We consider the following nonlinear parabolic equation:
\[ u_t - \text{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, t), \] where \( f: \Omega \times (0, T) \rightarrow \mathbb{R} \) and the exponent of nonlinearity \( p(\cdot) \) are given functions. By using a nonlinear operator theory, we prove the existence and uniqueness of weak solutions under suitable assumptions. We also give a two-dimensional numerical example to illustrate the decay of solutions.

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \). We consider the following initial and boundary value problem:
\[
\begin{align*}
  & u_t - \text{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, t), & u(x, t) = 0, & \text{on } \partial \Omega \times (0, T) \quad (P) \\
  & u(x, 0) = u_0(x), & u(x, t) = 0, & \text{on } \partial \Omega \times (0, T) \\
  & u(x, 0) = u_0(x), & \text{in } \Omega,
\end{align*}
\]
where \( f: \Omega \times (0, T) \rightarrow \mathbb{R} \) and \( u_0: \Omega \rightarrow \mathbb{R} \) are given functions. The exponent \( p(\cdot) \) is a given measurable function on \( \Omega \) such that
\[ 2 < p_1 \leq p(x) \leq p_2 < +\infty, \]
with\[ p_1 = \text{essinf}_{x \in \Omega} p(x), \quad p_2 = \text{esssup}_{x \in \Omega} p(x). \]
We also assume that \( p(\cdot) \) satisfies the log-Hölder continuity condition:
\[ |p(x) - p(y)| \leq -\frac{A}{\log |x - y|}, \]
\[ \forall x, y \in \Omega, \text{ with } |x - y| < \delta, \]
where \( A > 0 \) and \( 0 < \delta < 1 \) are constants. The term \( \text{div}(|\nabla u|^{p(x)-2}\nabla u) \) is called the \( p(\cdot) \)-Laplacian and denoted by \( \Delta_{p(\cdot)}u \).

The study of partial differential equations involving variable-exponent nonlinearities has attracted the attention of researchers in recent years. The interest in studying such problems is stimulated and motivated by their applications in elastic mechanics, fluid dynamics, nonlinear elasticity, electrorheological fluids, and so forth. In particular, parabolic equations involving the \( p(\cdot) \)-Laplacian are related to the field of image restoration and electrorheological fluids which are characterized by their ability to change the mechanical properties under the influence of the exterior electromagnetic field. The rigorous study of these physical problems has been facilitated by the development of the Lebesgue and Sobolev spaces with variable exponents.

Regarding parabolic problems with nonlinearities of variable-exponent type, many works have appeared. We note
here that most of the results deal with blow-up and global nonexistence. Let us mention some of these works. For instance, Alouini et al. [1] considered the following nonlinear heat equation:

\[ u_t (x, t) - \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) = u |u|^{p(x)-2} + f, \]

in a bounded domain in \( \Omega \subset \mathbb{R}^n \) (\( n \geq 1 \)) with a smooth boundary \( \partial \Omega \). Under appropriate conditions on the exponent functions \( m, p \) and for \( f = 0 \), they showed that any solution with nontrivial initial datum blows up in finite time. They also gave a two-dimensional numerical example to illustrate their result. Pinasco [2] studied the following problem:

\[
\begin{align*}
    u_t - \Delta u &= f(u), \quad \text{in } \Omega \times [0, T) \\
    u(x, t) &= 0, \quad \text{on } \partial \Omega \times [0, T) \\
    u(x, 0) &= u_0(x), \quad \text{in } \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a smooth boundary \( \partial \Omega \), and the source term is of the following form:

\[
f(u) = a(x) u^{p(x)}
\]

or

\[
f(u) = a(x) \int_{\Omega} g(y, t) dy,
\]

with \( p, q : \Omega \to (1, \infty) \) and the continuous function \( a : \Omega \to \mathbb{R} \) being given functions satisfying specific conditions. They established the local existence of positive solutions and proved that solutions with initial data sufficiently large blow up in finite time. Parabolic problems with sources like the ones in (5) appear in several branches of applied mathematics and have been used to model chemical reactions, heat transfer, or population dynamics.

