t_1}^{\max\{t,t_1\}} \frac{\beta(s)}{1+|\omega|(t-s)^\alpha}\mathrm{d}s \\
&\quad \le CM\int_{-\infty}^{t_1} \frac{1}{1+|\omega|(t-s)^\alpha}\mu(t) \\
&\quad \cdot\|F_2\left(s, v(s)+\omega_k(s), B\left(v(s)+\omega_k(s)\right)\right) \\
&\quad - F_2\left(s, v(s)+\omega_0(s), B\left(v(s)+\omega_0(s)\right)\right)\| \mathrm{d}s \\
&\quad + CM\sigma(t)\left(2\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+|||v|||\right)\right) \le \frac{\varepsilon}{3} \\
&\quad + \frac{2\varepsilon}{3} = \varepsilon
\end{aligned}
\tag{131}
$$

whenever $k \ge N$. Accordingly, Γ^2 is continuous on Θ_{k_0}.

In the sequel, we consider the compactness of Γ^2.

Set $B_r(X)$ for the closed ball with center at 0 and radius r in X, $V = \Gamma^2(\Theta_{k_0})$, and $z(t) = \Gamma^2(u(t))$ for $u(t) \in \Theta_{k_0}$. First, for all $\omega(t) \in \Theta_{k_0}$ and $t \in \mathbb{R}$,

$$\begin{aligned}
\mu(t)\left\|\left(\Gamma^2\omega\right)(t)\right\| &= \mu(t)\left\|\int_{-\infty}^{t} S_\alpha(t-s)\right. \\
&\quad \left.\cdot F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\,\mathrm{d}s\right\| \\
&\le CM\int_{-\infty}^{t}\frac{1}{1+|\omega|(t-s)^\alpha}\mu(t) \\
&\quad \cdot\left\|F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\right\|\mathrm{d}s \\
&\le CM\int_{-\infty}^{t}\frac{\beta(s)}{1+|\omega|(t-s)^\alpha}\mu(t)\left(\|v(s)\|+\|\omega(s)\|\right. \\
&\quad \left.+\|Bv(s)\|+\|B\omega(s)\|\right)\mathrm{d}s \\
&\le CM\int_{-\infty}^{t}\frac{\beta(s)}{1+|\omega|(t-s)^\alpha}\mu(t)\left(1+\|B\|_{\mathbb{L}(X)}\right) \\
&\quad \cdot\left(\|v(s)\|+\|\omega(s)\|\right)\mathrm{d}s \le CM\sigma(t)\left(1\right. \\
&\quad \left.+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+\|v(s)\|\right),
\end{aligned} \tag{132}$$

in view of $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ which follows from Lemma 23; one concludes that

$$\lim_{|t|\longrightarrow+\infty}\left(\Gamma^2\omega\right)(t) = 0 \quad \text{uniformly for } \omega(t) \in \Theta_{k_0}. \tag{133}$$

as

$$\begin{aligned}
\left(\Gamma^2\omega\right)(t) &= \int_{-\infty}^{t} S_\alpha(t-s)F_2(s, v(s) \\
&\quad +\omega(s), B(v(s)+\omega(s)))\,\mathrm{d}s = \int_0^{+\infty} S_\alpha(\tau) \\
&\quad \cdot F_2(t-\tau, v(t-\tau) \\
&\quad +\omega(t-\tau), B(v(t-\tau)+\omega(t-\tau)))\,\mathrm{d}\tau.
\end{aligned} \tag{134}$$

Hence, for given $\varepsilon_0 > 0$, one can choose a $\xi > 0$ such that

$$\left\|\int_{\xi}^{+\infty} S_\alpha(\tau)F_2(t-\tau, v(t-\tau)+\omega(t-\tau), B(v(t-\tau)+\omega(t-\tau)))\,\mathrm{d}\tau\right\| < \varepsilon_0. \tag{135}$$

