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Research Article

Multiple Solutions for Elliptic $(p(x), q(x))$-Kirchhoff-Type Potential Systems in Unbounded Domains

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In this paper, we establish the existence of at least three weak solutions for a parametric double eigenvalue quasi-linear elliptic $(p(x), q(x))$-Kirchhoff-type potential system. Our approach is based on a variational method, and a three critical point theorem is obtained by Bonano and Marano.

1. Introduction

The aim of this paper is to show the existence of at least three weak solutions for the following class of nonlocal quasi-linear elliptic systems in $\mathbb{R}^N$:

$$\begin{cases} -M_1\left(L_p(u)\right)\left(\Delta_{p(x)}u - a(x)|u|^{p(x)-2}u\right) = \lambda F_u(x,u,v), & \text{in } \mathbb{R}^N, \\ -M_2\left(L_q(v)\right)\left(\Delta_{q(x)}v - b(x)|v|^{q(x)-2}v\right) = \lambda F_v(x,u,v), & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 2$, p and $q \in C_*(\mathbb{R}^N) := \{r \in C(\mathbb{R}^N): 1 < r^- = \inf_{x\in\mathbb{R}^N} r(x) < r(x) < r^+ = \sup_{x\in\mathbb{R}^N} r(x) < N, \forall x \in \mathbb{R}^N\}$, λ is a positive real parameter, and a, $b \in L^\infty(\mathbb{R}^N)$ such that $a := \operatorname{ess\,inf}_{x\in\mathbb{R}^N} a(x) > 0$ and $b := \operatorname{ess\,inf}_{x\in\mathbb{R}^N} b(x) > 0$. M_1 and M_2 are bounded continuous functions, F belongs to $C^1(\mathbb{R}^N \times \mathbb{R}^2)$ and satisfies adequate growth assumptions, and F_u (respectively, F_v) denotes the partial derivative of F with respect to u (respectively, v). Here, we denote $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ the so-called $p(x)$-Laplacian operator, and for $u \in C_*(\mathbb{R}^N)$,

$$L_r(u) := \int_{\mathbb{R}^N} \frac{1}{r(x)}\left(|\nabla|u^{r(x)} + a(x)|u|^{r(x)}\right)dx. \tag{2}$$

System (1) is a generalization of the elliptic equation associated with the following Kirchhoff equation, introduced by Kirchhoff in [1]:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

where ρ, ρ_0, E, and L are constants. This equation extends classical D'Alembert's wave equation by considering the effects of the changes on the length of the strings during the vibrations. A distinguishing feature of equation (3) is that the equation contains a nonlocal coefficient $(\rho_0/h) + (E/2L)\int_0^L |\partial u/\partial x|^2 dx$ which depends on the average $(1/2L)\int_0^L |\partial u/\partial x|^2 dx$, and hence, the equation is no longer a pointwise equation. The parameters in equation (3) have the following meanings: E is Young's modulus of the material, ρ is the mass density, L is the length of the string, h is the area of cross section, and ρ_0 is the initial tension.

The $p(x)$-Laplacian operator possesses more complicated nonlinearities than p-Laplacian operator mainly due to the fact that it is not homogeneous. The study of various mathematical problems involving variable exponents has received a strong rise of interest in recent years. We can, for example, refer to [2–12]. This great interest may be justified

by their various physical applications. In fact, there are applications concerning nonlinear elasticity theory [13], electrorheological fluids [14, 15], stationary thermorheological viscous flows [16], and continuum mechanics [17]. It also has wide applications in different research fields, such as image processing model [18] and the mathematical description of the filtration process of an ideal barotropic gas through a porous medium [19].

The existence and multiplicity of solutions for the elliptic systems involving the $p(x)$-Kirchhoff model have been studied by many authors, where the nonlinear source F has different mixed growth conditions. We refer the reader to see [20–22] and the references therein for an overview on this subject. In connection to our context, the author obtained in [23] the existence and multiplicity of solutions for the vector-valued elliptic system:

$$\begin{cases} -M_1\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|u|^{p(x)}\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=\frac{\partial F}{\partial u}(x,u,v), & \text{in } \Omega,\\ -M_2\left(\int_{\mathbb{R}^N}\frac{1}{q(x)}|v|^{q(x)}\right)\operatorname{div}\left(|\nabla v|^{q(x)-2}\nabla v\right)=\frac{\partial F}{\partial v}(x,u,v), & \text{in } \Omega,\\ u=v=0, \quad \text{in } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, p and $q\in C_*(\Omega)=\{r\in C(\Omega)\colon 1<r^-=\inf_{x\in\Omega} r(x)<r(x)<r^+=\sup_{x\in\Omega} r(x)<N, \forall x\in\Omega\}$, and $M_1(t)$ and $M_2(t)$ are continuous functions such that $M_1(t)=M_2(t)$. The author applies a direct variational approach and the theory of variable exponent Sobolev spaces.

