Existence of Asymptotically Almost Automorphic Mild Solutions of Semilinear Fractional Differential Equations

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Received 21 December 2017; Revised 18 April 2018; Accepted 10 May 2018; Published 1 August 2018

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This paper is concerned with the existence of asymptotically almost automorphic mild solutions to a class of abstract semilinear fractional differential equations

\[ D_\alpha \tau x(t) = Ax(t) + D_{\alpha-1} \tau F(t, x(t), Bx(t)), \quad t \in \mathbb{R}, \]

where \( 1 < \alpha < 2 \), \( A \) is a linear densely defined operator of sectorial type on a complex Banach space \( X \) and \( B \) is a bounded linear operator defined on \( X \), \( F \) is an appropriate function defined on phase space, and the fractional derivative is understood in the Riemann-Liouville sense. Combining the fixed point theorem due to Krasnoselskii and a decomposition technique, we prove the existence of asymptotically almost automorphic mild solutions to such problems. Our results generalize and improve some previous results since the (locally) Lipschitz continuity on the nonlinearity \( F \) is not required. The results obtained are utilized to study the existence of asymptotically almost automorphic mild solutions to a fractional relaxation-oscillation equation.

1. Introduction

The almost periodic function introduced seminally by Bohr in 1925 plays an important role in describing the phenomena that are similar to the periodic oscillations which can be observed frequently in many fields, such as celestial mechanics, nonlinear vibration, electromagnetic theory, plasma physics, engineering, and ecosphere. The concept of almost automorphy, which is an important generalization of the classical almost periodicity, was first introduced in the literature [1–4] by Bochner in relation to some aspects of differential geometry. Since then, this pioneer work has attracted more and more attention and has been substantially extended in several different directions. Many authors have made important contributions to this theory (see, for instance, [5–17] and the references therein). Especially, in [5, 6], the authors gave an important overview about the theory of almost automorphic functions and their applications to differential equations.

As a natural extension of almost automorphy, the concept of asymptotic almost automorphy, which is the central issue to be discussed in this paper, was introduced in the literature [18] by N’Guérékata in the early eighties. Since then, this notion has found several developments and has been generalized into different directions. Until now, the asymptotically almost automorphic functions as well as the asymptotically almost automorphic solutions for differential systems have been investigated by many mathematicians; see [19] by Bugajewski and N’Guérékata, [20] by Diagana, Hernández, and dos Santos, and [21] by Ding, Xiao, and Liang for the asymptotically almost automorphic solutions to integrodifferential equations, see [22] by Zhao, Chang, and N’Guérékata for the asymptotically almost automorphic solutions to the nonlinear delay integral equations, and see [23] by Chang and Tang and [24] by Zhao, Chang, and Nieto for the asymptotically almost automorphic solutions to stochastic differential equations, and the existence of asymptotically almost automorphic solutions has become one of the most attractive topics in the qualitative theory of differential equations due to its significance and applications in physics, mathematical biology, control theory, and so on. We refer the reader to the monographs of N’Guérékata [25] for the recently theory and applications of asymptotically almost automorphic functions.
With motivation coming from a wide range of engineering and physical applications, fractional differential equations have recently attracted great attention of mathematicians and scientists. This kind of equations is a generalization of ordinary differential equations to arbitrary noninteger orders. Fractional differential equations find numerous applications in the field of viscoelasticity, feedback amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry, and biological sciences [26–32]. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; we only enumerate here the monographs of Kilbas et al. [26, 27], Diethelm [28], Hilfer [29], Podlubny [30], Miller [31], and Zhou [32] and the papers of Agarwal et al. [33, 34], Benchohra et al. [35, 36], El-Borai [37], Lakshmikantham et al. [38–41], Mophou et al. [42–45], N’Guérékata [46], and Zhou et al. [47–50] and the reference therein.

The study of almost periodic and almost automorphic type solutions to fractional differential equations was initiated by Araya and Lizama [11]. In their work, the authors investigated the existence and uniqueness of an almost automorphic mild solution of the semilinear fractional differential equation

\[ D_\alpha^t x(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2, \]  

(1)

where \( A \) is a generator of an \( \alpha \)-resolvent family and \( D_\alpha^t \) is the Riemann-Liouville fractional derivative. In [51], Cuevas and Lizama considered the fractional differential equation:

\[ D_\alpha^t x(t) = Ax(t) + D_\alpha^{t-1} F(t, x(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2, \]  

(2)

where \( A \) is a linear operator of sectoral negative type on a complex Banach space \( X \) and the fractional derivative is understood in the Riemann-Liouville sense. Under suitable conditions on \( F(t, x) \), the authors proved the existence and uniqueness of an almost automorphic mild solution to (2). Cao et al. [59] studied the existence and uniqueness of antiperiodic mild solution to (2). Cao et al. [59] studied the existence and uniqueness of weighted Stepanov-like pseudo almost automorphic mild solution to (2). In [60], Cuevas et al. showed sufficient conditions to ensure the existence and uniqueness of mild solution for (2) in the following classes of vector-valued function spaces: periodic functions, asymptotically periodic functions, pseudo periodic functions, almost periodic functions, asymptotically almost periodic functions, compact almost automorphic functions, compact almost automorphic functions, compact almost automorphic functions, pseudo compact almost automorphic functions, \( \omega \)-asymptotically \( \omega \)-periodic functions, decay functions, and mean decay functions.

Recently, Xia et al. [61] established some sufficient criteria for the existence and uniqueness of \((\mu, \nu)\)-pseudo almost automorphic solution to the semilinear fractional differential equation

\[ D_\alpha^t x(t) = Ax(t) + D_\alpha^{t-1} Bx(t), \quad t \in \mathbb{R}, \]  

(3)

where \( 1 < \alpha < 2, A \) is a sectorial operator of type \( \omega < 0 \) on a complex Banach space \( X \) and \( B \) is a bounded linear operator. The fractional derivative is understood in the Riemann-Liouville sense. Their discussion is divided into two cases, i.e., \( F : \mathbb{R} \times X \rightarrow X \), \( (t, x) \rightarrow F(t, x) \) is \((\mu, \nu)\)-pseudo almost automorphic and \( F : \mathbb{R} \times X \rightarrow X \), and \( (t, x) \rightarrow F(t, x) \) is Stepanov-like \((\mu, \nu)\)-pseudo almost automorphic. Kayitha et al. [62] studied weighted pseudo almost automorphic solutions of the fractional integrodifferential equation

\[ D_\alpha^t x(t) = Ax(t) + D_\alpha^{t-1} F(t, x(t), Bx(t)), \quad t \in \mathbb{R}, \]  

(4)

where \( 1 < \alpha < 2 \) and

\[ Kx(t) = \int_{-\infty}^{t} k(t-s) h(s, x(s)) \, ds, \]

(5)

\( A \) is a linear densely defined sectoral operator on a complex Banach space \( X \), \( F : \mathbb{R} \times X \times X \rightarrow X \), and \( (t, x, y) \rightarrow F(t, x, y) \) is a weighted pseudo almost automorphic function in \( t \in \mathbb{R} \) for each \( x, y \in X \) satisfying suitable conditions. The fractional derivative is understood in the Riemann-Liouville sense. Mophou [63] investigated the existence and uniqueness of weighted pseudo almost automorphic mild solution to the fractional differential equation:

\[ D_\alpha^t x(t) = Ax(t) + D_\alpha^{t-1} F(t, x(t), Bx(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2, \]  

(6)

where \( A : D(A) \subset X \rightarrow X \) is a linear densely operator of sectorial type on a complex Banach space \( X \), \( B : X \rightarrow X \) is a bounded linear operator and \( F : \mathbb{R} \times X \times X \rightarrow X \), and \( (t, x, y) \rightarrow F(t, x, y) \) is a weighted pseudo almost automorphic function in \( t \in \mathbb{R} \) for each \( x, y \in X \) satisfying suitable conditions. The fractional derivative \( D_\alpha^t \) is to be understood in Riemann-Liouville sense. Chang et al.
investigated some existence results of \(\mu\)-pseudo almost automorphic mild solutions to (6) assuming that \(F : \mathbb{R} \times X \times X \to X\) and \((t,x,y) \mapsto F(t,x,y)\) is a \(\mu\)-pseudo almost automorphic function in \(t \in \mathbb{R}\) for each \(x, y \in X\) satisfying suitable conditions. For more on the almost periodicity and almost automorphy for fractional differential equations and related issues, we refer the reader to [65–67] and others.

Equation (6) is motivated by physical problems. Indeed, due to their applications in fields of science where characteristics of anomalous diffusion are presented, type (6) equations are attracting increasing interest (cf. [68–70] and references therein). For example, anomalous diffusion in fractals [69] or in macroeconomics [71] has been recently well studied in the setting of fractional Cauchy problems like (6). For this reason, (6) has gotten a considerable attention in recent years (cf. [51–64]).

To the best of our knowledge, much less is known about the existence of asymptotically almost automorphic mild solutions to (6) when the nonlinearity \(F(t,x,y)\) as a whole loses the Lipschitz continuity with respect to \(x\) and \(y\). Motivated by the abovementioned works, the purpose of this paper is to establish some new existence results of asymptotically almost automorphic mild solutions to (6). In our results, the nonlinearity \(F : \mathbb{R} \times X \times X \to X\), \((t,x,y) \mapsto F(t,x,y)\) does not have to satisfy a (locally) Lipschitz condition (see Remark 22). However, in many papers (for instance, [11, 51–64]) on almost periodic type and almost automorphic type solutions to fractional differential equations, to be able to apply the well-known Banach contraction principle, a (locally) Lipschitz condition for the nonlinearity of corresponding fractional differential equations is needed. As can be seen, our results generalize those as well as related research and have more broad applications. In particular, as application and to illustrate our main results, we will examine some sufficient conditions for the existence of asymptotically almost automorphic mild solutions to the fractional relaxation-oscillation equation given by

\[
\frac{d^2}{dt^2} u(t,x) = \frac{1}{2 + \cos t + \cos \sqrt{2}t} \left[ \sin u(t,x) + u(t,x) \right] - pu(t,x) + \frac{\alpha}{t} \left[ \mu a(t) + \frac{v \cdot e^{-|t|} [u(t,x) + \sin u(t,x)]}{t} \right] , \quad t \in \mathbb{R} , \ x \in [0, \pi]
\]

with boundary conditions \(u(t,0) = u(t,\pi) = 0\), \(t \in \mathbb{R}\), where \(a(t) \in BC(\mathbb{R},\mathbb{R}^+)\) is a function and \(p, \mu, v\) are positive constants.

The rest of this paper is organized as follows. In Section 2, some concepts, the related notations, and some useful lemmas are introduced and established. In Section 3, we prove the existence of asymptotically almost automorphic mild solutions to such problems. The results obtained are utilized to study the existence of asymptotically almost automorphic mild solutions to a fractional relaxation-oscillation equation given in Section 4.

2. Preliminaries

This section is concerned with some notations, definitions, lemmas, and preliminary facts which are used in what follows.

From now on, let \((X, \| \cdot \|)\) and \((Y, \| \cdot \|_Y)\) be two Banach spaces and \(BC(\mathbb{R}, X)\) (resp., \(BC(\mathbb{R} \times Y, X)\)) is the space of all \(X\)-valued bounded continuous functions (resp., jointly bounded continuous functions \(F : \mathbb{R} \times Y \times X \to X\)). Furthermore, \(C_0(\mathbb{R}, X)\) (resp., \(C_0(\mathbb{R} \times Y, X)\)) is the closed subspace of \(BC(\mathbb{R}, X)\) (resp., \(BC(\mathbb{R} \times Y, X)\)) consisting of functions vanishing at infinity (vanishing at infinity uniformly in any compact subset of \(Y \times X\), in other words,

\[
\lim_{|t| \to +\infty} \|g(t,x,y)\| = 0 \quad \text{uniformly for } (x,y) \in \mathbb{K}, \tag{8}
\]

where \(\mathbb{K}\) is any compact subset of \(Y \times X\). Let also \(\mathbb{L}(X)\) be the Banach space of all bounded linear operators from \(X\) into itself endowed with the norm:

\[
\|T\|_{\mathbb{L}(X)} = \sup \{ \|Tx\| : x \in X , \|x\| = 1 \}. \tag{9}
\]

For a bounded linear operator \(A \in \mathbb{L}(X)\), let \(\rho(A)\) and \(D(A)\) stand for the resolvent and domain of \(A\), respectively.

