Variational Iteration Method and Differential Transformation Method for Solving the SEIR Epidemic Model

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1. Introduction

The whole world is experiencing hardship due to coronavirus (COVID-19), which was first identified in Wuhan, China, in the month of December 2019. It has been considered that COVID-19 originated from wild animals (bats [1]) and transmitted to humans as numerous infected patients claimed that they had been to a local wet market in Wuhan during the end of November [2]. Later, some investigators confirmed that the virus transmission occurs from person to person [3].

Mathematical models can simulate the effects of a disease at many levels, ranging from how the disease influences the interaction between cells in a single patient (within-host models) to how it spreads across several geographically separated populations (metapopulation models). Models simulating the disease spread within and among populations, such as those used to forecast the COVID-19 outbreak [4], are typically based on the SEIR model.

The SEIR model is based on the division of the population under study into four compartments: an individual can either be susceptible (S), exposed to the disease but not yet infectious (E), infectious (I), or recovered (R). The SEIR model can represent many human infectious diseases [5–9]. In this paper, we focus, analyze, and find a solution for the model of nonlinear ordinary differential equations (ODEs) describing the deadly and most parlous coronavirus (COVID-19). A mathematical model based on the four nonlinear ODEs is presented, and the corresponding numerical results are studied by applying the variational iteration method (VIM) and differential transformation method (DTM).
problems in electric circuit analysis. The DTM provides in a fast manner exact values of the $n^{th}$ derivative of an analytical function at a point in terms of known and unknown boundary conditions. This method constructs, for differential equations, an analytical solution in the form of a polynomial.

2. The SEIR Model

The SEIR model in epidemiology for the spread of an infectious disease is described by the following system of differential equations:

\[
\begin{align*}
\frac{dS}{dt}(t) &= -\beta S(t)I(t), \\
\frac{dE}{dt}(t) &= \beta S(t)I(t) - \alpha E(t), \\
\frac{dI}{dt}(t) &= \alpha E(t) - \gamma I(t), \\
\frac{dR}{dt}(t) &= \gamma I(t).
\end{align*}
\]

(1)

Here, $\beta$, $\alpha$, and $\gamma$ are positive parameters and $S$, $E$, $I$, and $R$ denote the fractions of the population that are susceptible, exposed, infectious, and recovered, respectively.

A schematic diagram of the disease transmission among the individuals is shown in Figure 1 using the SEIR model. For more information about the model refer to [18].

The SEIR model of the novel coronavirus (COVID-19) can be represented as follows:

(1) The rate of change in the number of susceptible people = the susceptible portion of the population $\times$ the average number of people infected by an infectious person over the average duration of infection $\times$ the number of people infected by infectious people $+$ the susceptible portion of the population $\times$ the rate of infectious animal source $-$ the number of exposed people over the average latency period $-$ percentage of population traveling out $\times$ the number of exposed people $-$ the death rate of the exposed people $\times$ the number of exposed people $-$ testing and therapy rate $\times$ the number of exposed people:

\[
\frac{dS}{dt}(t) = -\beta \frac{S(t)}{N} I(t) - \frac{Z}{N} S(t) + \left(\rho_t + \rho_e\right) - \left(\frac{\theta_t}{N} + \frac{\theta_e}{N}\right) S(t) + \nu N(t) - \mu S(t).
\]

(2)

(2) The rate of change in the number of exposed people = the susceptible portion of the population $\times$ the average number of people infected by an infectious person over the average duration of infection $\times$ the number of people infected by infectious people + the susceptible portion of the population $\times$ the rate of infectious animal source $-$ the number of exposed people over the average latency period $-$ percentage of population traveling out $\times$ the number of exposed people $-$ testing and therapy rate $\times$ the number of exposed people:

\[
\frac{dE}{dt}(t) = \beta \frac{S(t)}{N} I(t) + \frac{Z}{N} S(t) - \alpha E(t) - \left(\frac{\theta_t}{N} + \frac{\theta_e}{N}\right) E(t) - \mu E(t) - \sigma E(t).
\]

(3)

(3) The rate of change in the number of infected people = the number of exposed people over the average latency period $-$ the number of infected people over the average duration of infection $-$ percentage of population traveling out $\times$ the number of exposed people $-$ the death rate of the infected people $\times$ the number of infected people:

\[
\frac{dI}{dt}(t) = \alpha E(t) - \gamma I(t) - \left(\frac{\theta_t}{N} + \frac{\theta_e}{N}\right) I(t) - \mu I(t).
\]