On the contrary, by using the mountain pass theorem, the authors in [24] showed the existence of nontrivial solutions for system (1) when $(p,q)\in[C(\mathbb{R}^N)]^2$ $(N\geq 2)$, $M_1(t)$ and $M_2(t)$ are continuous functions such that $M_1(t)=M_2(t)$, $a(x)=b(x)=0$, $\lambda=1$, and $F\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$ verifies some mixed growth conditions.

The goal of this work is to establish the existence of a definite interval in which λ lies such that system (1) admits at least three weak solutions by applying the following very recent abstract critical point result of Bonanno and Marano [25], which is a more precise version of Theorem 3.2 of [26].

Lemma 1 (see [25], Theorem 3.6). *Let X be a reflexive real Banach space; $\Phi\colon X\longrightarrow\mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; and $\Psi\colon X\longrightarrow\mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0)=\Psi(0)=0. \tag{5}$$

Assume that there exist $e>0$ and $\overline{x}\in X$, with $e<\Phi(\overline{x})$, such that

(a_1) $\sup_{\Phi(x)}\leq e\Psi(x)/e<\Psi(\overline{x})/\Phi(\overline{x})$

(a_2) For each $\lambda\in\Lambda_e:=\,](\Phi(\overline{x})/\Psi(\overline{x})),(e/\sup_{\Phi(x)}\leq e\Psi(x))[$, the functional $\Phi-\lambda\Psi$ is coercive

Then, for each $\lambda\in\Lambda_e$, the functional $\Phi-\lambda\Psi$ has at least three distinct critical points in X.

The rest of the paper is organized as follows. Section 2 contains some basic preliminary knowledge of the variable exponent spaces and some results that we shall use here. Finally, in Section 3, we state and establish our main result.

2. Preliminaries and Basic Notations

First, we introduce the definitions of Lebesgue–Sobolev spaces with variable exponents. The details can be found in [27–29]. Denote $\mathscr{M}(\mathbb{R}^N)$ as the set of all measurable real functions on $\mathbb{R}^N$. Set

$$C_+\left(\mathbb{R}^N\right)=\left\{p\in C\left(\mathbb{R}^N\right)\colon \inf_{x\in\mathbb{R}^N} p(x)>1\right\}. \tag{6}$$

For any $p\in C_+(\mathbb{R}^N)$, we define

$$\begin{aligned} p^- &:= \inf_{x\in\mathbb{R}^N} p(x),\\ p^+ &:= \sup_{x\in\mathbb{R}^N} p(x). \end{aligned} \tag{7}$$

For any $p\in C_+(\mathbb{R}^N)$, we define the variable exponent Lebesgue space as

$$L^{p(x)}\left(\mathbb{R}^N\right)=\left\{u\in\mathscr{M}\left(\mathbb{R}^N\right)\colon \int_{\mathbb{R}^N}|u(x)|^{p(x)}dx<\infty\right\}, \tag{8}$$

endowed with the Luxemburg norm

$$|u|_{p(x)}:=|u|_{L^{p(x)}(\mathbb{R}^N)}=\inf\left\{\mu>0\colon \int_{\mathbb{R}^N}\left|\frac{u(x)}{\mu}\right|^{p(x)}dx\leq 1\right\}. \tag{9}$$

Let $a\in M(\mathbb{R}^N)$ be such that $a(x)>0$, for a.e $x\in\mathbb{R}^N$. Define the weighted variable exponent Lebesgue space $L_a^{p(x)}(\mathbb{R}^N)$:

$$L_a^{p(x)}\left(\mathbb{R}^N\right)=\left\{u\in\mathscr{M}\left(\mathbb{R}^N\right)\colon \int_{\mathbb{R}^N}a(x)|u(x)|^{p(x)}dx<\infty\right\}, \tag{10}$$

with the norm

$$|u|_{p(x),a(x)}:=|u|_{L_a^{p(x)}(\mathbb{R}^N)}=\inf\left\{\mu>0\colon \int_{\mathbb{R}^N}a(x)\left|\frac{u(x)}{\mu}\right|^{p(x)}dx\leq 1\right\}. \tag{11}$$

From now on, we suppose that $a\in L^\infty(\mathbb{R}^N)$ with $a:=\operatorname{ess\,inf}_{x\in\mathbb{R}^N} a(x)>0$. Then, obviously, $L_a^{p(x)}$ is a Banach space (see [30] for details).