First, let us recall some basic definitions and results on almost automorphic and asymptotically almost automorphic functions.

**Definition 1** ([Bochner] [1] (N’Guérékata) [6]). A continuous function \(F : \mathbb{R} \to X\) is said to be almost automorphic if for every sequence \(\{s_n\}\) of real numbers, there exists a subsequence \(\{s_{n_k}\}\) such that

\[
\Theta(t) = \lim_{n \to \infty} F(t + s_n) \tag{10}
\]

is well defined for each \(t \in \mathbb{R}\) and

\[
\lim_{n \to \infty} \Theta(t - s_n) = F(t) \quad \text{for each } t \in \mathbb{R}. \tag{11}
\]

Denote by \(AA(\mathbb{R}, X)\) the set of all such functions.

**Remark 2** (see [6]). By the point-wise convergence, the function \(\Theta(t)\) in Definition 1 is measurable but not necessarily continuous. Moreover, if \(\Theta(t)\) is continuous, then \(F(t)\) is uniformly continuous (cf., e.g., [17], Theorem 2.6), and if the convergence in Definition 1 is uniform on \(\mathbb{R}\), one gets almost periodicity (in the sense of Bochner and von Neumann). Almost automorphy is thus a more general concept than almost periodicity. There exists an almost automorphic function which is not almost periodic. The function \(F : \mathbb{R} \to \mathbb{R}\) given by

\[
F(t) = \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2}t} \right) \tag{12}
\]

is an example of such functions [72].

**Lemma 3** (see [5]). \(AA(\mathbb{R}, X)\) is a Banach space with the norm \(\|F\|_\infty = \sup_{t \in \mathbb{R}} \|F(t)\|\).
**Definition 4 (see [6]).** A continuous function $F : \mathbb{R} \times Y \times Y \rightarrow X$ is said to be almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where $K$ is any bounded subset of $Y \times Y$, if for every sequence of real numbers $\{s_n\}$, there exists a subsequence $\{s_{n_k}\}$ such that

$$
\lim_{n \to \infty} F(t + s_{n_k}, x, y) = \Theta(t, x, y)
$$

exists for each $t \in \mathbb{R}$ and each $(x, y) \in K$.

The collection of those functions is denoted by $AA(\mathbb{R} \times Y \times Y)$. Similar to Lemma 2.2 of [73] and Proposition 3.2 of [63], we have the following result on almost automorphic functions.

**Lemma 6.** Let $F : \mathbb{R} \times X \times X \rightarrow X$ be almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where $K$ is any bounded subset of $X \times X$, and assume that $F(t, x, y)$ is uniformly continuous on $K$ uniformly for $t \in \mathbb{R}$, that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_1, x_2, y_1, y_2 \in K$ and $\|x_1 - y_1\| + \|x_2 - y_2\| < \delta$ imply that

$$
\|F(t, x_1, x_2) - F(t, y_1, y_2)\| < \varepsilon \quad \forall t \in \mathbb{R}.
$$

Let $x, y : \mathbb{R} \rightarrow X$ be almost automorphic. Then the function $Y : \mathbb{R} \rightarrow X$ defined by $Y(t) = F(t, x(t), y(t))$ is almost automorphic.

**Proof.** Suppose that $\{s_n\}$ is a sequence of real numbers. Then by the definition of almost automorphic functions, we can extract a subsequence $\{t_{n_k}\}$ of $\{s_n\}$ such that

$$
\begin{align*}
(P_1) & \quad \lim_{n \to \infty} x(t + t_{n_k}) = \tilde{x}(t) \quad \text{for each } t \in \mathbb{R}, \\
(P_2) & \quad \lim_{n \to \infty} \tilde{x}(t - t_{n_k}) = x(t) \quad \text{for each } t \in \mathbb{R}, \\
(P_3) & \quad \lim_{n \to \infty} y(t + t_{n_k}) = \tilde{y}(t) \quad \text{for each } t \in \mathbb{R}, \\
(P_4) & \quad \lim_{n \to \infty} \tilde{y}(t - t_{n_k}) = y(t) \quad \text{for each } t \in \mathbb{R}, \\
(P_5) & \quad \lim_{n \to \infty} F(t + t_{n_k}, x, y) = \tilde{F}(t, x, y) \quad \text{for each } t \in \mathbb{R}, x, y \in X, \\
(P_6) & \quad \lim_{n \to \infty} \tilde{F}(t - t_{n_k}, x, y) = F(t, x, y) \quad \text{for each } t \in \mathbb{R}, x, y \in X.
\end{align*}
$$

Write

$$
\tilde{Y}(t) = \tilde{F}(t, \tilde{x}(t), \tilde{y}(t)), \quad t \in \mathbb{R}.
$$

Then

$$
\begin{align*}
\|Y(t + \tau_n) - \tilde{Y}(t)\| & = \|F(t + \tau_n, x(t + \tau_n), y(t + \tau_n)) - \tilde{F}(t, \tilde{x}(t), \tilde{y}(t))\| \\
& \leq \|F(t + \tau_n, x(t + \tau_n), y(t + \tau_n)) - F(t + \tau_n, \tilde{x}(t), \tilde{y}(t))\| + \|F(t + \tau_n, \tilde{x}(t), \tilde{y}(t)) - \tilde{F}(t, \tilde{x}(t), \tilde{y}(t))\|. 
\end{align*}
$$

(19)

Since $x(t)$ and $y(t)$ are almost automorphic, then $x(t), y(t)$ and $\tilde{x}(t)$, and $\tilde{y}(t)$ are bounded. Therefore we can choose a bounded subset $K \subset X \times X$, such that

$$
(x(t), y(t)) \in K,
$$

$$
(\tilde{x}(t), \tilde{y}(t)) \in K
$$

(20)

for each $t \in \mathbb{R}$.

By $(P_1)$, $(P_2)$, and the uniform continuity of $F(t, x, y)$ in $(x(t), y(t)) \in K$, we have

$$
\lim_{n \to \infty} \|F(t + \tau_n, x(t + \tau_n), y(t + \tau_n)) - F(t + \tau_n, \tilde{x}(t), \tilde{y}(t))\| = 0.
$$

(21)

Moreover, by $(P_3)$,

$$
\lim_{n \to \infty} \|F(t + \tau_n, \tilde{x}(t), \tilde{y}(t)) - \tilde{F}(t, \tilde{x}(t), \tilde{y}(t))\| = 0,
$$

(22)

so remembering the above triangle inequality, we deduce that

$$
\lim_{n \to \infty} \|Y(t + \tau_n) - \tilde{Y}(t)\| = 0 \quad \text{for each } t \in \mathbb{R}.
$$

(23)

Using the same argument we can prove that

$$
\lim_{n \to \infty} \|\tilde{Y}(t - \tau_n) - Y(t)\| = 0 \quad \text{for each } t \in \mathbb{R}.
$$

(24)

This proves that $Y(t)$ is almost automorphic by the definition. □

**Remark 7.** If $F(t, x, y)$ satisfies a Lipschitz condition with respect to $x$ and $y$ uniformly in $t \in \mathbb{R}$, i.e., for each pair $x_1, x_2, y_1, y_2 \in X$,

$$
\|F(t, x_1, x_2) - F(t, y_1, y_2)\| \leq L (\|x_1 - y_1\| + \|x_2 - y_2\|)
$$

(25)

uniformly in $t \in \mathbb{R}$, where $L > 0$ is called the Lipschitz constant for the function $F(t, x, y)$, then $F(t, x, y)$ is uniformly continuous on $K$ uniformly for $t \in \mathbb{R}$, where $K$ is any bounded subset of $X \times X$. 

International Journal of Differential Equations
Remark 8. If $F(t, x, y)$ satisfies a local Lipschitz condition with respect to $x$ and $y$ uniformly in $t \in \mathbb{R}$, i.e., for each pair $x_1, x_2, y_1, y_2 \in X, t \in \mathbb{R}$,

\[ \|F(t, x_1, x_2) - F(t, y_1, y_2)\| \leq L(t) (\|x_1 - y_1\| + \|x_2 - y_2\|), \]

(26)

where $L(t) \in BC(\mathbb{R}, \mathbb{R}^+)$, then $F(t, x, y)$ is uniformly continuous on $K$ uniformly for $t \in \mathbb{R}$, where $K$ is any bounded subset of $X \times X$.

Definition 9 (see [6]). A continuous function $F : \mathbb{R} \rightarrow X$ is said to be asymptotically almost automorphic if it can be decomposed as $F(t) = G(t) + \Phi(t)$, where

\[ G(t) \in AA(\mathbb{R}, X), \]

\[ \Phi(t) \in C_0(\mathbb{R}, X). \]

Denote by $AAA(\mathbb{R}, X)$ the set of all such functions.

Remark 10. The function $F : \mathbb{R} \rightarrow \mathbb{R}$ defined by

\[ F(t) = G(t) + \Phi(t) \]

(28)

\[ = \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2t}} \right) + e^{-|t|} \]

is an asymptotically almost automorphic function with

\[ G(t) = \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2t}} \right) \in AA(\mathbb{R}, \mathbb{R}), \]

\[ \Phi(t) = e^{-|t|} \in C_0(\mathbb{R}, \mathbb{R}). \]

Lemma 11 (see [6]). $AAA(\mathbb{R}, X)$ is also a Banach space with the supremum norm \( \| \cdot \|_{\infty} \).

Definition 12 (see [6]). A continuous function $F : \mathbb{R} \times Y \times Y \rightarrow X$ is said to be asymptotically almost automorphic if it can be decomposed as $F(t, x, y) = G(t, x, y) + \Phi(t, x, y)$, where

\[ G(t, x, y) \in AA(\mathbb{R} \times Y \times Y, X), \]

\[ \Phi(t, x, y) \in C_0(\mathbb{R} \times Y \times Y, X). \]

(30)

Denote by $AAA(\mathbb{R} \times Y \times Y, X)$ the set of all such functions.

Remark 13. The function $F : \mathbb{R} \times X \times X \rightarrow X$ given by

\[ F(t, x, y) = G(t, x, y) + \Phi(t, x, y) \]

(29)

\[ = \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2t}} \right) \sin (x) + y] + e^{-|t|} [x + \sin (y)] \]

is asymptotically almost automorphic in $t \in \mathbb{R}$ uniformly for all $(x, y) \in K$, where $K$ is any bounded subset of $X \times X = L^2[0, \pi]$ and

\[ G(t, x, y) = \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2t}} \right) \sin (x) + y] \]

\[ \in AA(\mathbb{R} \times X \times X, X), \]

(32)

\[ \Phi(t, x, y) = e^{-|t|} [x + \sin (y)] \in C_0(\mathbb{R} \times X \times X, X). \]

Next we give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 14 (see [26]). The fractional integral of order $\alpha > 0$ with the lower limit $t_0$ for a function $f$ is defined as

\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f(s) \, ds, \quad t > t_0, \alpha > 0 \]

(33)

provided that the right-hand side is point-wise defined on $[t_0, \infty)$, where $\Gamma$ is the Gamma function.

Definition 15 (see [26]). Riemann-Liouville derivative of order $\alpha > 0$ with the lower limit $t_0$ for a function $f : [t_0, \infty) \rightarrow \mathbb{R}$ can be written as

\[ D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t - s)^{n - \alpha} f(s) \, ds, \]

(34)

\[ t > t_0, \quad n - 1 < \alpha < n. \]

The first and maybe the most important property of Riemann-Liouville fractional derivative is that, for $t > t_0$ and $\alpha > 0$, one has $D_t^\alpha (I^\alpha f(t)) = f(t)$, which means that Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order $\alpha$.