Recently, Shangerganesh et al. [3] studied the following fourth-order degenerate parabolic equation:

\[ u_t + \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) = f - g,
\]

in a bounded domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 1 \)) with a smooth boundary \( \partial \Omega \), and proved the existence and uniqueness of weak solutions of (7) by using the difference and variation methods under suitable assumptions on \( f, g \) and the exponents \( p \).

Equation (P) is a nonlinear diffusion equation which has been used to study image restoration and electrorheological fluids (see [4–11]). In particular, Bendahmane et al. [12] proved the well-posedness of a solution, for \( L^1 \)-data. Akagi and Matsuura [13] gave the well-posedness for \( L^2 \) initial datum and discussed the long-time behaviour of the solution using the subdifferential calculus approach. In our paper, we give an alternative proof of the well-posedness of (P) which is simpler than that in [13] using a theory of nonlinear evolution equations. In addition, we give a numerical example in 2D to illustrate the decay result obtained in [13].

This paper consists of three sections in addition to the introduction. In Section 2, we recall the definitions of the variable-exponent Lebesgue spaces, \( L^{p(\cdot)}(\Omega) \), the Sobolev spaces, \( W^{1,p(\cdot)}(\Omega) \), as well as some of their properties. We also state, without proof, a proposition to be used in the proof of our main result. In Section 3, we state and prove the well-posedness of solution to our problem. In Section 4, we give a numerical verification of the decay result.

2. Preliminaries

We present some preliminary facts about the Lebesgue and Sobolev spaces with variable exponents (see [1, 14–16]). Let \( p : \Omega \to [1, \infty] \) be a measurable function, where \( \Omega \) is a domain of \( \mathbb{R}^n \). We define the Lebesgue space with a variable-exponent \( p(\cdot) \) by

\[ L^{p(\cdot)}(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable in } \Omega : \varrho_{L^{p(\cdot)}(\Omega)}(\lambda u) < \infty, \text{ for some } \lambda > 0 \}, \]

where

\[ \varrho_{L^{p(\cdot)}(\Omega)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx \]

is called a modular. Equipped with the Luxembourg-type norm,

\[ \|u\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(\lambda u) \leq 1 \}, \]

\( L^{p(\cdot)}(\Omega) \) is a Banach space (see [10]).

**Lemma 1** (Hölder’s inequality [10]). Let \( p, q, s \geq 1 \) be measurable functions defined on \( \Omega \) such that

\[ \frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \]

for a.e. \( y \in \Omega \). If \( f \in L^{p(\cdot)}(\Omega) \) and \( g \in L^{q(\cdot)}(\Omega) \), then \( fg \in L^{s(\cdot)}(\Omega) \) and

\[ \|fg\|_{L^{s(\cdot)}(\Omega)} \leq 2 \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{q(\cdot)}(\Omega)}. \]

**Lemma 2** (see [10]). Let \( p \) be a measurable function on \( \Omega \). Then,

(a) \( \|f\|_{L^{p(\cdot)}(\Omega)} \leq 1 \) if and only if \( \varrho_{L^{p(\cdot)}(\Omega)}(f) \leq 1; \)

(b) for \( f \in L^{p(\cdot)}(\Omega) \), if \( \|f\|_{L^{p(\cdot)}(\Omega)} \leq 1 \), then \( \varrho_{L^{p(\cdot)}(\Omega)}(f) \leq \|f\|_{L^{p(\cdot)}(\Omega)}; \)

and if \( \|f\|_{L^{p(\cdot)}(\Omega)} \geq 1 \), then \( \varrho_{L^{p(\cdot)}(\Omega)}(f) \leq \|f\|_{L^{p(\cdot)}(\Omega)}; \)

(c) \( \|f\|_{L^{p(\cdot)}(\Omega)} \leq 1 + \varrho_{L^{p(\cdot)}(\Omega)}(f). \)