On the contrary, the variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is defined as follows:

$$W^{1,p(x)}\left(\mathbb{R}^N\right)=\left\{u\in L^{p(x)}\left(\mathbb{R}^N\right)\colon |\nabla u|\in L^{p(x)}\left(\mathbb{R}^N\right)\right\} \tag{12}$$

and is endowed with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\mathbb{R}^N). \tag{13}$$

Next, the weighted variable exponent Sobolev space $W_a^{1,p(x)}(\mathbb{R}^N)$ is defined as

$$W_a^{1,p(x)}(\mathbb{R}^N) := \left\{u \in L_a^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L_a^{p(x)}(\mathbb{R}^N)\right\} \tag{14}$$

and is endowed with the norm

$$\|u\|_a := \inf\left\{\mu > 0 : \int_{\mathbb{R}^N} \left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}, \quad \forall u \in W_a^{1,p(x)}(\mathbb{R}^N). \tag{15}$$

Note that $\|\cdot\|_a$ and $\|\cdot\|_{1,p(x)}$ are equivalent norms in $W_a^{1,p(x)}(\mathbb{R}^N)$. Moreover, when $p^- > 1$, it is well known that $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$, and $W_a^{1,p(x)}(\mathbb{R}^N)$ are separable, reflexive, and uniformly convex Banach spaces.

Now, we display some facts that we shall use later.

Proposition 1 (see [27, 28]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1. \tag{16}$$

Moreover, for any $(u, v) \in L^{p(x)}(\Omega) \times L^{p'(x)}(\Omega)$, we have

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}. \tag{17}$$

Proposition 2 (see [27, 28]). *Denote $\rho(u) := \int_{\mathbb{R}^N}|u|^{p(x)}dx$, for all $u \in L^{p(x)}(\mathbb{R}^N)$. We have*

$$\min\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\} \le \rho(u) \le \max\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\}, \tag{18}$$

and the following implications are true:

(i) $|u|_{p(x)} < 1$ *(resp.* $= 1, > 1) \Longleftrightarrow \rho(u) < 1$ *(resp.* $= 1, > 1)$

(ii) $|u|_{p(x)} > 1 \Longrightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$

(iii) $|u|_{p(x)} < 1 \Longrightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$

Denote $\rho_a(u) := \int_{\mathbb{R}^N}(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)})dx$, for all $u \in W_a^{1,p(x)}(\mathbb{R}^N)$. From Proposition 2, we have

$$\|u\|_a^{p^-} \le \rho_a(u) \le \|u\|_a^{p^+}, \quad \text{if } \|u\|_a \ge 1, \tag{19}$$

$$\|u\|_a^{p^+} \le \rho_a(u) \le \|u\|_a^{p^-}, \quad \text{if } \|u\|_a \le 1. \tag{20}$$

Proposition 3 (see [31]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x), q(x) < \infty$ almost everywhere in $\mathbb{R}^N$. If $u \in L^{q(x)}(\mathbb{R}^N)$, $u \ne 0$, then we have*

$$|u|_{p(x)q(x)} \le 1 \Longrightarrow |u|_{p(x)q(x)}^{p^-} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|_{p(x)q(x)}^{p^+},$$
$$|u|_{p(x)q(x)} \ge 1 \Longrightarrow |u|_{p(x)q(x)}^{p^+} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|_{p(x)q(x)}^{p^-}. \tag{21}$$

In particular, if $p(x) = p$ is constant, then

$$\left||u|^p\right|_{q(x)} = |u|_{pq(x)}^p. \tag{22}$$

For all $x \in \mathbb{R}^N$, denote

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & \text{for } p(x) < N, \\ +\infty, & \text{for } p(x) \ge N, \end{cases} \tag{23}$$

the critical Sobolev exponent of $p(x)$.