It is important to define sectorial operator for the definition of mild solution of any fractional abstract equations. So, let us now give the definitions of sectorial linear operators and their associated solution operators.

Definition 16 ([74] sectorial operator). A closed and linear operator $A$ is said to be sectorial of type $\omega$ and angle $\theta$ if there exist $0 < \theta < \pi/2, M > 0$, and $\omega \in \mathbb{R}$ such that its resolvent $\rho(A)$ exists outside the sector $\omega + S_\theta = \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$ and

\[ \| (\lambda - A)^{-1} \| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + S_\theta. \]

(35)

Sectorial operators are well studied in the literature, usually for the case $\omega = 0$. For a recent reference including several examples and properties we refer the reader to [74]. Note that an operator $A$ is sectorial of type $\omega$ if and only if $\omega I - A$ is sectorial of type $0$.

Definition 17 (see [75]). Let $A$ be a closed and linear operator with domain $D(A)$ defined on a Banach space $X$. We call $A$
the generator of a solution operator if there are \( \omega \in \mathbb{R} \) and a strongly continuous function \( S_\alpha : \mathbb{R}^+ \rightarrow L(X) \) such that \( \{\lambda^\alpha : \Re \lambda > \omega\} \subseteq \rho(A) \) and

\[
\lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} \, d\lambda,
\]

where \( y \) is a suitable path lying outside the sector \( \omega + \Sigma_\theta \) (cf. [74]).

Very recently, Cuesta in [74](Theorem 1) has proved that if \( A \) is a sectorial operator of type \( \omega < 0 \) for some \( M > 0 \) and \( 0 \leq \theta < \pi(1-\alpha/2) \), then there exists \( C > 0 \) such that

\[
\|S_\alpha(t)\|_{L(X)} \leq \frac{CM}{1 + |\omega|^\alpha t^\alpha} \quad \text{for} \quad t \geq 0.
\]

In this case, \( S_\alpha(t) \) is the solution operator generated by \( A \).

Note that if \( A \) is sectorial of type \( \omega \) with \( 0 \leq \theta \leq \pi(1-\alpha/2) \), then \( A \) is the generator of a solution operator given by

\[
S_\alpha(t) = \frac{1}{2\pi i} \int_y e^{-\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} \, d\lambda,
\]

where \( y \) is a suitable path lying outside the sector \( \omega + \Sigma_\theta \) (cf. [74]).

In this section, we study the existence of asymptotically almost automorphic mild solutions for the semilinear fractional differential equations of the form

\[
D^\alpha x(t) = Ax(t) + D^\alpha F(t, x(t), Bx(t)),
\]

where \( A : D(A) \subset X \rightarrow X \) is a linear densely defined operator of sectorial type of \( \omega < 0 \) on a complex Banach space \( X \), \( B : X \rightarrow X \) is a bounded linear operator and \( F : \mathbb{R} \times X \times X \rightarrow X \) is a given function to be specified later. The fractional derivative \( D^\alpha \) is to be understood in Riemann-Liouville sense.

We recall the following definition that will be essential for us.

**Definition 20** (see [63]). Assume that \( A \) generates an integrable solution operator \( S_\alpha(t) \). A continuous function \( x : \mathbb{R} \rightarrow X \) satisfying the integral equation

\[
x(t) = \int_{-\infty}^t S_\alpha(t-s) F(\sigma, x(\sigma), Bx(\sigma)) \, d\sigma, \quad t \in \mathbb{R}
\]

is called a mild solution on \( \mathbb{R} \) to (39).

In the proofs of our results, we need the following auxiliary result.

**Lemma 21.** Given \( Y(t) \in AA(\mathbb{R}, X) \) and \( Z(t) \in C_0(\mathbb{R}, X) \), let

\[
\Phi_1(t) := \int_{-\infty}^t S_\alpha(t-s) Y(s) \, ds,
\]

\[
\Phi_2(t) := \int_{-\infty}^t S_\alpha(t-s) Z(s) \, ds
\]

Then \( \Phi_1(t) \in AA(\mathbb{R}, X), \Phi_2(t) \in C_0(\mathbb{R}, X) \).

**Proof.** Firstly, note that

\[
\int_0^\infty \frac{1}{1 + |\omega| s^\alpha} \, ds = \frac{|\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \quad \text{for} \quad 1 < \alpha < 2.
\]

Then

\[
\|\Phi_1(t)\| = \left\| \int_{-\infty}^t S_\alpha(t-s) Y(s) \, ds \right\| \leq CM \|Y\|_\infty \int_0^\infty \frac{1}{1 + |\omega| t^\alpha} \, d\omega
\]

\[
= CM \frac{|\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \|Y\|_\infty,
\]

which implies that \( \Phi_1(t) \) is well defined and continuous on \( \mathbb{R} \).

Since \( Y(t) \in AA(\mathbb{R}, X) \), then for any \( \varepsilon > 0 \) and every sequence of real numbers \( \{s_n\} \), there exist a subsequence \( \{s_{n_k}\} \), a function \( \tilde{Y}(t) \), and \( N \in \mathbb{N} \) such that

\[
\|Y(s + s_n) - \tilde{Y}(s)\| < \varepsilon
\]

for each \( n > N \) and every \( s \in \mathbb{R} \).
Define

\[ \widetilde{\Phi}_1 (t) = \int_{-\infty}^{t} T(t-s) \widetilde{Y}(s) \, ds. \]  

(45)

Then

\[ \left\| \Phi_1 (t+s_n) - \widetilde{\Phi}_1 (t) \right\| = \left\| \int_{-\infty}^{t+s_n} S_\alpha(t+s_n-s) Y(s) \, ds \right\| 
\]

\[ - \int_{-\infty}^{t} S_\alpha(t-s) Y(s) \, ds 
\]

\[ = \left\| \int_{-\infty}^{+\infty} S_\alpha(s) Y(t+s_n-s) \, ds \right\| 
\]

\[ - \int_{0}^{+\infty} S_\alpha(s) Y(t-s) \, ds \]  

(46)

\[ \leq CM \int_{0}^{+\infty} \frac{1}{1 + |\omega|} \left\| Y(s+s_n) - \widetilde{Y}(s) \right\| \, ds \]

\[ \leq CM |\omega|^{-1/\alpha} \frac{\pi \varepsilon}{\alpha \sin (\pi/\alpha)} \]  

for each \( n > N \) and every \( t \in \mathbb{R} \). This implies that

\[ \widetilde{\Phi}_1 (t) = \lim_{n \to \infty} \Phi_1 (t+s_n) \]  

(47)

is well defined for each \( t \in \mathbb{R} \).

By a similar argument one can obtain

\[ \lim_{n \to \infty} \widetilde{\Phi}_1 (t-s_n) = \Phi_1 (t) \quad \text{for each} \ t \in \mathbb{R} \]  

(48)

Thus \( \Phi_1 (t) \in AA(\mathbb{R}, X) \).

Since \( Z(t) \in C_0(\mathbb{R}, X) \), one can choose an \( N_1 > 0 \) such that \( \|Z(t)\| < \varepsilon \) for all \( t > N_1 \). This enables us to conclude that, for all \( t > N_1 \),

\[ \left\| \Phi_2 (t) \right\| \leq \left\| \int_{-\infty}^{N_1} S_\alpha(t-s) Z(s) \, ds \right\| 
\]

\[ + \left\| \int_{N_1}^{t} S_\alpha(t-s) Z(s) \, ds \right\| \]

\[ \leq CM \|Z\|_{\infty} \int_{-\infty}^{N_1} \frac{1}{1 + |\omega| (t-s)\alpha} \, ds \]

\[ + \varepsilon CM \int_{N_1}^{t} \frac{1}{1 + |\omega| (t-s)\alpha} \, ds \]

\[ \leq CM \|Z\|_{\infty} \frac{1}{|\omega|} \int_{-\infty}^{N_1} \frac{1}{(t-s)\alpha} \, ds 
\]

\[ + CM |\omega|^{-1/\alpha} \frac{\pi \varepsilon}{\alpha \sin (\pi/\alpha)} \]  

which implies

\[ \lim_{t \to +\infty} \left\| \Phi_2 (t) \right\| = 0. \]  

(50)

On the other hand, from \( Z(t) \in C_0(\mathbb{R}, X) \) it follows that there exists an \( N_2 > 0 \) such that \( \|Z(t)\| < \varepsilon \) for all \( t < -N_2 \). This enables us to conclude that, for all \( t < -N_2 \),

\[ \left\| \Phi_2 (t) \right\| \leq \left\| \int_{-\infty}^{t} S_\alpha(t-s) Z(s) \, ds \right\| \]

\[ \leq \int_{-\infty}^{t} \left\| S_\alpha(t-s) \right\| \|Z(s)\| \, ds \]

\[ \leq CM \int_{-\infty}^{t} \frac{1}{1 + |\omega| (t-s)\alpha} \, ds \]

\[ = CM |\omega|^{-1/\alpha} \frac{\pi \varepsilon}{\alpha \sin (\pi/\alpha)} \]  

which implies

\[ \lim_{t \to -\infty} \left\| \Phi_2 (t) \right\| = 0. \]  

(52)

Now we are in position to state and prove our first main result. To prove our main result, let us introduce the following assumptions:

\( (H_1) \) \( F(t,x,y) = F_1(t,x,y) + F_2(t,x,y) \in AAA(\mathbb{R} \times X \times X, X) \) with

\[ F_1 (t, x, y) \in AA(\mathbb{R} \times X \times X, X), \]

\[ F_2 (t, x, y) \in C_0(\mathbb{R} \times X \times X) \]  

(53)

and there exists a constant \( L > 0 \) such that, for all \( t \in \mathbb{R} \) and \( x_1, x_2, y_1, y_2 \in X \),

\[ \left\| F_1 (t,x_1,x_2) - F_1 (t,y_1,y_2) \right\| \]

\[ \leq L \left( \|x_1 - y_1\| + \|x_2 - y_2\| \right). \]  

(54)

\( (H_2) \) There exist a function \( \beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+) \) and a nondecreasing function \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that, for all \( t \in \mathbb{R} \) and \( x, y \in X \) with \( \|x\| + \|y\| \leq r \),

\[ \left\| F_2 (t, x, y) \right\| \leq \beta(t) \Phi (r) \]  

and \( \lim_{r \to \infty} \frac{\Phi (r)}{r} = \rho_1 \).

\[ \lim_{r \to \infty} \frac{\Phi (r)}{r} = \rho_1 \].

(55)

Remark 22. Assuming that \( F(t,x,y) \) satisfies the assumption \( (H_1) \), it is noted that \( F(t,x,y) \) does not have to meet the
Lipschitz continuity with respect to $x$ and $y$. Such class of asymptotically almost automorphic functions $F(t,x,y)$ are more complicated than those with Lipschitz continuity with respect to $x$ and $y$ and little is known about them.

Let $\beta(t)$ be the function involved in assumption $(H_2)$. Define

$$\sigma(t) = \int_{-\infty}^{t} \frac{\beta(s)}{1 + |\omega|(t-s)^\alpha} ds, \quad t \in \mathbb{R}. \tag{56}$$

Lemma 23. $\sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$. 