**Lemma 3** (see [10]). If (1) holds, then

\[ \min \left\{ \|u\|_{L^{p_1(\cdot)}(\Omega)}, \|u\|_{L^{p_2(\cdot)}(\Omega)} \right\} \leq \varrho_{L^{p(\cdot)}(\Omega)}(u) \]

\[ \leq \max \left\{ \|u\|_{L^{p_1(\cdot)}(\Omega)}, \|u\|_{L^{p_2(\cdot)}(\Omega)} \right\}, \]

for any \( u \in L^{p(\cdot)}(\Omega) \).
We next define the variable-exponent Sobolev space \( W^{1, p(\cdot)}(\Omega) \) as follows:

\[
W^{1, p(\cdot)}(\Omega) = \left\{ u \in L^{1, p(\cdot)}(\Omega) \mid \text{such that } \nabla u \text{ exists, } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.
\] (14)

This space is a Banach space with respect to the norm \( \|u\|_{p(\cdot)} = \|u\|_{p(\cdot), \Omega} + \|\nabla u\|_{p(\cdot), \Omega} \). Furthermore, \( W^{1, p(\cdot)}_0(\Omega) \) is the closure of \( C_0^{\infty}(\Omega) \) in \( W^{1, p(\cdot)}(\Omega) \). The dual of \( W^{1, p(\cdot)}_0(\Omega) \) is defined as \( W^{-1, p'(\cdot)}(\Omega) \), by the same way as the usual Sobolev spaces where \( 1/p(x) + 1/p'(x) = 1 \).

**Lemma 4** (see [10]). Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) and \( p(\cdot) \) satisfies (1) and (3), and then

\[
\|u\|_{p(\cdot)} \leq C \|u\|_{p(\cdot)}^{\infty}, \quad \forall u \in W^{1, p(\cdot)}_0(\Omega),
\] (15)

where the positive constant \( C \) depends on \( p(\cdot) \) and \( \Omega \). In particular, the space \( W^{1, p(\cdot)}_0(\Omega) \) has an equivalent norm given by \( \|u\|_{W^{1, p(\cdot)}_0(\Omega)} = \|u\|_{p(\cdot), \Omega} \).

**Lemma 5** (see [10]). If \( p(\cdot) \in C(\Omega), q : \Omega \to [1, \infty) \) is a continuous function and

\[
\text{essinf}_{x \in \Omega} (p^*(x) - q(x)) > 0
\]

with \( p^*(x) = \begin{cases} np(x) & \text{if } p_2 < n \\ \sup_{x \in \Omega} (n - p(x)) & \text{if } p_2 \geq n. \end{cases} \) (16)

Then the embedding \( W^{1, p(\cdot)}_0(\Omega) \hookrightarrow L^q(\Omega) \) is continuous and compact.

**Definition 6** (see [17]). Let \( V \) be a separable Banach space and \( H \) be a Hilbert space such that \( V \subset H \subset V' \) with continuous embedding and \( V \) is dense in \( H \). Let \( A : V \to V' \) be a nonlinear operator.

(1) \( A \) is said to be monotone if \( \langle A(u) - A(v), u - v \rangle_{V', V} \geq 0 \). If, in addition, we have

\[
\langle A(u) - A(v), u - v \rangle_{V', V} \neq 0, \quad \forall u \neq v,
\] (17)

then \( A \) is said to be strictly monotone.

(2) \( A \) is said to be bounded, if \( A(S) \) is bounded in \( V' \), whenever \( S \) is bounded in \( V \).

(3) \( A \) is said to be hemicontinuous, if the real function

\[
\lambda \mapsto \langle A(u + \lambda v), w \rangle
\]

is continuous from \( \mathbb{R} \) to \( \mathbb{R} \), for any fixed \( u, v, w \in V \).

We end this section with a proposition which is exactly like Theorem 7.1 [17].