Thus we get

$$z(t) \in \xi\overline{c\left(\left\{S_\alpha(\tau)F_2(\lambda, v(\lambda)+\omega(\lambda), B(v(\lambda)+\omega(\lambda))) : 0 \le \tau \le \xi,\ t-\xi \le \lambda \le \xi,\ \|\omega\| \le k_0\right\}\right)} + B_{\varepsilon_0}\left(\Theta_{k_0}\right), \tag{136}$$

where $c(K)$ denotes the convex hull of K. Using the fact that $S_\alpha(\cdot)$ is strongly continuous, we infer that

$$\begin{aligned}
K = \{&S_\alpha(\tau)F_2(\lambda, v(\lambda)+\omega(\lambda), B(v(\lambda)+\omega(\lambda))) : 0 \\
&\le \tau \le \xi,\ t-\xi \le \lambda \le \xi,\ \|\omega\| \le k_0\}
\end{aligned} \tag{137}$$

is a relatively compact set and $V \subset \xi\overline{c(K)} + B_{\varepsilon_0}(\Theta_{k_0})$, which implies that V is a relatively compact subset of Θ_{k_0}.

Next, we verify the equicontinuity of the set $\{(\Gamma^2\omega)(t) : \omega(t) \in \Theta_{k_0}\}$, given $\varepsilon_1 > 0$. In view of (114), together with the continuity of $\{S_\alpha(t)\}_{t>0}$, there exists an $\eta > 0$ such that, for all $\omega(t) \in \Omega_{k_0}$ and $t_2 \ge t_1$ with $t_2 - t_1 < \eta$,

$$\begin{aligned}
&\int_{t_1}^{t_2}\left\|S_\alpha(t_2-s)F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\right\|\mathrm{d}s < \frac{\varepsilon_1}{4}, \\
&\int_{t_1-\eta}^{t_1}\left\|\left[S_\alpha(t_2-s)-S_\alpha(t_1-s)\right]F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\right\|\mathrm{d}s < \frac{\varepsilon_1}{4}.
\end{aligned} \tag{138}$$

Also, one can choose a $k > 0$ such that

$$\begin{aligned}
&\int_{t_1-k}^{t_1-\eta}\left\|\left[S_\alpha(t_2-s)-S_\alpha(t_1-s)\right]F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\right\|\mathrm{d}s < \frac{\varepsilon_1}{4} \\
&\left(1+\|B\|_{\mathbb{L}(X)}\right)\left(k_0+\|v\|\right)\sup_{s\in[-\infty, t_1-k]}\left\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\right\|\int_{-\infty}^{t_1-k}\beta(s)\,\mathrm{d}s < \frac{\varepsilon_1}{4},
\end{aligned} \tag{139}$$

which implies that, for all $\omega(t) \in \Omega_{k_0}$ and $t_2 \geq t_1$,

$$\begin{aligned}&\int_{-\infty}^{t_1-k} \left\| \left[S_\alpha(t_2-s) - S_\alpha(t_1-s)\right] F_2(s, v(s)+\omega(s), B(v(s)+\omega(s))) \right\| \mathrm{d}s \\ &\quad \leq (1+\|B\|_{\mathbb{L}(X)})(k_0+\|v\|) \sup_{s\in[-\infty,t_1-k]} \left\|S_\alpha(t_2-s) - S_\alpha(t_1-s)\right\| \int_{-\infty}^{t_1-k} \beta(s)\,\mathrm{d}s < \frac{\varepsilon_1}{4}.\end{aligned} \tag{140}$$