Proof. Since $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$, one can choose a $T_1 > 0$ such that $\|\beta(t)\| < \varepsilon$ for all $t > T_1$. This enables us to conclude that, for all $t > T_1$,

$$\|\sigma(t)\| \leq \left\| \int_{-\infty}^{T_1} \frac{\beta(s)}{1 + |\omega|(t-s)^\alpha} ds \right\| + \varepsilon \int_{T_1}^{t} \frac{1}{1 + |\omega|(t-s)^\alpha} ds$$

$$\leq \frac{\|\beta\|_{\mathcal{L}}}{|\omega|} \left( \int_{-\infty}^{T_1} \frac{1}{(t-s)^\alpha} ds + \varepsilon \int_{T_1}^{t} \frac{1}{(t-s)^\alpha} ds \right) + \frac{|\omega|^{1-\alpha} \pi \varepsilon}{\alpha \sin(\pi/\alpha)}$$

$$\leq \frac{\|\beta\|_{\mathcal{L}}}{|\omega|} \left( \frac{1}{(t-T_1)^{\alpha-1}} + \frac{|\omega|^{1-\alpha} \pi \varepsilon}{\alpha \sin(\pi/\alpha)} \right),$$

which implies

$$\lim_{t \to +\infty} \|\sigma(t)\| = 0. \tag{58}$$

On the other hand, from $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ it follows that there exists a $T_2 > 0$ such that $\|\beta(t)\| < \varepsilon$ for all $t < -T_2$. This enables us to conclude that, for all $t < -T_2$,

$$\|\sigma(t)\| = \left\| \int_{-\infty}^{t} \frac{\beta(s)}{1 + |\omega|(t-s)^\alpha} ds \right\|$$

$$\leq \varepsilon \int_{-\infty}^{t} \frac{1}{1 + |\omega|(t-s)^\alpha} ds + \frac{|\omega|^{1-\alpha} \pi \varepsilon}{\alpha \sin(\pi/\alpha)},$$

which implies

$$\lim_{t \to -\infty} \|\sigma(t)\| = 0. \tag{60}$$

Proof. The proof is divided into the following five steps.

Step 1. Define a mapping $\Lambda$ on $AA(\mathbb{R}, X)$ by

$$(\Lambda v)(t) = \int_{-\infty}^{t} S_\alpha(t-s) F_1(s, v(s), Bv(s)) ds, \quad t \in \mathbb{R} \tag{62}$$

and prove $\Lambda$ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

Firstly, since the function $s \to F_1(s, v(s), Bv(s))$ is bounded in $\mathbb{R}$ and

$$|[\Lambda v](t)| \leq \int_{-\infty}^{t} \|S_\alpha(t-s)\| \|F_1(s, v(s), Bv(s))\| ds$$

$$\leq CM \int_{-\infty}^{t} \frac{1}{1 + |\omega|(t-s)^\alpha} \|F_1(s, v(s), Bv(s))\| ds$$

$$\leq CM \|F_1\|_{\mathcal{L}} \frac{1}{1 + |\omega|(t-s)^\alpha} ds$$

$$= \frac{CML |\omega|^{-1/\alpha} \pi \|F_1\|_{\mathcal{L}}}{\alpha \sin(\pi/\alpha)},$$

this implies that $(\Lambda v)(t)$ exists. Moreover from $F_1(t, x, y) \in AA(\mathbb{R} \times X \times X)$ satisfying (54), together with Lemma 6 and Remark 7, it follows that

$$F_1(\cdot, v(\cdot), Bv(\cdot)) \in AA(\mathbb{R}, X)$$

for every $v(\cdot) \in AA(\mathbb{R}, X). \tag{64}$$

This together with Lemma 21, implies that $\Lambda$ is well defined and maps $AA(\mathbb{R}, X)$ into itself.

In the sequel, we verify that $\Lambda$ is continuous.

Let $v_n(t), v(t)$ be in $AA(\mathbb{R}, X)$ with $v_n(t) \to v(t)$ as $n \to \infty$; then one has

$$\left[ \int_{-\infty}^{t} S_\alpha(t-s) \right] \left[ F_1(s, v_n(s), Bv_n(s)) - F_1(s, v(s), Bv(s)) \right] ds \leq L \int_{-\infty}^{t} \|S_\alpha(t-s)\| ds$$

$$\leq CML \int_{-\infty}^{t} \frac{1}{1 + |\omega|(t-s)^\alpha} \|F_1(s, v(s), Bv(s)) - F_1(s, v_n(s), Bv_n(s))\| ds$$

$$\leq CM \|F_1\|_{\mathcal{L}} \frac{1}{1 + |\omega|(t-s)^\alpha} \left( 1 + \|B\|_{\mathcal{L}(X)} \right) \|v_n(s) - v(s)\| ds \leq CML (1 + \|B\|_{\mathcal{L}(X)}) \|v_n(s) - v(s)\| ds$$

$$\to 0$$

as $n \to \infty$.

Theorem 24. Assume that $A$ is sectorial of type $\omega < 0$. Let $F : \mathbb{R} \times X \times X \to X$ satisfy the hypotheses $(H_1)$ and $(H_2)$.

Put $p_2 = \sup_{t \in \mathbb{R}} \sigma(t)$. Then (39) has at least one asymptotically almost automorphic mild solution provided that

$$\frac{CML (1 + \|B\|_{\mathcal{L}(X)}) |\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} + CM (1 + \|B\|_{\mathcal{L}(X)}) p_1 p_2 < 1. \tag{61}$$
\[
\cdot \int_{-\infty}^{t} \frac{1}{1 + |\omega(t - s)|^\alpha} ds = \frac{CML (1 + \|B\|_{L(\mathbb{X})}) |\omega|^{-1/\alpha} \pi}{\alpha \sin (\pi/\alpha)} \|v_n - v\|_\infty .
\]

Therefore, as \( n \to \infty \) and \( \Lambda v_n \to \Lambda v \), hence \( \Lambda \) is continuous.

Next, we prove that \( \Lambda \) is a contraction on \( AA(\mathbb{R}, \mathbb{X}) \) and has a unique fixed point \( v(t) \in AA(\mathbb{R}, \mathbb{X}) \).

In fact, let \( v_1(t), v_2(t) \) be in \( AA(\mathbb{R}, \mathbb{X}) \), and similar to the above proof of the continuity of \( \Lambda \), one has

\[
\|[\Lambda v_1](t) - [\Lambda v_2](t)\| 
\leq \frac{CML (1 + \|B\|_{L(\mathbb{X})}) |\omega|^{-1/\alpha} \pi}{\alpha \sin (\pi/\alpha)} \|v_1 - v_2\|_\infty ,
\]

which implies

\[
\|[\Lambda v_1](t) - [\Lambda v_2](t)\|_\infty 
\leq \frac{CML (1 + \|B\|_{L(\mathbb{X})}) |\omega|^{-1/\alpha} \pi}{\alpha \sin (\pi/\alpha)} \|v_1 - v_2\|_\infty .
\]

Together with (61), this proves that \( \Lambda \) is a contraction on \( AA(\mathbb{R}, \mathbb{X}) \). Thus, Banach’s fixed point theorem implies that \( \Lambda \) has a unique fixed point \( v(t) \in AA(\mathbb{R}, \mathbb{X}) \).

**Step 2.**

Set

\[
\Omega_r = \{ \omega(t) \in C_0(\mathbb{R}, \mathbb{X}) : \|\omega(t)\| \leq r \} .
\]

For the above \( v(t) \), define \( \Gamma = \Gamma^1 + \Gamma^2 \) on \( C_0(\mathbb{R}, \mathbb{X}) \) as

\[
\Gamma^1(\omega)(t) = \int_{-\infty}^{t} S_\alpha (t - s) \cdot \left[ F_1 (s, v(s) + \omega(s)) + B(\omega(s) + \omega(s)) \right] ds,
\]

\[
\Gamma^2(\omega)(t) = \int_{-\infty}^{t} S_\alpha (t - s) \cdot F_2 (s, v(s) + \omega(s)) ds
\]

and prove that \( \Gamma \) maps \( \Omega_k \) into itself, where \( k_0 \) is a given constant.

Firstly, from (54) it follows that, for all \( s \in \mathbb{R} \) and \( \omega(s) \in X \),

\[
\left[ F_1 (s, v(s) + \omega(s)) + B(\omega(s) + \omega(s)) \right] - F_1 (s, v(s), Bv(s)) \leq L \left[ \|\omega(s)\| + \|B\omega(s)\| \right]
\]

\[
\leq L (1 + \|B\|_{L(\mathbb{X})}) \|\omega(s)\| ,
\]

which implies that

\[
F_1 (\cdot, v(\cdot) + \omega(\cdot)) + B(\omega(\cdot) + \omega(\cdot)) - F_1 (\cdot, v(\cdot), Bv(\cdot)) \in C_0 (\mathbb{R}, \mathbb{X}) \quad \text{for every } \omega(\cdot) \in C_0 (\mathbb{R}, \mathbb{X}) .
\]

According to (55), one has

\[
\left\| F_2 (s, v(s) + \omega(s), B(v(s) + \omega(s))) \right\| \leq \beta(s)
\]

\[
\cdot \Phi \left( \|\omega(s) + B\omega(s)\| + \sup_{s \in \mathbb{R}} \|v(s) + Bv(s)\| \right)
\]

\[
\leq \beta(s) \Phi \left( (1 + \|B\|_{L(\mathbb{X})}) \|\omega(s)\| \right)
\]

\[
+ (1 + \|B\|_{L(\mathbb{X})}) \sup_{s \in \mathbb{R}} \|v(s)\| = \beta(s)
\]

\[
\leq \beta(s) \Phi \left( (1 + \|B\|_{L(\mathbb{X})}) \|\omega(s)\| + \sup_{s \in \mathbb{R}} \|v(s)\| \right)
\]

(72)

(73)

Those, together with Lemma 21, yield that \( \Gamma \) is well defined and maps \( C_0(\mathbb{R}, \mathbb{X}) \) into itself.

On the other hand, in view of (55) and (61) it is not difficult to see that there exists a constant \( k_0 > 0 \) such that

\[
\frac{CML (1 + \|B\|_{L(\mathbb{X})}) |\omega|^{-1/\alpha} \pi}{\alpha \sin (\pi/\alpha)} k_0
\]

\[
+ CML \Phi \left( (1 + \|B\|_{L(\mathbb{X})}) \left( k_0 + \sup_{s \in \mathbb{R}} \|v(s)\| \right) \right)
\]

\[
\leq k_0 .
\]

This enables us to conclude that, for any \( t \in \mathbb{R} \) and \( \omega(t) \in \Omega_{k_0} \),

\[
\left\| \left( \Gamma^1 \omega_1 \right)(t) + \left( \Gamma^2 \omega_2 \right)(t) \right\| \leq \int_{-\infty}^{t} S_\alpha (t - s) \cdot \left[ F_1 (s, v(s) + \omega_1(s)) + B(\omega_1(s) + \omega_1(s)) \right] ds
\]

\[
- F_1 (s, v(s), Bv(s)) \right\| ds \geq \int_{-\infty}^{t} S_\alpha (t - s) \cdot F_2 (s, v(s) + \omega_2(s)) \right\| ds
\]

\[
\leq \int_{-\infty}^{t} S_\alpha (t - s) \cdot F_1 (s, v(s) + \omega_1(s)) \right\| ds
\]

\[
+ \omega_1(s) + B(\omega_1(s) + \omega_1(s)) \right\| ds
\]

\[
\leq \int_{-\infty}^{t} S_\alpha (t - s) \cdot F_2 (s, v(s) + \omega_2(s)) \right\| ds
\]

\[
\leq \int_{-\infty}^{t} S_\alpha (t - s) \cdot F_1 (s, v(s) + \omega_1(s)) \right\| ds
\]

\[
\leq \int_{-\infty}^{t} S_\alpha (t - s) \cdot F_2 (s, v(s) + \omega_2(s)) \right\| ds
\]

\[
\leq CM \int_{-\infty}^{t} \frac{1}{1 + |\omega(t - s)|^\alpha} \|\omega_1(s)\| ds
\]
which implies that \( (\Gamma^1 \omega_1)(t) + (\Gamma^2 \omega_2)(t) \in \Omega_{k_2} \). Thus \( \Gamma \) maps \( \Omega_{k_2} \) into itself.