**Proposition 7.** Let \( u_0 \in H \) and \( f \in L^p((0, T), V') \). Suppose that \( A : V \to V' \) is a bounded monotone and hemicontinuous (nonlinear) operator satisfying, for some \( \alpha, \beta > 0 \) and for some \( p > 1 \),

\[
\langle A(v), v \rangle \geq \alpha \|v\|^p_p - \beta, \quad \forall v \in V.
\] (19)

Then the following problem

\[
u_t + A(u) = f, \quad u(0, 0) = u_0,
\] (20)

has a unique weak solution:

\[
u \in L^p((0, T), V) \quad \text{with} \quad \nu_t \in L^{p'(0, T), V'}, \quad (21)
\]

where \( 1/p + 1/p' = 1 \).

### 3. Well-Posedness

In this section, we state and prove the well-posedness of our problem.

**Theorem 8.** Let \( u_0 \in L^2(\Omega), f \in L^{p(\cdot)}((0, T), W^{-1, p'(\cdot)}(\Omega)) \). Assume that (1) and (3) hold. Then (P) has a unique weak solution:

\[
u \in L^p((0, T), W^{1, p(\cdot)}(\Omega)) \cap L^\infty((0, T), L^2(\Omega)),
\]

\[
u_t \in L^{p'(0, T), (0, T), W^{-1, p'(\cdot)}(\Omega)}, \quad (22)
\]

where \( 1/p + 1/p' = 1 \).

**Proof.** We verify the conditions of Proposition 7. Let \( V = W^{1, p(\cdot)}_0(\Omega) \), and equip it with the norm,

\[
\|u\|_{V^{p(\cdot)}} = \|u\|_{p(\cdot), \Omega}.
\] (23)

So, \( V' = W^{-1, p'(\cdot)}(\Omega) \). Define \( A : V \to V' \) by

\[
A(u) = -\text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right).
\] (24)

**Boundedness of \( A \).** For all \( u, v \in V \),

\[
|\langle A(u), v \rangle| = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \leq \int_\Omega |\nabla u|^{p(x)-1} |\nabla v|.
\] (25)

Hölder inequality gives

\[
\int_\Omega |\nabla u|^{p(x)-1} |\nabla v| \leq 2 \|\nabla u|^{p(x)-1} \|_{p'(\cdot)} \|v\|_{V'}.
\] (26)

Combining (25) and (26), we obtain

\[
\|A(u)\|_{V'} \leq 2 \|\nabla u|^{p(x)-1} \|_{p'(\cdot)}.
\] (27)

Then, Lemma 2 implies

\[
\|\nabla u|^{p(x)-1} \|_{p'(\cdot)} \leq \left(1 + \theta_{p(\cdot)}(\nabla u) \right).
\] (28)
Combining (27) and (28), we arrive at
\[
\| A(u) \|_p \leq 2 \left( 1 + \theta_{p(\cdot)}(V u) \right). \tag{29}
\]
Let \( S \in V \) such that \( \| u \|_p \leq M \), for all \( u \in S \). That is, \( \| V u \|_{p(\cdot)} \leq M \).

If \( M \leq 1 \), then Lemma 2 implies \( \theta_{p(\cdot)}(V u) \leq 1 \) and (29) gives \( \| A(u) \|_p \leq 4 < +\infty \).

If \( M > 1 \), then \( \theta_{p(\cdot)}(V u) \leq M^p \). Thus, (29) implies \( \| A(u) \|_p \leq 2(1 + M^p) < +\infty \).

Hence \( A \) is bounded.

**Monotonicity of \( A \).** Let \( u, v \in V \).

\[
\langle A(u) - A(v), u - v \rangle_{V' \times V} = \int \text{div} \left( |\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v \right) (u - v) \, dx.
\]

By using the inequality,
\[
\left( |a|^{p(x)-2} a - |b|^{p(x)-2} b \right) \cdot (a - b) \geq 0,
\]
for all \( a, b \in \mathbb{R}^n \) and a.e. \( x \in \Omega \). Thus we obtain \( \langle A(u) - A(v), u - v \rangle \geq 0 \).