Then one has

$$\begin{aligned}&\left\|\left(\Gamma^2\omega\right)(t_2) - \left(\Gamma^2\omega\right)(t_1)\right\| \\ &\quad = \left\| \int_{-\infty}^{t_2} S_\alpha(t_2-s) F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\,\mathrm{d}s - \int_{-\infty}^{t_1} S_\alpha(t_1-s) F_2(s, v(s)+\omega(s), B(v(s)+\omega(s)))\,\mathrm{d}s \right\| \\ &\quad \leq \int_{t_1}^{t_2} \left\| S_\alpha(t_2-s) F_2(s, v(s)+\omega(s), B(v(s)+\omega(s))) \right\| \mathrm{d}s \\ &\quad + \int_{t_1-\eta}^{t_1} \left\| \left[S_\alpha(t_2-s) - S_\alpha(t_1-s)\right] F_2(s, v(s)+\omega(s), B(v(s)+\omega(s))) \right\| \mathrm{d}s \\ &\quad + \int_{-\infty}^{t_1-k} \left\| \left[S_\alpha(t_2-s) - S_\alpha(t_1-s)\right] F_2(s, v(s)+\omega(s), B(v(s)+\omega(s))) \right\| \mathrm{d}s \\ &\quad + \int_{t_1-k}^{t_1-\eta} \left\| \left[S_\alpha(t_2-s) - S_\alpha(t_1-s)\right] F_2(s, v(s)+\omega(s), B(v(s)+\omega(s))) \right\| \mathrm{d}s < \varepsilon_1,\end{aligned} \tag{141}$$

which implies the equicontinuity of the set $\{(\Gamma^2\omega)(t) : \omega(t) \in \Theta_{k_0}\}$.

Now an application of Lemma 18 justifies the compactness of Γ^2.

Step 5. Show that (39) has at least one asymptotically almost automorphic mild solution.

The proof is similar to the proof in Step 5 of Theorem 24. □

Taking $A = -\rho^\alpha I$ with $\rho > 0$ in (39), Theorem 29 gives the following corollary.

Corollary 30. *Let $F : \mathbb{R} \times X \times X \longrightarrow X$ satisfy (H_1') and (H_2') with $L(t) \in BC(\mathbb{R}, \mathbb{R}^+)$. Moreover the integral $\int_{-\infty}^{t} \max\{L(s), \beta(s)\}\mathrm{d}s$ exists for all $t \in \mathbb{R}$. Then (39) has at least one asymptotically almost automorphic mild solution.*

4. Applications

In this section we give an example to illustrate the above results.

Consider the following fractional relaxation-oscillation equation:

$$\begin{aligned}\partial_t^\alpha u(t,x) &= \partial_x^2 u(t,x) - p u(t,x) \\ &\quad + \partial_t^{\alpha-1}\left[\mu a(t) \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\right. \\ &\quad \times [\sin u(t,x) + u(t,x)] \\ &\quad \left. + \nu e^{-|t|}[u(t,x) + \sin u(t,x)]\right], \\ &\qquad t \in \mathbb{R},\ x \in [0,\pi], \\ u(t,0) &= u(t,\pi) = 0, \quad t \in \mathbb{R},\end{aligned} \tag{142}$$

where $a(t) \in BC(\mathbb{R}, \mathbb{R}^+)$ is a function and p, μ, and ν are positive constants.

Take $X = L^2([0,\pi])$ and define the operator A by

$$A\varphi := \varphi'' - p\varphi, \quad \varphi \in D(A), \tag{143}$$

where

$$D(A) := \left\{\varphi \in X : \varphi'' \in X,\ \varphi(0) = \varphi(\pi)\right\} \subset X. \tag{144}$$

It is well known that $Bu = u''$ is self-adjoint, with compact resolvent, and is the infinitesimal generator of an analytic semigroup on X. Hence, $pI - B$ is sectorial of type $\omega = -p < 0$. Let

$$\begin{aligned}F_1(t, x(\xi), y(\xi)) &:= \mu a(t) \\ &\quad \cdot \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)[\sin x(\xi) + y(\xi)], \\ F_2(t, x(\xi), y(\xi)) &:= \nu e^{-|t|}[x(\xi) + \sin y(\xi)].\end{aligned} \tag{145}$$