Proposition 4 (see [27, 31]). *Let $p \in C_+^{0,1}(\mathbb{R}^N)$, the space of Lipschitz-continuous functions defined on $\mathbb{R}^N$. There exists a positive constant c such that*

$$|u|_{p^*(x)} \le c\|u\|_a, \quad \forall u \in W_a^{1,p(x)}(\mathbb{R}^N). \tag{24}$$

Proposition 5 (see [27, 31]). *Assume that $p \in C(\mathbb{R}^N)$ satisfies $p(x) > 1$ for each $x \in \mathbb{R}^N$. If $q \in C(\mathbb{R}^N)$ is such that $1 < q(x) < p^*(x)$, for each $x \in \mathbb{R}^N$, then there exists a continuous and compact embedding $W^{1,p(x)}(\mathbb{R}^N) \longrightarrow L^{q(x)}(\mathbb{R}^N)$.*

In the following, we shall use the product space

$$X := W_a^{1,p(x)}(\mathbb{R}^N) \times W_b^{1,q(x)}(\mathbb{R}^N), \tag{25}$$

equipped with the norm

$$\|(u, v)\| := \max\{\|u\|_a, \|v\|_b\}, \quad \forall (u, v) \in X, \tag{26}$$

where $\|\cdot\|_a$ (respectively, $\|\cdot\|_b$) is the norm in $W_a^{1,p(x)}(\mathbb{R}^N)$ (respectively, $W_b^{1,q(x)}(\mathbb{R}^N)$) defined above. We denote X^* as the dual space of X equipped with the usual dual norm.

Definition 1. $(u, v) \in X$ is called a weak solution of system (1) if

$$M_1\left(L_p(u)\right)\int_{\mathbb{R}^N}\left(|\nabla u(x)|^{p(x)-2}\nabla u\nabla\varphi + a(x)|u|^{p(x)-2}u\varphi\right)dx$$
$$+M_2\left(L_q(u)\right)\int_{\mathbb{R}^N}\left(|\nabla v(x)|^{q(x)-2}\nabla v\nabla\psi + b(x)|v|^{q(x)-2}v\psi\right)dx$$
$$-\lambda\int_{\mathbb{R}^N}F_u(x,u,v)\phi dx - \lambda\int_{\mathbb{R}^N}F_v(x,u,v)\psi dx = 0, \tag{27}$$

for all $(\varphi,\psi)\in X$, where $L_r(u)$ is defined in (2).

We denote E_λ as the energy functional associated with problem (1):

$$E_\lambda(\cdot) := \Phi(\cdot) - \lambda\Psi(\cdot), \tag{28}$$

where $\Phi,\Psi\colon X\longrightarrow\mathbb{R}$ are defined as follows:

$$\Phi(u,v) = \Phi_1\left(L_p(u)\right) + \Phi_2\left(L_q(v)\right),$$
$$\Psi(\mathrm{u},\mathrm{v}) = \int_{\mathrm{R}^{\mathrm{N}}}\mathrm{F}(\mathrm{x},\mathrm{u},\mathrm{v})d\mathrm{x}, \tag{29}$$

where

$$\Phi_1\left(L_p(u)\right) = \widehat{M_1}\left(L_p(u)\right),$$
$$\Phi_2\left(L_q(v)\right) = \widehat{M_2}\left(L_q(v)\right). \tag{30}$$

for any $w=(u,v)$ in X, with

$$\widehat{M_i}(t) := \int_0^t M_i(s)ds,\quad \text{for all } t\ge 0,\ (i=1,2). \tag{31}$$

Note that we have the following formula:

$$F(x,u,v) = \int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)ds + F(x,0,0). \tag{32}$$

It is well know that $E_\lambda\in C^1(X,\mathbb{R})$ and that critical points of E_λ correspond to weak solutions of problem (1).

2.1. Hypotheses. In this paper, we use the following assumptions:

(H1) $F\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$ and $F(x,0,0)=0$.

(H2) There exist positive functions a_i and b_i $(i=1,2)$ such that

$$\left|\frac{\partial F}{\partial u}(x,u,v)\right| \le a_1(x)|u|^{\mu_1-1} + a_2(x)|v|^{\mu_2-1},$$
$$\left|\frac{\partial F}{\partial v}(x,u,v)\right| \le b_1(x)|u|^{\nu_1-1} + b_2(x)|v|^{\nu_2-1}, \tag{33}$$

where $1<\mu_1,\mu_2,\nu_1,\nu_2<\inf(p(x),q(x))$ and $p(x)$, $q(x)>N/2$, for all $x\in\mathbb{R}^N$, and the weight functions a_1,b_2 (respectively, a_2,b_1) belong to the generalized Lebesgue spaces $L^{\alpha_i}(\mathbb{R}^N)$ (respectively, $L^\beta(\mathbb{R}^N)$), with

$$\alpha_1(x) = \frac{p(x)}{p(x)-1},$$
$$\alpha_2(x) = \frac{q(x)}{q(x)-1}, \tag{34}$$
$$\beta(x) = \frac{p^*(x)q^*(x)}{p^*(x)q^*(x)-p^*(x)-q^*(x)}.$$

(H3) $M_i\colon\mathbb{R}^+\longrightarrow\mathbb{R}$ are continuous and increasing functions such that $0<m_0\le M_i(t)\le m_1$, for all $t\ge 0$, $(i=1,2)$.