Hence, \( A \) is monotone.

To verify (19), we note that, for all \( u \in V \), we have
\[
\langle A(u), u \rangle = \int_\Omega |\nabla u|^{p(x)} \, dx = \theta_{p(\cdot)}(V u).
\]

If \( \| u \|_V = \| V u \|_{p(\cdot)} > 1 \), then by Lemma 3, we get
\[
\theta_{p(\cdot)}(V u) \geq \| V u \|_{p(\cdot)}^p.
\]

Combining (32) and (33), we easily see that
\[
\langle A(u), u \rangle \geq \| u \|_{V}^p - 1.
\]

If \( \| u \|_V = \| V u \|_{p(\cdot)} \leq 1 \), then by Lemma 2, we obtain
\[
\langle A(u), u \rangle = \theta_{p(\cdot)}(V u) \geq \| u \|_{V}^p - 1 \geq \| u \|_{V}^p - 1.
\]

Therefore, we have
\[
\langle A(u), u \rangle \geq \| u \|_{V}^p - 1, \quad \forall u \in V.
\]

**Hemicontinuity of \( A \).** Let \( u, v, w \in V \) be fixed. Let
\[
g(\lambda) = \langle A(u + \lambda v), w \rangle = \int_\Omega \left( |\nabla u + \lambda \nabla v|^{p(x)-2} (\nabla u + \lambda \nabla v) \cdot \nabla w \right) dx.
\]

Let \( \lambda_k \to \lambda \) (real) and consider
\[
g(\lambda_k) = \int_\Omega \left( |\nabla u + \lambda_k \nabla v|^{p(x)-2} (\nabla u + \lambda_k \nabla v) \cdot \nabla w \right) dx.
\]

Since
\[
|\nabla u + \lambda \nabla v|^{p(x)-2} (\nabla u + \lambda \nabla v) \cdot \nabla w,
\]
\[
\to |\nabla u + \lambda \nabla v|^{p(x)-2} (\nabla u + \lambda \nabla v) \cdot \nabla w,
\]
for a.e. \( x \in \Omega \) and
\[
|\nabla u + \lambda \nabla v|^{p(x)-1} |\nabla w| \leq C \left( |\nabla u|^{p(x)-1} |\nabla w| + |\nabla v|^{p(x)-1} |\nabla w| \right) \in L^1(\Omega),
\]
where \( C = \max\{|2^{p(x)-2}, 2^{p(x)-2}(1 + |M|^{p(x)-1})| > 0 \), then, by the classical dominated convergence theorem,
\[
g(\lambda_k) \to g(\lambda) \quad \text{as} \quad k \to \infty.
\]

Hence, \( A \) is hemicontinuous.

Therefore, conditions of Proposition 7 are satisfied and problem \((P_*)\) has a unique solution.

\[\square\]

### 4. Numerical Study

In this section, we present some numerical results and applications of the problem:

\[
u_t - \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) = 0, \quad \text{in} \ Q = \Omega \times (0, T)
\]

\[
u = 0, \quad \text{on} \ \partial Q = \partial \Omega \times (0, T),
\]

\[
u(x, y, 0) = u_0(x, y), \quad \text{in} \ \Omega,
\]

which is a well-posed problem due to Theorem 8. Our objective is to provide a numerical verification of the following decay result:

**Proposition 9** (see [13]). Assume that (1) and (3) hold. Then the solution of \((P_*)\) satisfies the following:

(i) If \( p_2 = 2 \), then there exists a constant \( c_2 > 0 \) such that
\[
\| u(t) \|_2 \leq \| u_0 \|_2 e^{-c_2 t}, \quad \forall t \geq 0.
\]

(ii) If \( p_2 > 2 \), there exists a constant \( c_1 > 0 \) and \( t_1 \geq 0 \) such that
\[
\| u(t) \|_2 \leq \| u_0 \|_2 (1 + c_1 t)^{-1/(p_2-2)}, \quad \forall t \geq t_1.
\]

We consider two applications to illustrate numerically an exponential decay for the case \( p(x, y) = 2 \) and a polynomial decay for an exponent function \( p(\cdot) \) satisfying conditions (1)–(3).