**Step 3.** Show that \( \Gamma^1 \) is a contraction on \( \Omega_{k_2} \).

In fact, for any \( \omega_1(t), \omega_2(t) \in \Omega_{k_2} \) and \( t \in \mathbb{R} \), from (54) it follows that

\[
\| [F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) - F_1(s, v(s), Bv(s))] - [F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s)))] \| \leq L \| \omega_1(s) - \omega_2(s) \| 
\]

for all \( s \in (-\infty, t_1] \) as \( k \rightarrow +\infty \) and

\[
\| F_2(\cdot, v(\cdot) + \omega_1(\cdot), B(\cdot)) - F_2(\cdot, v(\cdot) + \omega_0(\cdot), B(\cdot) + \omega_0(\cdot)) \| 
\]

\[
\leq 2|\Phi((1 + \|B\|_{L(X)}) (k_0 + \|v\|_{\infty}))\beta(\cdot) | 
\]

\( \in L^1(-\infty, t_1] \).

Hence, by the Lebesgue dominated convergence theorem we deduce that there exists an \( N > 0 \) such that

\[
CM \int_{-\infty}^{t_1} \frac{1}{1 + |\omega| (t-s)}\| F_2(s, v(s) + \omega_k(s), B(v(s) + \omega_k(s))) - F_2(s, v(s) + \omega_0(s), B(v(s) + \omega_0(s))) \| \, ds \leq \frac{\varepsilon}{3} 
\]

and

\[
\| [F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) - F_1(s, v(s), Bv(s))] - [F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s)))] \| \leq L \| \omega_1(s) - \omega_2(s) \| 
\]

for all \( s \in (-\infty, t_1] \) as \( k \rightarrow +\infty \) and

\[
\| F_2(\cdot, v(\cdot) + \omega_1(\cdot), B(\cdot)) - F_2(\cdot, v(\cdot) + \omega_0(\cdot), B(\cdot) + \omega_0(\cdot)) \| 
\]

\[
\leq 2|\Phi((1 + \|B\|_{L(X)}) (k_0 + \|v\|_{\infty}))\beta(\cdot) | 
\]

\( \in L^1(-\infty, t_1] \).

Thus

\[
\| (\Gamma^1 \omega_1)(t) - (\Gamma^1 \omega_2)(t) \| \leq \frac{\varepsilon}{3CM} \]
whenever \( k \geq N \). Thus

\[
\left\| \left( \Gamma^2 \omega \right)(t) - \left( \Gamma^2 \omega \right)(t_1) \right\| = \left\| \int_{t_1}^t S_\alpha (t - s) F_2 (s, v(s) + \omega (s)) \, ds \right\|
\]

and in view of \( \sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+) \), which follows from Lemma 23, one concludes that

\[
\lim_{[t] \to +\infty} \left( \Gamma^2 \omega \right)(t) = 0 \quad \text{uniformly for } \omega(t) \in \Omega_{k_0}. \tag{85}
\]

As

\[
\left\| \left( \Gamma^2 \omega \right)(t) \right\| = \left\| \int_{-\infty}^t S_\alpha (t - s) F_2 (s, v(s) + \omega (s)) \, ds \right\|
\]

and

\[
\left\| \left( \Gamma^2 \omega \right)(t) \right\| \leq CM \sigma(t) \Phi \left( (1 + \|B\|_{L(X)})(k_0 + \|\omega\|_{\infty}) \right), \tag{84}
\]

Hence, given \( \varepsilon_0 > 0 \), one can choose a \( \xi > 0 \) such that

\[
\left\| \int_{\xi}^{+\infty} S_\alpha (\tau) F_2 (\tau - t, v(t)) \right\| < \varepsilon_0. \tag{87}
\]

Thus we get

\[
z(t) \in \xi \mathcal{C} \left( \left\{ S_\alpha (\tau) F_2 (\lambda, v(\lambda) + \omega(\lambda), B(v(\lambda) + \omega(\lambda)) : 0 \leq \tau \leq \xi, t - \xi \leq \lambda \leq \xi, \|\omega\|_{\infty} \leq r \right\} \right) + B_{\varepsilon_0}(X). \tag{88}
\]
\[ \omega(t) \in \Omega_{k_i} \]

Now an application of Lemma 18 justifies the compactness of \( \Gamma \).

**Step 5.** Show that (39) has at least one asymptotically almost automorphic mild solution.

Firstly, the complete continuity of \( \Gamma \), together with the results of Steps 2 and 3 as well as Lemma 19, yields that \( \Gamma \) has at least one fixed point \( \omega(t) \in \Omega_{k_i} \); furthermore \( \omega(t) \in C_0(\mathbb{R}, X) \).

Then, consider the following coupled system of integral equations:

\[
\begin{align*}
\nu(t) &= \int_{-\infty}^{t} S_\alpha (t-s) \mathcal{F}_1(s, \nu(s), B\nu(s)) \, ds, \quad t \in \mathbb{R}, \\
\omega(t) &= \int_{-\infty}^{t} S_\alpha (t-s) \\
&\quad \cdot \left[ \mathcal{F}_1(s, \nu(s) + \omega(s), B(\nu(s) + \omega(s))) - \mathcal{F}_2(s, \nu(s), B\nu(s)) \right] \, ds.
\end{align*}
\]

(91)

\[ t \in \mathbb{R}. \]

From the result of Step 1, together with the above fixed point \( \omega(t) \in C_0(\mathbb{R}, X) \), it follows that

\[ (\nu(t), \omega(t)) \in AA(\mathbb{R}, X) \times C_0(\mathbb{R}, X) \]

(92)

is a solution to system (91). Thus

\[ x(t) = \nu(t) + \omega(t) \in AAA(\mathbb{R}, X) \]

(93)

and it is a solution to the integral equation

\[ x(t) = \int_{-\infty}^{t} S_\alpha (t-s) F(s, x(s), Bx(s)) \, ds, \quad t \in \mathbb{R}; \]

(94)

that is, \( x(t) \) is an asymptotically almost automorphic mild solution to (39).

Taking \( \Lambda = -\rho^\alpha I \) with \( \rho > 0 \) in (39), the above theorem follows the following corollary.

**Corollary 25.** Let \( F : \mathbb{R} \times X \times X \rightarrow X \) satisfy (H1) and (H2). Put \( \rho^\alpha = \sup_{t \in \mathbb{R}} \sigma(t) \). Then (39) admits at least one asymptotically almost automorphic mild solution whenever

\[
\frac{CL(1 + \|B\|_{L(X)})\rho \pi}{\alpha \sin(\pi/\alpha)} + C(1 + \|B\|_{L(X)})\rho_1 \rho_2 < 1.
\]

(95)

Remark 26. It is interesting to note that the function \( \alpha \to \text{as} \frac{\pi}{\alpha} \) is increasing from \( 0 \) to \( 2/\pi \) in the interval \( 1 < \alpha < 2 \). Therefore, with respect to condition (61), the class of admissible terms \( F(t, x(t), Bx(t)) \) is the best in the case \( \alpha = 2 \) and the worst in the case \( \alpha = 1 \).

Theorem 24 can be extended to the case of \( F_t(x, y) \) being locally Lipschitz continuous with respect to \( x \) and \( y \), where we have the following result.

\[ (H'_1) \quad F(t, x, y) = F_1(t, x, y) + F_2(t, x, y) \in AAA(\mathbb{R} \times X \times X, X) \]

(96)

and for all \( x_1, x_2, y_1, y_2 \in X, t \in \mathbb{R}, \)

\[
\| F_1(t, x_1, x_2) - F_1(t, y_1, y_2) \| \leq L(t) \| x_1 - y_1 \| + \| x_2 - y_2 \|,
\]

(97)

where \( L(t) \) is a function on \( \mathbb{R} \).

**Theorem 27.** Assume that \( A \) is sectorial of type \( \omega < 0 \). Let \( F : \mathbb{R} \times X \times X \rightarrow X \) satisfy the hypotheses (H1) and (H2) with \( L(t) \in BC(\mathbb{R}, \mathbb{R}^+ \). Put \( \rho^\alpha = \sup_{t \in \mathbb{R}} \sigma(t) \). Let \( \|L\| = \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} L(s) \, ds \). Then (39) has at least one asymptotically almost automorphic mild solution provided that

\[
\frac{CML\|L\|\|B\|_{L(X)}\rho \pi}{\alpha \sin(\pi/\alpha)} + CM\rho \rho_2 (1 + \|B\|_{L(X)}) < 1.
\]

(98)

**Proof.** The proof is divided into the following five steps.

**Step 1.** Define a mapping \( \Lambda \) on \( AA(\mathbb{R}, X) \) by (62) and prove that \( \Lambda \) has a unique fixed point \( v(t) \in AA(\mathbb{R}, X) \).

Firstly, similar to the proof in Step 1 of Theorem 24, we can prove that (\( \Lambda v \))(t) exists. Moreover from \( F(t, x(t), y(t)) \in AA(\mathbb{R} \times X \times X, X) \) satisfying (97), together with Lemma 6 and Remark 8, it follows that

\[ F_1(t, v(t), Bv(t)) \in AA(\mathbb{R}, X) \]

(99)

for every \( v(t) \in AA(\mathbb{R}, X) \).

This, together with Lemma 21, implies that \( \Lambda \) is well defined and maps \( AP(\mathbb{R}, X) \) into itself.

In the sequel, we verify that \( \Lambda \) is continuous.

Let \( v_n(t) \in AA(\mathbb{R}, X) \) with \( v_n(t) \to v(t) \) as \( n \to \infty \); then one has

\[
\| [\Lambda v_n] (t) - [\Lambda v] (t) \| = \int_{-\infty}^{t} S_\alpha (t-s) \cdot \left[ F_1(s, v_n(s), Bv_n(s)) - F_1(s, v(s), Bv(s)) \right] \, ds
\]

\[ - \int_{-\infty}^{t} L(s) \| S_\alpha (t-s) \| \, ds \leq \int_{-\infty}^{t} L(s) \| S_\alpha (t-s) \| \, ds
\]

\[ - \int_{-\infty}^{t} L(s) \| S_\alpha (t-s) \| \, ds
\]
\[
\leq CM \int_{-\infty}^{t} \frac{L(s)}{1 + |\omega|(t-s)^{\alpha}} \left(1 + \|B\|_{L(X)}\right) \left\|v_n(s) - v(s)\right\| ds \\
- v(s)\| ds \leq CM \left(1 + \|B\|_{L(X)}\right) \\
\cdot \left(\sum_{m=0}^{\infty} \frac{1}{t-(m+1)} L(s) ds\right) \left\|v_n - v\right\|_{\infty} \\
\leq CM \left(1 + \|B\|_{L(X)}\right) \\
\left(\sum_{m=0}^{\infty} \frac{1}{t-(m+1)} L(s) ds\right) \left\|v_n - v\right\|_{\infty} \\
\leq CM \frac{L\|\|\omega\|^{-1/\alpha}}{\alpha \sin(\pi/\alpha)} \left(1 + \|B\|_{L(X)}\right) \left\|v_n - v\right\|_{\infty}. 
\] (100)

Therefore, as \( n \to \infty \) and \( \Lambda v_n \to \Lambda v \), hence \( \Lambda \) is continuous.

Next, we prove that \( \Lambda \) is a contraction on \( AA(\mathbb{R}, X) \) and has a unique fixed point \( v(t) \in AA(\mathbb{R}, X) \).

In fact, for \( v_1(t), v_2(t) \) in \( AA(\mathbb{R}, X) \), similar to the above proof of the continuity of \( \Lambda \), one has

\[
\left\|(\Lambda v_1)(t) - (\Lambda v_2)(t)\right\| \\
\leq CM \frac{L\|\|\omega\|^{-1/\alpha}}{\alpha \sin(\pi/\alpha)} \left(1 + \|B\|_{L(X)}\right) \left\|v_1 - v_2\right\|_{\infty}, 
\] (101)

which implies that

\[
\left\|(\Lambda v_1)(t) - (\Lambda v_2)(t)\right\|_{\infty} \\
\leq CM \frac{L\|\|\omega\|^{-1/\alpha}}{\alpha \sin(\pi/\alpha)} \left(1 + \|B\|_{L(X)}\right) \left\|v_1 - v_2\right\|_{\infty}. 
\] (102)

Hence, by (98), together with the contraction principle, \( \Lambda \) has a unique fixed point \( v(t) \in AA(\mathbb{R}, X) \).