(H4) There exist $e>0$ and $(w_1,w_2)\in X$ such that the following conditions are satisfied:

(C1) $m_0/p^+\min\left\{\|w_1\|_a^{p^-},\|w_1\|_a^{p^+}\right\} + m_0/q^+\min\left\{\|w_1\|_b^{q^-},\|w_1\|_b^{q^+}\right\} > e$,

(C2) $1/e\int_{\mathbb{R}^N}\sup_{(\xi_1,\xi_2)\in K(se/m_0)}F(x,\xi_1,\xi_2)dx < (1/m_1(\max\{\|w_1\|_a^{p^-},\|w_1\|_a^{p^+}\} + \max\{\|w_2\|_b^{q^-},\|w_2\|_b^{q^+}\}))\int_{\mathbb{R}^N}F(x,w_1,w_2)dx$,

where

$$K(t) := \left\{(\xi_1,\xi_2)\in\mathbb{R}^2\colon \min\left\{\left|\xi_1\right|_{p^*(x)}^{(p^*)^-},\left|\xi_1\right|_{p^*(x)}^{(p^*)^+}\right\} + \min\left\{\left|\xi_2\right|_{q^*(x)}^{(q^*)^-},\left|\xi_2\right|_{q^*(x)}^{(q^*)^+}\right\}\le t\right\}, \tag{35}$$

$$s = \min\left\{p^+\min\left\{c_{p(x)}^{(p^*)^-},c_{p(x)}^{(p^*)^+}\right\}, q^+\min\left\{c_{q(x)}^{(q^*)^-},c_{q(x)}^{(q^*)^+}\right\}\right\}, \tag{36}$$

with $t>0$ and $c_{p(x)}$ and $c_{q(x)}$ representing the constants defined in Proposition 4.

3. The Main Results

We will use the three critical point theorem obtained by Bonano and Marano together with the following lemmas to get our main results.

Lemma 2. *The functional Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous, coercive whose Gâteaux derivative admits a continuous inverse on X^*.*

Proof. It is well known that the functional Φ is well defined and is continuously Gâteaux differentiable functional whose derivative at the point $(u,v)\in X$ is the functional $\Phi'(u,v)$ given by

$$\langle\Phi'(u,v),(\varphi,\psi)\rangle = \langle\Phi_1'(u),\varphi\rangle + \langle\Phi_2'(v),\psi\rangle, \tag{37}$$

where

$$\begin{aligned}\langle \Phi_1'(u),\varphi\rangle &= M_1\big(L_p(u)\big)\int_{\mathbb{R}^N}\big(|\nabla u(x)|^{p(x)-2}\nabla u\nabla\varphi \\ &\quad + a(x)|u|^{p(x)-2}u\varphi\big)\mathrm{d}x, \\ \langle \Phi_2'(v),\psi\rangle &= M_2\big(L_q(u)\big)\int_{\mathbb{R}^N}\big(|\nabla v(x)|^{q(x)-2}\nabla v\nabla\psi \\ &\quad + b(x)|v|^{q(x)-2}v\psi\big)\mathrm{d}x,\end{aligned} \tag{38}$$

for every $(\varphi,\psi)\in X$ and $L_r(u)$ is defined in (19).

Let us show that Φ is coercive. By using (19) and (20), we have for all $(u,v)\in X$,

$$\begin{aligned}\Phi(u,v) &= \widehat{M_1}\big(L_p(u)\big)+\widehat{M_2}\big(L_q(v)\big) \\ &\geq \frac{m_0}{p^+}\int_{\mathbb{R}^N}\big(|\nabla u(x)|^{p(x)}\mathrm{d}x + a(x)|u(x)|^{p(x)}\big) \\ &\quad + \frac{m_0}{q^+}\int_{\mathbb{R}^N}\big(|\nabla v(x)|^{q(x)}\mathrm{d}x + b(x)|v(x)|^{q(x)}\big)\mathrm{d}x \\ &\geq \frac{m_0}{p^+}\min\big\{\|u\|_a^{p^-},\|u\|_a^{p^+}\big\} + \frac{m_0}{q^+}\min\big\{\|v\|_b^{q^-},\|v\|_b^{q^+}\big\}.\end{aligned} \tag{39}$$

This shows that $\Phi(u,v)\longrightarrow +\infty$ as $\|(u,v)\|\longrightarrow +\infty$, that is, Φ is coercive on X.