For this purpose, we introduce a numerical scheme for \((P_*)\), prove its convergence in Section 4.1, and show the decay results in Section 4.2.
4.1. Numerical Method. In this part, we present a linearized numerical scheme to obtain the numerical results of the system \( P_h \) and confirm the decay results. The system is fully discretized through a finite difference method for the time variable and a finite element Galerkin method for the space variable. Useful background about the numerical and error analysis of these methods is found in [18]. More interestingly in [19], Li and Wang introduced a numerical scheme to solve strongly nonlinear parabolic systems and proved unconditional error estimates of the scheme. Our problem \( P_h \) is highly nonlinear due to the presence of the gradient and nonlinear exponent in the diffusivity coefficient, which can be zero inside the spatial domain. Below, we introduce our numerical scheme for the purpose of confirming the decay results.

The parabolic equation
\[
  u_t - \text{div} \left( |\nabla u|^{p(x,y)-2} \nabla u \right) = 0, \quad \text{in } Q = \Omega \times (0,T)
\]
is discretized using finite differences for the time derivative and a finite element method for the \( p(\cdot)\)-Laplacian term. For this, we divide the time interval \([0,T]\) into \( N \) equal subintervals by
\[
t_n = n\tau, \quad \tau = \frac{T}{N} \quad (45)
\]
and denote by
\[
  u^n(x,y) = u(x,y,t_n), \quad n = 0, 1, \ldots, N. \quad (46)
\]
The term \( u_t \) is approximated using the first-order forward finite difference formula:
\[
  u_t^{n+1} = \frac{u^{n+1} - u^n}{\tau}. \quad (47)
\]

Semidiscrete Problem. A linear semidiscrete formulation of \( P_h \) takes the following form: given \( p \) and \( u_0 \), find \( \{u^n\}, n=1,\ldots,N \) such that
\[
  u_t^{n+1} - \text{div} \left( |\nabla u^n|^{p(x,y)-2} \nabla u^{n+1} \right) = 0, \quad \text{in } \Omega \quad (48)
\]
\[
  u^{n+1} = 0, \quad \text{on } \partial \Omega \quad (48)
\]
\[
  u^0 = u_0(x,y), \quad \text{in } \Omega. \quad (48)
\]

This problem is elliptic and admits a unique solution [20], for every \( n = 0, 1, \ldots, N \). Also, the Rothe approximation \( u^{[n]} \) to the exact solution \( u \) given by
\[
  u^{[n]}(x,y,t) = u^{n-1}(x,y) + (t - t_{n-1}) \frac{u^{n} - u^{n-1}}{\tau}, \quad (49)
\]
is well defined and \( u^{[n]} \to u \) in \( L^2(\Omega) \) as \( \tau \to 0 \), see [1].

Full-Discrete Problem. The variable \( u^{[n]} \) is discretized in space by a finite element method. For this, let \( \Omega_h \) be a triangulation of \( \Omega \) with a maximal element size \( h \). Let also \( v_h \) be a test function in the linear Lagrangian space \( P_1(\Omega_h) \) such that \( v_h = 0 \) on \( \partial \Omega_h \).