Then it is easy to verify that $F_1, F_2 : \mathbb{R} \times X \times X \longrightarrow X$ are continuous and $F_1(t, x, y) \in AA(\mathbb{R} \times X \times X, X)$ satisfying

$$\begin{aligned} &\left\|F_1(t, x_1, y_1) - F_1(t, x_2, y_2)\right\|_2^2 \\ &\quad \le \int_0^\pi \mu^2 \left| a(t) \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)\right|^2 \\ &\quad \cdot \left|[\sin x_1(s) + y_1(s)] - [\sin x_2(s) + y_2(s)]\right| \mathrm{d}s \\ &\quad \le \mu^2 a^2(t) \left|\sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)\right|^2 \\ &\quad \cdot \left(\|x_1 - x_2\|_2^2 + \|y_1 - y_2\|_2^2\right), \end{aligned} \tag{146}$$

that is,

$$\begin{aligned} &\left\|F_1(t, x_1, y_1) - F_1(t, x_2, y_2)\right\|_2 \\ &\quad \le \mu a(t) \left(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2\right) \\ &\qquad \forall t \in \mathbb{R},\ x_1, y_1, x_2, y_2 \in X; \end{aligned} \tag{147}$$

furthermore

$$\begin{aligned} &\left\|F_1(t, x_1, y_1) - F_1(t, x_2, y_2)\right\|_2 \\ &\quad \le \mu \|a\|_\infty \left(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2\right) \\ &\qquad \forall t \in \mathbb{R},\ x_1, y_1, x_2, y_2 \in X. \end{aligned} \tag{148}$$

And

$$\begin{aligned} \left\|F_2(t, x, y)\right\|_2^2 &\le \int_0^\pi \nu^2 e^{-2|t|} \left|x(s) + \sin y(s)\right| \mathrm{d}s \\ &\le \nu^2 e^{-2|t|} \left(\|x\|_2^2 + \|y\|_2^2\right), \end{aligned} \tag{149}$$

that is,

$$\begin{aligned} \left\|F_2(t, x, y)\right\|_2 &\le \nu e^{-|t|} \left(\|x\|_2 + \|y\|_2\right) \\ &\qquad \forall t \in \mathbb{R},\ x, y \in X, \end{aligned} \tag{150}$$

which implies $F_2(t, x, y) \in C_0(\mathbb{R} \times X \times X, X)$. Furthermore

$$\begin{aligned} F(t, x, y) &= F_1(t, x, y) + F_2(t, x, y) \\ &\in AAA(\mathbb{R} \times X \times X, X). \end{aligned} \tag{151}$$

Thus, (142) can be reformulated as the abstract problem (39) and the assumptions (H_1) and (H_2) hold with

$$\begin{aligned} L &= \mu \|a\|_\infty, \\ \Phi(r) &= r, \\ \beta(t) &= \nu e^{-|t|}, \\ \rho_1 &= 1, \\ \rho_2 &\le \nu, \end{aligned} \tag{152}$$

the assumption (H_1') holds with $L(t) = \mu a(t)$, and the assumption (H_2') holds.

In consequence, the fractional relaxation-oscillation equation (142) has at least one asymptotically almost automorphic mild solutions if either

$$\frac{\mu CM \|a\|_\infty \pi |p|^{-1/\alpha}}{\alpha \sin(\pi/\alpha)} + CM\nu < \frac{1}{2} \tag{153}$$

(Theorem 24) or

$$\frac{\mu CM \|a\| \pi |p|^{-1/\alpha}}{\alpha \sin(\pi/\alpha)} + CM\nu < \frac{1}{2} \tag{154}$$

(Theorem 27), where $\|a\| = \sup_{t\in\mathbb{R}} \int_t^{t+1} a(s)\mathrm{d}s$ or the integral

$$\int_{-\infty}^t \max\left\{\mu a(s), \nu e^{-|t|}\right\} \mathrm{d}s \tag{155}$$

exists for all $t \in \mathbb{R}$ (Theorem 29).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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