Now, in order to show that the operator Φ': $X\longrightarrow X^*$ is strictly monotone, it suffices to prove that Φ is strictly convex.

For $r\in C_*(\mathbb{R}^N)$, the functional L_r: $W_a^{1,r(x)}(\mathbb{R}^N)\longrightarrow\mathbb{R}$ defined in (2) is clearly a Gâteaux derivative at any $u\in W_a^{1,r(x)}(\mathbb{R}^N)$, and his derivative is given by

$$\langle L_r'(u),\varphi\rangle = \int_{\mathbb{R}^N}\big(|\nabla u(x)|^{r(x)-2}\nabla u\nabla\varphi + a(x)|u|^{r(x)-2}u\varphi\big)\mathrm{d}x, \tag{40}$$

for all $\varphi\in W_a^{1,r(x)}(\mathbb{R}^N)$.

Taking into account the inequality (see, e.g., Chapter I in [32]) for $\gamma>1$, there exists a positive constant C_γ such that

$$\langle |\alpha|^{\gamma-2}\alpha - |\beta|^{\gamma-2}\beta,\alpha-\beta\rangle \geq \begin{cases} C_\gamma|\alpha-\beta|^\gamma, & \text{if } \gamma\geq 2, \\ C_\gamma\dfrac{|\alpha-\beta|^2}{(|\alpha|+|\beta|)^{2-\gamma}},\ (\alpha,\beta)\neq(0,0), & \text{if } 1<\gamma<2,\end{cases} \tag{41}$$

for any $\alpha,\beta\in\mathbb{R}^N$. Therefore, we have

$$\langle L_p'(u_1)-L_p'(u_2),u_1-u_2\rangle > 0, \tag{42}$$

for all $u_1\neq u_2\in W_a^{1,p(x)}(\mathbb{R}^N)$ which means that L_p' is strictly monotone. So, by ([33], Proposition 25.10), L_p is strictly convex. Moreover, since the Kirchhoff function M_1 is nondecreasing, $\widehat{M_1}$ is convex in $[0,+\infty[$. Thus, for every $u_1,u_2\in W_a^{1,p(x)}(\mathbb{R}^N)$ with $u_1\neq u_2$ and every $]s,t\in 0,1[$ with $s+t=1$, one has

$$\begin{aligned}\widehat{M_1}(L(su_1+tu_2)) < \widehat{M_1}((sL(u_1)+tL(u_2))) \leq s\widehat{M_1}(L(u_1)) \\ + t\widehat{M_1}(L(u_2)).\end{aligned} \tag{43}$$

This shows that Φ_1 is strictly convex in $W_a^{1,p(x)}(\mathbb{R}^N)$. Similarly, we have that Φ_2 is strictly convex in $W_b^{1,q(x)}(\mathbb{R}^N)$. Hence, Φ is strictly convex in X, and so $\Phi'=\Phi_1'+\Phi_2'$ is strictly monotone.

It is clear that Φ' is an injection since Φ' is a strictly monotone operator in X. Moreover, since we have

$$\begin{aligned}&\lim_{\|(u,v)\|\longrightarrow+\infty}\frac{\langle\Phi'(u,v),(u,v)\rangle}{\|(u,v)\|}\geq \\ &\lim_{\|(u,v)\|\longrightarrow+\infty}\frac{m_0\Big(\int_{\mathbb{R}^N}\big(|\nabla u|^{p(x)}+a(x)|u|^{p(x)}\big)\mathrm{d}x + \int_{\mathbb{R}^N}\big(|\nabla v|^{q(x)}+b(x)|v|^{q(x)}\big)\mathrm{d}x\Big)}{\|(u,v)\|} = +\infty,\end{aligned} \tag{44}$$

then, we deduce that Φ' is coercive (see (19)). Thus, Φ' is a surjection. Now, since Φ' is hemicontinuous in X, then by applying (Proposition 4.2, [22]), we conclude that Φ' admits a continuous inverse on X^*. Moreover, the monotonicity of

Φ' on X^* ensures that Φ is sequentially lower semi-continuously on X (see [33], Proposition 25. 20). The proof of the lemma is complete. □

Lemma 3 (see [8]). *Under assumptions* $(H1)$ *and* $(H2)$, *the functional* Ψ *is well defined and is of class* C^1 *on* X. *Moreover, its derivative is given by*

$$\Psi^1(u,v)(\varphi,\psi) = \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)\varphi + \frac{\partial F}{\partial v}(x,u,v)\psi \mathrm{d}x, \quad \forall (u,v), (\varphi,\psi) \in X. \tag{45}$$

Moreover, Ψ^1 *is compact from* X *to* X^*.