The semidiscrete problem is then written in a weak form to define the full-discrete problem: given \( p_h, u^n \in P_1(\Omega_h) \), find \( u_h^{n+1} \in P_1(\Omega_h) \) such that
\[
  \int_{\Omega_h} \left( \frac{u_h^{n+1} - u_h^n}{\tau} v_h + |\nabla u_h^n|^{p(x,y)-2} \nabla u_h^{n+1} \cdot \nabla v_h \right) d\Omega_h = 0, \quad \forall v_h \in P_1(\Omega_h). \quad (50)
\]

For \( p_h \geq 2 \), the above problem has a unique solution \( u_h^{n+1} \in H^2_0(\Omega_h) \) for every nontrivial \( u_h^n \in H^1(\Omega_h) \). This follows from the Lax-Milgram Lemma, and the Galerkin approximation \( u_h^{n+1} \) converges to \( u^{n+1} \) in \( H^1(\Omega_h) \) as \( h \to 0 \); see [18].

4.2. Numerical Results. In this subsection, we present the following numerical applications of \( P_h \):

(1) Exponential decay: for \( p_h(x,y) = 2 \), we show, for some \( c > 0 \), that
\[
  \|u^n_h\|_{L^2(\Omega_h)} \leq e^{-ct}, \quad \forall t \geq 0. \quad (51)
\]

(2) Polynomial decay: for \( p_h(x,y) = (1/5)|x|^2 + 2.5 \), we show, for some \( c > 0 \) and \( t_0 > 0 \), that
\[
  \|u^n_h\|_{L^2(\Omega_h)} \leq (1 + ct)^{-1/(p_2-2)}, \quad \forall t \geq t_0. \quad (52)
\]

Here, \([\cdot]\) denote the greatest integer function.

In both applications, we set the following parameters:
\[
  T = 100, \quad \tau = 0.1, \quad h = 0.1, \quad \Omega_h = [-5,5] \times [-5,5]. \quad (53)
\]

Figure 1 shows the mesh used for \( \Omega_h \), which involves 23702 triangles and 12052 vertices.
Figure 2: Initial condition: $u_0^h(x, y) = e^{-0.5(x^2+y^2)}$.

Figure 3: Numerical solutions for Application 1.
The initial condition is taken to be $u_0^h(x, y) = e^{-0.5(x^2+y^2)}$ and projected into $P_1(\Omega_h)$; see Figure 2.

The numerical results are obtained using the noncommercial software, FreeFem++ [21].

**Application 1.** $p_h(x, y) = 2$ satisfies the required conditions (1)–(3). Figure 3 shows the numerical solutions for $t = 5$, $t = 10$, $t = 20$, and $t = 50$.

With $c = 0.1$, Figure 4(a) shows that $u_n^h$ decays exponentially as

$$g_1(t) = \frac{\|u_n^h\|}{\|u_0^h\|} \leq e^{-0.1t}, \quad 0 \leq t \leq T.$$  \hspace{1cm} (54)

This is also confirmed by Figure 4(b) that shows the ratio $y = g_1(t)/e^{-0.1t}$ is less than one and remains decreasing for a large value of $T$.

**Application 2.** The exponent function $p_h(x, y) = (1/5)|x|^2 + 2.5$, in Figure 5, satisfies the required conditions (1)–(3) as

(i) $p_1 = 2.5$, $p_2 = 7.5$;

(ii) $|p(x, y) - p(x_0, y_0)| = (1/5)|x|^2 - |x_0|^2| \leq - (20\sqrt{2}\log(1/\delta))/\log|x - y|$ for $|x - y| < \delta$ with $0 < \delta < 1$.

Figure 6 shows the numerical solutions for $t = 5$, $t = 10$, $t = 20$, and $t = 50$.

In this case, the solution $u_n^h$ has a polynomial decay. With $c = 1$, Figure 7(a) shows that

$$g_2(t) = \frac{\|u_n^h\|}{\|u_0^h\|} \leq (1 + t)^{-2/11}, \quad 0 \leq t \leq T.$$  \hspace{1cm} (55)

This is also confirmed by Figure 7(b), which shows that the ratio $y = g_2(t)/(1 + t)^{-2/11}$ remains less than one and decreasing until $T$.

We conclude that the numerical results in above applications verify Proposition 9.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
Figure 6: Numerical solutions for Application 2.

Figure 7: Polynomial decay.
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References