Step 2. Set

\[
\Omega_r := \{|w(t)| \in C_0(\mathbb{R}, X) : \|w(t)\| \leq r\}. 
\] (103)

For the above \( v(t) \), define \( \Gamma := \Gamma_1 + \Gamma_2 \) on \( C_0(\mathbb{R}, X) \) as (69) and prove that \( \Gamma \) maps \( \Omega_k \) into itself, where \( k_0 \) is a constant.

Firstly, from (97), it follows that, for all \( s \in \mathbb{R}, \omega(s) \in X \),

\[
\|F_1(s, v(s) + \omega(s), B(v(s) + \omega(s)))\| \\
- F_1(s, v(s), Bv(s)) \leq L(s) \left\|\omega(s)\right\| + \|B\left(\omega(s)\right)\| 
\] (104)

which together with \( L(s) \in BC(\mathbb{R}, \mathbb{R}^+) \) implies that

\[
F_1(v, \omega) \in C_0(\mathbb{R}, X) \\
\] (105)

for every \( \omega \in C_0(\mathbb{R}, X) \).

According to (55), one has

\[
\|F_2(s, v(s) + \omega(s), B(v(s) + \omega(s)))\| \leq \beta(s) \\
\cdot \left\|\|\omega(s)\| + B\omega(s)\right\| + \sup_{s \in \mathbb{R}} \|v(s) + Bv(s)\| 
\] (106)

\[
\leq \beta(s) \cdot \left(1 + \|B\|_{L(X)}\right) \left\|\omega(s)\right\| \\
+ (1 + \|B\|_{L(X)}) \sup_{s \in \mathbb{R}} \|v(s)\| \leq \beta(s) 
\]

for all \( s \in \mathbb{R} \) and \( \omega(s) \in X \) with \( \|\omega(s)\| \leq r \); then

\[
F_2(v, \omega) \in C_0(\mathbb{R}, X) \\
as \beta(s) \in C_0(\mathbb{R}, \mathbb{R}^+). 
\]

Those, together with Lemma 21, yield that \( \Gamma \) is well defined and maps \( C_0(\mathbb{R}, X) \) into itself.

On the other hand, in view of (55) and (98) it is not difficult to see that there exists a constant \( k_0 > 0 \) such that

\[
CM \frac{L\|\|\omega\|^{-1/\alpha}}{\alpha \sin(\pi/\alpha)} \left(1 + \|B\|_{L(X)}\right) k_0 \\
\] (107)

\[
+ C\rho \cdot \Phi \left(1 + \|B\|_{L(X)}\right) \left(\|\omega(s)\| + \sup_{s \in \mathbb{R}} \|v(s)\|\right) 
\] (108)

\[
\leq k_0. 
\]

This enables us to conclude that, for any \( t \in \mathbb{R} \) and \( \omega_1(t) \), \( \omega_2(t) \in \Omega_{k_0} \),

\[
\|F_1(t, \omega_1(t))\| + \left(\|\omega_2(t)\| + \|B\omega_2(t)\|\right) ds \leq \int_{-\infty}^{t} S_{\alpha}(t-s) \|\omega_1(s)\| \] (109)

\[
+ \|B\omega_1(s)\| ds + CM \int_{-\infty}^{t} \frac{\beta(s)}{1 + |\omega|(t-s)^{\alpha}} \Phi \left(\|\omega_2(s)\| + \|B\omega_2(s)\| + \|v(s)\| + \|Bv(s)\|\right) ds \] (110)

\[
\leq CM \int_{-\infty}^{t} \frac{L(s)}{1 + |\omega|(t-s)^{\alpha}} \left(1 + \|B\|_{L(X)}\right) \] (111)

\[
\cdot \|\omega_1(s)\| ds + CM \int_{-\infty}^{t} \frac{\beta(s)}{1 + |\omega|(t-s)^{\alpha}} \Phi \left(1 + \|B\|_{L(X)}\right) \left(\|\omega_2(s)\| + \|v(s)\|\right) ds \leq CM \left(1 
\]
which implies that $(\Gamma^1 \omega_1)(t) + (\Gamma^2 \omega_2)(t) \in \Omega_{k_0}$. Thus $\Gamma$ maps $\Omega_{k_0}$ into itself.

**Step 3.** Show that $\Gamma^1$ is a contraction on $\Omega_{k_0}$.

In fact, for any $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$, from (97) it follows that

\[
\begin{align*}
\| [F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) & - F_1(s, v(s), Bv(s))] - [F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s))) & - F_1(s, v(s), Bv(s))] \| \\ & \leq L(s) \| \omega_1(s) - \omega_2(s) \|,
\end{align*}
\]  

which implies that $(\Gamma^1 \omega_1)(t) + (\Gamma^2 \omega_2)(t) \in \Omega_{k_0}$. Thus $\Gamma$ maps $\Omega_{k_0}$ into itself.

**Step 3.** Show that $\Gamma^1$ is a contraction on $\Omega_{k_0}$.

In fact, for any $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$, from (97) it follows that

\[
\begin{align*}
\| [F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) & - F_1(s, v(s), Bv(s))] - [F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s))) & - F_1(s, v(s), Bv(s))] \| \\ & \leq L(s) \| \omega_1(s) - \omega_2(s) \|,
\end{align*}
\]  

which implies that $(\Gamma^1 \omega_1)(t) + (\Gamma^2 \omega_2)(t) \in \Omega_{k_0}$. Thus $\Gamma$ maps $\Omega_{k_0}$ into itself.

**Step 3.** Show that $\Gamma^1$ is a contraction on $\Omega_{k_0}$.

In fact, for any $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$, from (97) it follows that

\[
\begin{align*}
\| [F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) & - F_1(s, v(s), Bv(s))] - [F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s))) & - F_1(s, v(s), Bv(s))] \| \\ & \leq L(s) \| \omega_1(s) - \omega_2(s) \|,
\end{align*}
\]  

which implies that $(\Gamma^1 \omega_1)(t) + (\Gamma^2 \omega_2)(t) \in \Omega_{k_0}$. Thus $\Gamma$ maps $\Omega_{k_0}$ into itself.

**Step 3.** Show that $\Gamma^1$ is a contraction on $\Omega_{k_0}$.

In fact, for any $\omega_1(t), \omega_2(t) \in \Omega_{k_0}$ and $t \in \mathbb{R}$, from (97) it follows that

\[
\begin{align*}
\| [F_1(s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) & - F_1(s, v(s), Bv(s))] - [F_1(s, v(s) + \omega_2(s), B(v(s) + \omega_2(s))) & - F_1(s, v(s), Bv(s))] \| \\ & \leq L(s) \| \omega_1(s) - \omega_2(s) \|,
\end{align*}
\]  

which implies that $(\Gamma^1 \omega_1)(t) + (\Gamma^2 \omega_2)(t) \in \Omega_{k_0}$. Thus $\Gamma$ maps $\Omega_{k_0}$ into itself.

**Step 4.** Show that $\Gamma^2$ is completely continuous on $\Omega_{k_0}$.

The proof is similar to the proof in Step 4 of Theorem 24.

**Step 5.** Show that (39) has at least one asymptotically almost automorphic mild solution.
The proof is similar to the proof in Step 5 of Theorem 24.

Taking $A = -\rho^a I$ with $\rho > 0$ in (39), Theorem 27 gives the following corollary.

**Corollary 28.** Let $F : \mathbb{R} \times X \times X \to X$ satisfy $(H'_1)$ and $(H'_2)$ with $L(t) \in BC(\mathbb{R}, \mathbb{R}^r)$. Put $\rho_\alpha := \sup_{t \in \mathbb{R}} \rho(t)$. Let $||F|| = \sup_{t \in \mathbb{R}} \int_{t_0}^{t_1} L(s)ds$. Then (39) admits at least one asymptotically almost automorphic mild solution whenever

$$C \|L\|\rho \alpha (1 + \|B\|_{L(\mathbb{X})}) + C\rho_1 \rho_2 (1 + \|B\|_{L(\mathbb{X})}) < 1.$$  \hspace{1cm} (113)

Now we consider a more general case of equations introducing a new class of functions $L(t)$. We have the following result.

$(H'_3)$ There exists a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^r)$ such that, for all $t \in \mathbb{R}$ and $x, y \in X$,

$$\|F_2 (t, x, y)\| \leq \beta(t) (\|x\| + \|y\|).$$ \hspace{1cm} (114)

**Theorem 29.** Assume that $A$ is sectorial of type $\omega < 0$. Let $F : \mathbb{R} \times X \times X \to X$ satisfy the hypotheses $(H'_1)$ and $(H'_2)$ with $L(t) \in BC(\mathbb{R}, \mathbb{R}^r)$. Moreover the integral $\int_{-\infty}^t \max \{L(s), \beta(s)\}ds$ exists for all $t \in \mathbb{R}$. Then (39) has at least one asymptotically almost automorphic mild solution.

**Proof.** The proof is divided into the following five steps.

**Step 1.** Define a mapping $\Lambda$ on $AA(\mathbb{R}, X)$ by (62) and prove that $\Lambda$ has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

Firstly, similar to the proof in Step 1 of Theorem 27, we can prove that $\Lambda$ is well defined and maps $AP(\mathbb{R}, X)$ into itself; moreover $\Lambda$ is continuous.

Next, we prove that $\Lambda$ is a contraction on $AA(\mathbb{R}, X)$ and has a unique fixed point $v(t) \in AA(\mathbb{R}, X)$.

In fact, for $v_1(t), v_2(t)$ is in $AA(\mathbb{R}, X)$ and defines a new norm

$$\|\lambda \alpha \| := \sup_{t \in \mathbb{R}} \{\mu(t) \|x(t)\|\},$$ \hspace{1cm} (115)

where $\mu(t) := e^{-k \int_{-\infty}^t \max \{L(s), \beta(s)\}ds}$ and $k$ is a fixed positive constant. Let $C_\alpha := \sup_{t \in \mathbb{R}} |S_\alpha(t)|$; then we have

$$\mu(t) \|\Lambda v_1(t) - \Lambda v_2(t)\| = \mu(t) \left\| \int_{-\infty}^t S_\alpha(t - \sigma) \left[ F_1 (\sigma, v_1 (\sigma), B v_1 (\sigma)) - F_1 (\sigma, v_2 (\sigma), B v_2 (\sigma)) \right] d\sigma \right\|$$

$$\leq C_\alpha \int_{-\infty}^t \mu(t) L(\sigma) \|v_1 (\sigma) - v_2 (\sigma)\| + \|B v_1 (\sigma) - B v_2 (\sigma)\| d\sigma = C_\alpha \int_{-\infty}^t \mu(t) \mu(\sigma) L(\sigma)$$

$$\frac{\mu(\sigma)^{-1} (1 + \|B\|_{L(\mathbb{X})}) \|v_1 (\sigma) - v_2 (\sigma)\| d\sigma}{\mu(\sigma)^{-1} (1 + \|B\|_{L(\mathbb{X})}) \|v_1 (\sigma) - v_2 (\sigma)\| d\sigma} \leq C_\alpha \left(1 + \frac{\|B\|_{L(\mathbb{X})}}{k}\right) \|v_1 - v_2\|$$

$$\leq C_\alpha \left(1 + \frac{\|B\|_{L(\mathbb{X})}}{k}\right) \|v_1 - v_2\|$$

which implies that

$$\|A x(t) - A y(t)\| \leq C_\alpha \left(1 + \frac{\|B\|_{L(\mathbb{X})}}{k}\right) \|x - y\|.$$ \hspace{1cm} (117)

Hence $\Lambda$ has a unique fixed point $x \in AA(\mathbb{R}, X)$ when $k$ is greater than $C_\alpha (1 + \|B\|_{L(\mathbb{X})})$.