Theorem 1. *Under assumptions* $(H1)-(H4)$, *system (1) admits at least three distinct weak solutions in* X *for each*

$$\lambda \in \left] \frac{m_1\left(\max\left\{\|w_1\|_a^{p^-}, \|w_1\|_a^{p^+}\right\} + \max\left\{\|w_2\|_b^{q^-}, \|w_2\|_b^{q^+}\right\}\right)}{\int_{\mathbb{R}^N} F(x, w_1(x), w_1(x))\mathrm{d}x}, \frac{e}{\int_{\mathbb{R}^N} \sup_{(\xi_1,\xi_2)\in K(se/m_0)} F(x,\xi_1,\xi_2)\mathrm{d}x} \right[. \tag{46}$$

Proof. By Lemma 2, Φ is coercive, and by the definitions of Φ and Ψ and from hypothesis (H_1), we have $\Phi(0,0) = \Psi(0,0) = 0$. Moreover, the required hypothesis $\Phi(\overline{x}) > e$ follows from condition (C1) and the definition of Φ by choosing $\overline{x} = (w_1, w_2)$. On the contrary, by applying Proposition 4 for $(u,v) \in X$, we have

$$\frac{1}{s}\left(\min\left\{|u|_{p^*}^{p^-}, |u|_{p^*}^{p^+}\right\} + \min\left\{|v|_{s^*}^{q^-}, |v|_{q^*}^{q^+}\right\}\right) \le \frac{1}{p^+}\min\left\{\|u\|_a^{p^-}, \|u\|_a^{p^+}\right\} + \frac{1}{q^+}\min\left\{\|v\|_b^{q^-}, \|v\|_b^{q^+}\right\}, \tag{47}$$

with $s = \min\left\{p^+ \min\left\{c_{p(x)}^{(p^*)^-}, c_{p(x)}^{(p^*)^+}\right\}, q^+ \min\left\{c_{q(x)}^{(q^*)^-}, c_{q(x)}^{(q^*)^+}\right\}\right\}$, defined in (36). Now, from (47), we obtain for $e > 0$

$$\begin{aligned}
\Phi^{-1}(]-\infty, e[) &= \{x = (u,v) \in X : \Phi(u,v) \le e\} \\
&\subseteq \left\{(u,v) \in X : \frac{m_0}{p^+}\min\left\{\|u\|_a^{p^-}, \|u\|_a^{p^+}\right\} + \frac{m_0}{q^+}\min\left\{\|v\|_b^{q^-}, \|v\|_b^{q^+}\right\} \le e\right\} \\
&\subseteq \left\{(u,v) \in X : \min\left\{|u|_{p^*}^{p^-}, |u|_{p^*}^{p^+}\right\} + \min\left\{|v|_{s^*}^{q^-}, |v|_{q^*}^{q^+}\right\} \le \frac{se}{m_0}\right\} \\
&= K\left(\frac{se}{m_0}\right),
\end{aligned} \tag{48}$$

where $K(\cdot)$ is defined in (35). Then,

$$\begin{aligned}
\sup_{(u,v)\in\Phi^{-1}(]-\infty,e[)} \Psi(u) &= \sup_{(u,v)\in\Phi^{-1}(]-\infty,e[)} \int_{\mathbb{R}^N} F(x,u,v)\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \sup_{(\xi_1,\xi_2)\in K(se/m_0)} F(x,\xi_1,\xi_2)\mathrm{d}x.
\end{aligned} \tag{49}$$

Therefore, from condition (C2), we have

$$\begin{aligned}\sup_{(u,v)\in\Phi^{-1}(]-\infty,e[)}\Psi(u)&\le e\frac{\int_{\mathbb{R}^N}F(x,w_1(x),w_1(x))\mathrm{d}x}{m_1\left(\max\left\{\|w_1\|_a^{p^-},\|w_1\|_a^{p^+}\right\}+\max\left\{\|w_2\|_b^{q^-},\|w_2\|_b^{q^+}\right\}\right)}\\&\le e\frac{\Psi(w_1,w_2)}{\Phi(w_1,w_2)},\end{aligned}\tag{50}$$

from which condition (a_1) of Lemma 1 follows.

To show that the functional $E_\lambda=\Phi-\lambda\Psi$ is coercive, we use inequality (3.8). For all $(u,v)\in X$, we have in virtue of (H1) and (H2)

$$\begin{aligned}E_\lambda(u,v)&=\widehat{M_1}\left(L_p(u)\right)+\widehat{M_2}\left(L_q(v)\right)-\lambda\int_{\mathbb{R}^N}F(x,u(x),v(x))\mathrm{d}x\\&\ge\frac{m_0}{p^+}\int_{\mathbb{R}^N}\left(|\nabla u(x)|^{p(x)}\mathrm{d}x+a(x)|u(x)|^{p(x)}\right)\\&+\frac{m_0}{q^+}\int_{\mathbb{R}^N}\left(|\nabla v(x)|^{q(x)}\mathrm{d}x+b(x)|v(x)|^{q(x)}\right)\mathrm{d}x\\&-\int_{\mathbb{R}^N}\left(\int_0^u\frac{\partial F}{\partial s}(x,s,v)\mathrm{d}s+\int_0^v\frac{\partial F}{\partial s}(x,0,s)\mathrm{d}s+F(x,0,0)\mathrm{d}x\right)\\&\ge\frac{m_0}{p^+}\rho_a(u)+\frac{m_0}{q^+}\rho_b(v)-\int_{\mathbb{R}^N}\left(a_1(x)|u|^{\mu_1}+a_2(x)|v|^{\mu_2-1}|u|+b_2(x)|v|^{\gamma_2}\right)\mathrm{d}x\\&\ge\frac{m_0}{p^+}\rho_a(u)+\frac{m_0}{q^+}\rho_b(v)-\left(\left|a_1\right|_{\alpha_1(x)}\left||u|^{\mu_1}\right|_{p(x)}+\left|a_2\right|_{\beta(x)}\left||v|^{\mu_2-1}\right|_{q^*(x)}\left|u\right|_{p^*(x)}\right.\\&\left.+\left|b_2\right|_{\alpha_2(x)}\left||v|^{\gamma_2}\right|_{q(x)}\right).\end{aligned}\tag{51}$$

Using Young's inequality, we obtain

$$\begin{aligned}E_\lambda(u,v)&\ge\frac{m_0}{p^+}\|u\|_a^{p^-}+\frac{m_0}{q^+}\|v\|_b^{q^-}-\lambda\left(\left|a_1\right|_{\alpha_1(x)}\|u\|_{p(x)}^{\mu_1}\right.\\&\left.+\left|a_2\right|_{\beta(x)}\left(\frac{\mu_2-1}{\mu_2}\|v\|_{q(x)}^{\mu_2}+\frac{1}{\mu_2}\|u\|_{p(x)}^{\mu_2}\right)+\left|b_2\right|_{\alpha_2(x)}\|v\|_{q(x)}^{\gamma_2}\right)\\&\ge\frac{m_0}{p^+}\|u\|_a^{p^-}+\frac{m_0}{q^+}\|v\|_b^{q^-}-c\left(\left|a_1\right|_{\alpha_1(x)}\|u\|_{p(x)}^{\mu_1}\right.\\&\left.+\left|a_2\right|_{\beta(x)}\|v\|_{q(x)}^{\mu_2}+\left|a_2\right|_{\beta(x)}\|u\|_{p(x)}^{\mu_2}+\left|b_2\right|_{\alpha_2(x)}\|v\|_{q(x)}^{\gamma_2}\right).\end{aligned}\tag{52}$$

This shows that $\Phi-\lambda\Psi\longrightarrow+\infty$ as $\|(u,v)\|_X\longrightarrow\infty$ since we have $1<\mu_1,\mu_2,\nu_1,\nu_2<\inf(p(x),q(x))$, that is, $\Phi-\lambda\Psi$ is coercive on X, for every parameter λ, in particular, for every $\lambda\in\Lambda_e:=\,](\Phi(w_1,w_2)/\Psi(w_1,w_2)),(e/\sup_{\Phi(u,v)}\le e\Psi(u,v))[$. Then, condition (a_2) in Lemma 1 also holds. Now, all the hypotheses of Lemma 1 are satisfied. Note that the solutions of the equation $\Phi'(u,v)-\lambda\Psi^1(u,v)=0$ are exactly the weak solutions of (1). Thus, for each

$$\lambda \in \left] \frac{m_1\left(\max\left\{\|w_1\|_a^{p^-}, \|w_1\|_a^{p^+}\right\} + \max\left\{\|w_2\|_b^{q^-}, \|w_2\|_b^{q^+}\right\}\right)}{\int_{\mathbb{R}^N} F(x, w_1(x), w_1(x))\mathrm{d}x}, \frac{e}{\int_{\mathbb{R}^N} \sup_{(\xi_1,\xi_2)\in K(se/m_0)} F(x, \xi_1, \xi_2)\mathrm{d}x} \right[, \quad (53)$$

system (1) admits at least three weak solutions in X. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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