**Step 2.** Set $\Theta := \{\omega(t) \in C_0(\mathbb{R}, X) : \|\omega(t)\| \leq r\}$. For the above $v(t)$, define $\Gamma := I^{1+} + \Gamma_0$ on $C_0(\mathbb{R}, X)$ as (69) and prove that $\Gamma$ maps $\Theta_0$ into itself, where $\Theta_0$ is a given constant.

Firstly, from (97) it follows that, for all $s \in \mathbb{R}, \omega(s) \in X$,

$$F_1 (s, v(s) + \omega(s), B (v(s) + \omega(s))) - F_1 (s, v(s), B v(s)) \leq L(s) \|\omega(s)\| + \|B \omega(s)\|,$$ \hspace{1cm} (118)

$$\leq L(s) (1 + \|B\|_{L(\mathbb{X})}) \|\omega(s)\| + \|B \omega(s)\|,$$

which together with $L(s) \in BC(\mathbb{R}, \mathbb{R}^r)$ implies that

$$F_1 (\cdot, v(\cdot) + \omega(\cdot), B (v(\cdot) + \omega(\cdot))) - F_1 (\cdot, v(\cdot), B v(\cdot))$$

$$\in C_0(\mathbb{R}, X)$$

for every $\omega(\cdot) \in C_0(\mathbb{R}, X)$. According to (114), one has

$$\|F_2 (s, v(s) + \omega(s), B (v(s) + \omega(s)))\| \leq \beta(s)$$

$$\cdot \left(\|\omega(s)\| + \|B \omega(s)\| + \|v(s)\| + \|B v(s)\|\right) \leq \beta(s)$$

$$\cdot \left(1 + \|B\|_{L(\mathbb{X})}\right) \|\omega(s)\| + \|v(s)\|$$

$$\leq \beta(s) \left(1 + \|B\|_{L(\mathbb{X})}\right) \|\omega(s)\| + \|v(s)\|.$$ \hspace{1cm} (120)
for all $t \in \mathbb{R}$ and $\omega(s) \in X$ with $\|\omega(s)\| \leq r$; then

$$F_2 (\cdot, v(\cdot) + \omega(\cdot), B(v(\cdot) + \omega(\cdot))) \in C_0 (\mathbb{R}, X)$$

as $\beta(\cdot) \in C_0 (\mathbb{R}, \mathbb{R}^+)$. \hfill (121)

Those, together with Lemma 21, yield that $\Gamma$ is well defined and maps $C_0 (\mathbb{R}, X)$ into itself.

On the other hand, it is not difficult to see that there exists a constant $k_0 > 0$ such that

$$\frac{2C_\alpha (1 + \|B\|_{L(X)})}{k}k_0 + \frac{C_\alpha (1 + \|B\|_{L(X)})}{k} \|v(s)\| \leq k_0$$

(122)

when $k$ is large enough. This enables us to conclude that, for any $t \in \mathbb{R}$ and $\omega_1(t), \omega_2(t) \in \Theta_{k_0}$,

$$\mu(t) \left( (1^{\alpha_1} \omega_1)(t) + (1^{\alpha_2} \omega_2)(t) \right) \leq \mu(t) \left( \int_{-\infty}^{t} S_\alpha(t-s) \right)$$

$$\cdot [F_1 (s, v(s) + \omega_1(s), B(v(s) + \omega_1(s)))$$

$$- F_1 (s, v(s), Bv(s))] \mathrm{ds} + \mu(t) \left( \int_{-\infty}^{t} S_\alpha(t-s) \right)$$

$$\leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\| \mathrm{ds}$$

$$+ C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_2(s)\| + \|B\omega_2(s)\| + \|Bv(s)\| \mathrm{ds}$$

$$= C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\| \mathrm{ds}$$

$$= C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\| \mathrm{ds}$$

$$= C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\| \mathrm{ds}$$

$$\leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$+ \left( 1 + \|B\|_{L(X)} \right) \|\omega_2(s)\| + \|B\omega_2(s)\| + \|Bv(s)\|$$

$$\leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$+ \left( 1 + \|B\|_{L(X)} \right) \|\omega_2(s)\| + \|B\omega_2(s)\| + \|Bv(s)\|$$

$$\mu(t) \|\omega_1(t) - \omega_2(t)\| \leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$+ \|B\omega_2(s)\| + \|Bv(s)\|$$

$$\leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

which implies that $(1^{\alpha_1} \omega_1)(t) + (1^{\alpha_2} \omega_2)(t) \in \Theta_{k_0}$. Thus $\Gamma$ maps $\Theta_{k_0}$ into itself.

**Step 3.** Show that $1^{\alpha_1}$ is a contraction on $\Theta_{k_0}$.

In fact, for any $\omega_1(t), \omega_2(t) \in \Theta_{k_0}$ and $t \in \mathbb{R}$, from (97) it follows that

$$\left[ [F_1 (s, v(s) + \omega_1(s), B(v(s) + \omega_1(s)))$$

$$- F_1 (s, v(s), Bv(s))] \right] \mathrm{ds} \leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$\leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$\mu(t) \left( 1^{\alpha_1} \omega_1(t) - 1^{\alpha_2} \omega_2(t) \right) = \mu(t) \left( \int_{-\infty}^{t} S_\alpha(t-s) \left( [F_1 (s, v(s) + \omega_1(s), B(v(s) + \omega_1(s))) - F_1 (s, v(s), Bv(s))] \right) \mathrm{ds} \right)$$

$$+ \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$\leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$\leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$\mu(t) \left( 1^{\alpha_1} \omega_1(t) - 1^{\alpha_2} \omega_2(t) \right) \leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

Thus

$$\|\omega_1(t) - \omega_2(t)\| \leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

$$\mu(t) \|\omega_1(t) - \omega_2(t)\| \leq C_\alpha \left( 1 + \|B\|_{L(X)} \right) \|\omega_1(s)\| + \|B\omega_1(s)\| + \|Bv(s)\|$$

which implies that $(1^{\alpha_1} \omega_1)(t) + (1^{\alpha_2} \omega_2)(t) \in \Theta_{k_0}$. Thus $\Gamma$ maps $\Theta_{k_0}$ into itself.
\[ -\omega_2(\sigma) \| d\sigma = C_\alpha \int_{-\infty}^{t} (1 + \|B\|_{L(X)}) \| \omega_1(\sigma) - \omega_2(\sigma) \| d\sigma \leq C_\alpha (1 + \|B\|_{L(X)}) \| \omega_1 \]
\[ -\omega_2 \| \int_{-\infty}^{t} \mu(t) \mu(\sigma) L(\sigma) \mu(\sigma)^{-1} (1 + \|B\|_{L(X)}) \| \omega_1(\sigma) - \omega_2(\sigma) \| d\sigma \leq C_\alpha (1 + \|B\|_{L(X)}) \| \omega_1 \]
\[ \leq C_\alpha \int_{-\infty}^{t} ke^{-k} \int_{-\infty}^{t} \mu(t) L(\sigma) \| \omega_1(\sigma) - \omega_2(\sigma) \| d\sigma \leq C_\alpha \int_{-\infty}^{t} \frac{1}{k} \| \omega_1(\sigma) - \omega_2(\sigma) \| d\sigma \]
\[ = \frac{C_\alpha (1 + \|B\|_{L(X)})}{k} \| \omega_1 - \omega_2 \| , \]
\[ \text{which implies} \]
\[ \| \left( \Gamma^1 \omega_1 \right)(t) - \left( \Gamma^1 \omega_2 \right)(t) \| \leq \frac{\varepsilon}{3CM} \]
\[ \text{Thus, when } k \text{ is greater than } C_\alpha (1 + \|B\|_{L(X)}), \text{ one obtains the conclusion.} \]

**Step 4.** Show that \( \Gamma^2 \) is completely continuous on \( \Theta_\kappa \).

Given \( \varepsilon > 0 \). Let \( \omega_n \stackrel{\varepsilon \rightarrow 0}{\longrightarrow} \omega_0 \in \Theta_\kappa \) as \( n \rightarrow +\infty \). Since \( \sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+) \), one may choose a \( t_1 > 0 \) big enough such that, for all \( t \geq t_1 \),
\[ \left( 1 + \|B\|_{L(X)} \right) (k_0 + \| v \|) \sigma(t) < \frac{\varepsilon}{3CM} . \]

Also, in view of \( (H'_1) \), we have
\[ F_2 \left( s, v(s) + \omega_k(s), B \left( v(s) + \omega_k(s) \right) \right) \]
\[ \rightarrow F_2 \left( s, v(s) + \omega_0(s), B \left( v(s) + \omega_0(s) \right) \right) \]
for all \( s \in (-\infty, t_1] \) as \( k \rightarrow +\infty \) and
\[ \mu(\cdot) F_2 \left( s, v(\cdot) + \omega_n(\cdot), B \left( v(\cdot) + \omega_n(\cdot) \right) \right) \]
\[ - F_2 \left( s, v(\cdot) + \omega_0(\cdot), B \left( v(\cdot) + \omega_0(\cdot) \right) \right) \| \leq \mu(\cdot) \]
\[ \cdot \beta(\cdot) \left( \| \omega_n(\cdot) + v(\cdot) + B \omega_n(\cdot) + B v(\cdot) \| \right) \leq \beta(\cdot) \]
\[ \| \omega_0(\cdot) \| + \| v(\cdot) \| + \| \omega_0(\cdot) \| + \| B \omega_0(\cdot) + B v(\cdot) \| \leq \beta(\cdot) \]
\[ \cdot \left( \| \omega_0(\cdot) + v(\cdot) + B \omega_0(\cdot) + B v(\cdot) \| + \| \omega_0(\cdot) \| + \| \omega_0(\cdot) \| + \| B \omega_0(\cdot) + B v(\cdot) \| \right) \leq \beta(\cdot) \]
\[ \cdot \left( 2 \left( 1 + \|B\|_{L(X)} \right) (k_0 + \| v \|) \right) \in L^1 \left( -\infty, t_1 \right] . \]

Hence, by the Lebesgue dominated convergence theorem we deduce that there exists an \( N > 0 \) such that
\[ CM \int_{-\infty}^{t_1} \frac{1}{1 + |\sigma(t) - s|^\alpha} \mu(t) \]
\[ \cdot \| F_2 \left( s, v(s) + \omega_k(s), B \left( v(s) + \omega_k(s) \right) \right) \]
\[ \text{whenever } k \geq N. \]

Accordingly, \( \Gamma^2 \) is continuous on \( \Theta_\kappa \).

In the sequel, we consider the compactness of \( \Gamma^2 \).
Set \( B_r(X) \) for the closed ball with center at 0 and radius \( r \) in \( X, V = \Gamma^2(\Theta_k) \), and \( z(t) = \Gamma^2(u(t)) \) for \( u(t) \in \Theta_k \). First, for all \( \omega(t) \in \Theta_k \) and \( t \in \mathbb{R} \), we have

\[
\mu(t) \left\| \left( t^2 \omega \right)(t) \right\| = \mu(t) \left\| \int_{-\infty}^{t} S_\alpha(t-s) \right\| \cdot \| F_2(s, v(s) + \omega(s), B(v(s) + \omega(s))) \| ds \leq CM \int_{-\infty}^{t} \frac{1}{1 + |\omega| (t-s)^2} \mu(t) \left\| \omega \right\| + \left\| B_\omega(s) \right\| \left\| \omega \right\| ds \leq CM \int_{-\infty}^{t} \frac{1}{1 + |\omega| (t-s)^2} \mu(t) \left( 1 + \left\| B_\omega(s) \right\| \right) \leq CM \int_{-\infty}^{t} \frac{\beta(s)}{1 + |\omega| (t-s)^2} \mu(t) \left( 1 + \left\| B_\omega(s) \right\| \right)
\]

in view of \( \sigma(t) \in C_0(\mathbb{R}, \mathbb{R}^+) \) which follows from Lemma 23; one concludes that

\[
\lim_{|t| \to +\infty} \left( t^2 \omega \right)(t) = 0 \quad \text{uniformly for } \omega(t) \in \Theta_k.
\]

Hence, for given \( \varepsilon_0 > 0 \), one can choose a \( \xi > 0 \) such that

\[
\left\| \int_{\xi}^{+\infty} S_\alpha(\tau) F_2(t-\tau, v(t-\tau) + \omega(t-\tau), B(v(t-\tau) + \omega(t-\tau))) d\tau \right\| < \varepsilon_0.
\]

Thus we get

\[
z(t) \in \xi^{c_c} \left( \left\{ S_\alpha(\tau) F_2(\lambda, v(\lambda) + \omega(\lambda), B(v(\lambda) + \omega(\lambda)) : 0 \leq \tau \leq \xi, t - \xi \leq \lambda \leq \xi, \left\| \omega \right\| \leq k_0 \right\} \right) + B_{\xi} \left( \Theta_k \right),
\]

where \( c(K) \) denotes the convex hull of \( K \). Using the fact that \( S_\alpha(\cdot) \) is strongly continuous, we infer that

\[
K = \left\{ S_\alpha(\tau) F_2(\lambda, v(\lambda) + \omega(\lambda), B(v(\lambda) + \omega(\lambda)) : 0 \leq \tau \leq \xi, t - \xi \leq \lambda \leq \xi, \left\| \omega \right\| \leq k_0 \right\}
\]

is a relatively compact set and \( V \subset \xi^{c_c}(K) + B_{\xi} \left( \Theta_k \right) \), which implies that \( V \) is a relatively compact subset of \( \Theta_k \).

Next, we verify the equicontinuity of the set \( \{ (t^2 \omega)(t) : \omega(t) \in \Theta_k \} \), given \( \varepsilon_1 > 0 \). In view of (114), together with the continuity of \( S_\alpha(\cdot) \), there exists an \( \eta > 0 \) such that, for all \( \omega(t) \in \Omega_k \) and \( t_2 \geq t_1 \) with \( t_2 - t_1 < \eta \),

\[
\int_{t_1}^{t_2} \left\| S_\alpha(t_2-s) F_2(s, v(s) + \omega(s), B(v(s) + \omega(s))) \right\| ds < \frac{\varepsilon_1}{4},
\]

Also, one can choose a \( k > 0 \) such that

\[
\int_{t_1-k}^{t_1-\eta} \left\| S_\alpha(t_2-s) - S_\alpha(t_1-s) \right\| F_2(s, v(s) + \omega(s), B(v(s) + \omega(s))) ds < \frac{\varepsilon_1}{4},
\]

\[
(1 + \left\| B_\omega(s) \right\|) \sup_{s \in [-\infty, t_1-k]} \left\| S_\alpha(t_2-s) - S_\alpha(t_1-s) \right\| \int_{-\infty}^{t_1-k} \beta(s) ds < \frac{\varepsilon_1}{4},
\]

\[
(1 + \left\| B_\omega(s) \right\|) \sup_{s \in [-\infty, t_1-k]} \left\| S_\alpha(t_2-s) - S_\alpha(t_1-s) \right\| \int_{-\infty}^{t_1-k} \beta(s) ds < \frac{\varepsilon_1}{4}.
\]
which implies that, for all $\omega(t) \in \Omega_{k}$ and $t_{2} \geq t_{1}$,

$$
\int_{t_{2} - t_{1}}^{t_{2} - t_{k}} \left\| \left[ S_{\alpha}(t_{2} - s) - S_{\alpha}(t_{1} - s) \right] F_{2}(s, v(s) + \omega(s), B(v(s) + \omega(s))) \right\| \, ds \\
\leq (1 + \| B \|_{L(X)}) \left( k_{0} + \| v \| \right) \sup_{s \in [-\infty, t_{2} - t_{k}]} \left\| S_{\alpha}(t_{2} - s) - S_{\alpha}(t_{1} - s) \right\| \int_{-\infty}^{t_{1} - k} \beta(s) \, ds < \frac{\epsilon_{1}}{4}.
$$

(140)

Then one has

$$
\left\| (\Gamma^{2} \omega)(t_{2}) - (\Gamma^{2} \omega)(t_{1}) \right\| \\
\leq \int_{t_{2} - t_{1}}^{t_{1} - k} \left\| S_{\alpha}(t_{2} - s) F_{2}(s, v(s) + \omega(s), B(v(s) + \omega(s))) \right\| \, ds \\
+ \int_{t_{1} - k}^{t_{1}} \left\| S_{\alpha}(t_{1} - s) F_{2}(s, v(s) + \omega(s), B(v(s) + \omega(s))) \right\| \, ds \\
+ \int_{t_{2} - s}^{t_{1} - k} \left\| S_{\alpha}(t_{2} - s) - S_{\alpha}(t_{1} - s) \right\| F_{2}(s, v(s) + \omega(s), B(v(s) + \omega(s))) \, ds \\
+ \int_{t_{1} - k}^{t_{1}} \left\| S_{\alpha}(t_{1} - s) - S_{\alpha}(t_{1} - s) \right\| F_{2}(s, v(s) + \omega(s), B(v(s) + \omega(s))) \, ds < \epsilon_{1},
$$

(141)

which implies the equicontinuity of the set $\{ (\Gamma^{2} \omega)(t) : \omega(t) \in \Theta_{k} \}$.

Now an application of Lemma 18 justifies the compactness of $\Gamma^{2}$.

**Step 5.** Show that (39) has at least one asymptotically almost automorphic mild solution.

The proof is similar to the proof in Step 5 of Theorem 24.

Taking $A = -\rho^{2}I$ with $\rho > 0$ in (39), Theorem 29 gives the following corollary.

**Corollary 30.** Let $F : \mathbb{R} \times X \times X \rightarrow X$ satisfy $(H'_{1})$ and $(H'_{2})$ with $L(t) \in BC(\mathbb{R}, \mathbb{R}^{+})$. Moreover the integral

$$
\int_{-\infty}^{t} \max \{ L(s), \beta(s) \} \, ds \text{ exists for all } t \in \mathbb{R}. \text{ Then (39) has at least one asymptotically almost automorphic mild solution.}
$$

**4. Applications**

In this section we give an example to illustrate the above results.

Consider the following fractional relaxation-oscillation equation:

$$
\frac{\partial^{\alpha} u(t, x)}{\partial t^{\alpha}} = \frac{\partial^{2} u(t, x)}{\partial t^{2}} - pu(t, x) \\
+ \alpha^{n-1} \left( \mu a(t) \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2}t} \right) \right).
$$

(142)

where $a(t) \in BC(\mathbb{R}, \mathbb{R}^{+})$ is a function and $p, \mu, \nu$ are positive constants.

Take $X = L^{2}([0, \pi])$ and define the operator $A$ by

$$
A\phi = \phi'' - p\phi, \quad \phi \in D(A),
$$

where

$$
D(A) = \{ \phi \in X : \phi'' \in X, \phi(0) = \phi(\pi) \} \subset X.
$$

(143)

It is well known that $Bu = u''$ is self-adjoint, with compact resolvent, and is the infinitesimal generator of an analytic semigroup on $X$. Hence, $pI - B$ is sectorial of type $\omega = -p < 0$. Let

$$
F_{1}(t, x(\xi), y(\xi)) = \mu a(t) \cdot \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2}t} \right) [\sin x(\xi) + y(\xi)],
$$

(145)

$$
F_{2}(t, x(\xi), y(\xi)) = \nu e^{-|t|} [x(\xi) + \sin y(\xi)].
$$
Then it is easy to verify that $F_1, F_2 : \mathbb{R} \times X \times X \rightarrow X$ are continuous and $F_1(t, x, y) \in AA(\mathbb{R} \times X \times X, X)$ satisfying

$$
\| F_1(t, x_1, y_1) - F_1(t, x_2, y_2) \|_2 \\
\leq \int_0^\tau \mu^2 \left| a(t) \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2}t} \right) \right|^2 \\
\cdot \left| \left[ \sin x_1(s) + y_1(s) \right] - \left[ \sin x_2(s) + y_2(s) \right] \right| ds \\
\leq \mu^2 a^2(t) \left| \sin \left( \frac{1}{2 + \cos t + \cos \sqrt{2}t} \right) \right|^2 \\
\cdot \left( \| x_1 - x_2 \|_2^2 + \| y_1 - y_2 \|_2^2 \right),
$$

that is,

$$
\| F_1(t, x_1, y_1) - F_1(t, x_2, y_2) \|_2 \\
\leq \mu a(t) \left( \| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 \right) \\
\forall t \in \mathbb{R}, x_1, x_2, y_1, y_2 \in X; \tag{147}
$$

furthermore

$$
\| F_1(t, x_1, y_1) - F_1(t, x_2, y_2) \|_2 \\
\leq \mu \| a \|_{\infty} \left( \| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 \right) \\
\forall t \in \mathbb{R}, x_1, x_2, y_1, y_2 \in X. \tag{148}
$$

And

$$
\| F_2(t, x, y) \|_2^2 \leq \int_0^\tau v^2 e^{-2|t|} |x(s) + \sin y(s)| \ ds \\
\leq v e^{-2|t|} \left( \| x \|_2^2 + \| y \|_2^2 \right), \tag{149}
$$

that is,

$$
\| F_2(t, x, y) \|_2 \leq v e^{-|t|} \left( \| x \|_2 + \| y \|_2 \right) \\
\forall t \in \mathbb{R}, x, y \in X, \tag{150}
$$

which implies $F_2(t, x, y) \in \mathcal{C}_0(\mathbb{R} \times X \times X, X)$. Furthermore

$$
F(t, x, y) = F_1(t, x, y) + F_2(t, x, y) \\
\in AAA(\mathbb{R} \times X \times X, X). \tag{151}
$$

Thus, (142) can be reformulated as the abstract problem (39) and the assumptions $(H_1)$ and $(H_2)$ hold with

$$
L = \mu \| a \|_{\infty}, \\
\Phi(t) = r, \\
\beta(t) = v e^{-|t|}, \\
\rho_1 = 1, \\
\rho_2 \leq \nu,
$$

the assumption $(H'_1)$ holds with $L(t) = \mu a(t)$, and the assumption $(H'_2)$ holds.

In consequence, the fractional relaxation-oscillation equation (142) has at least one asymptotically almost automorphic mild solution if either

$$
\frac{\mu \| a \|_{\infty} \pi \left| p \right|^{-1/\alpha}}{\alpha \sin (\pi/\alpha)} + CM_\nu < \frac{1}{2} 
$$

(Theorem 24) or

$$
\frac{\mu \| a \|_{\infty} \pi \left| p \right|^{-1/\alpha}}{\alpha \sin (\pi/\alpha)} + CM_\nu < \frac{1}{2} \tag{154}
$$

(Theorem 27), where $\| a \| = \sup_{t \in \mathbb{R}} \int_0^t a(s) \ ds$ or the integral

$$
\int_{-\infty}^t \max \{ \mu a(s), v e^{-|s|} \} \ ds \tag{155}
$$

exists for all $t \in \mathbb{R}$ (Theorem 29).

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This research was supported by the NNSF of China (no. 11561009) and (no. 41665006), the Guangdong Province Natural Science Foundation (no. 2015A030313896), the Characteristic Innovation Project (Natural Science) of Guangdong Province (no. 2016KTSCX094), the Science and Technology Program Project of Guangzhou (no. 201707010230), and the Guangxi Province Natural Science Foundation (no. 2016GXNSFAA380240).

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