(4)

(4) The rate of change in the number of recovered people $=$ the number of infected people over the average duration of infection $-$ the death rate of the recovered people $\times$ the number of recovered people $+$ testing and therapy rate $\times$ the number of exposed people Figure 2:

\[
\frac{dR}{dt}(t) = \gamma I(t) - \mu R(t) + \sigma E(t).
\]

(5)

The transitions between model classes can now be expressed by the following system of first-order differential equations (Table 1):

![Figure 1: SEIR compartmental model.](image-url)
\[
\frac{dS}{dt} = -\frac{\beta S(t)}{N} I(t) - \frac{Z}{N} S(t) + \left(\frac{\rho_I}{N} + \frac{\rho_E}{N}\right) S(t) + \nu N(t) - \mu S(t),
\]
\[
\frac{dE}{dt} = \frac{\beta S(t)}{N} I(t) + \frac{Z}{N} S(t) - aE(t) - \left(\frac{\theta_I}{N} + \frac{\theta_E}{N}\right) E(t) - \mu E(t) - \sigma E(t),
\]
\[
\frac{dI}{dt} = aE(t) - \gamma I(t) - \left(\frac{\theta_I}{N} + \frac{\theta_E}{N}\right) I(t) - \mu I(t),
\]
\[
\frac{dR}{dt} = \gamma I(t) - \mu R(t) + \sigma E(t),
\]

with the initial conditions
\[
S(0) = S_0,
E(0) = E_0,
I(0) = I_0,
R(0) = R_0.
\]

where \(L\) is a linear operator, \(N\) is a nonlinear operator, and \(F(t)\) is a known analytical function. We can construct a correction functional according to the variational method as follows:
\[
U_{n+1}(t) = U_n(t) + \int_0^t \lambda \left[ LU_n(s) + N \ddot{U}_n(s) - F(s) \right] ds,
\]

where \(\lambda\) is the general Lagrange multiplier [19], which can be identified optimally via the variational theory, \(U_n\) is the \(n\)th approximate solution, and \(\ddot{U}_n\) denotes a restricted variation, which means \(\delta N \ddot{U}_n = 0\). Successive approximations, \(U_{n+1}\), will be obtained by applying the obtained Lagrange multiplier and a properly chosen initial approximation \(U_0\). Consequently, the solution is given by \(U = \lim_{n \to \infty} U_n\). For
solving equation (6) by means of the VIM, we construct the correctional functional as follows:

\[
\begin{align*}
S_{n+1}(t) &= S_n(t) + \int_0^t \lambda_1(r) \left\{ \frac{dS_n(r)}{dr} + \beta \frac{S_n(r)}{N} I_n(r) + \frac{Z S_n(r)}{N} - (\rho_I + \rho_E) \left( \frac{\rho_I}{N} + \frac{\rho_E}{N} \right) S_n(r) - \gamma N_n(r) + \mu S_n(r) \right\} dr, \\
E_{n+1}(t) &= E_n(t) + \int_0^t \lambda_2(r) \left\{ \frac{dE_n(r)}{dr} - \beta \frac{S_n(r)}{N} I_n(r) - \frac{Z S_n(r)}{N} + \left( \alpha + \left( \frac{\rho_I}{N} + \frac{\rho_E}{N} \right) + \mu + \sigma \right) E_n(r) \right\} dr, \\
I_{n+1}(t) &= I_n(t) + \int_0^t \lambda_3(r) \left\{ \frac{dI_n(r)}{dr} - \alpha E_n(r) + \left( \gamma + \frac{\rho_I}{N} + \frac{\rho_E}{N} + \mu \right) I_n(r) \right\} dr, \\
R_{n+1}(t) &= R_n(t) + \int_0^t \lambda_4(r) \left\{ \frac{dR_n(r)}{dr} - \gamma I_n(r) + \mu \tilde{R}_n(r) - \sigma \tilde{E}_n(r) \right\} dr.
\end{align*}
\]

(10)

Here, \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \) are general Lagrange multipliers. Making the above correctional functional stationary with respect to \( S_n(t), E_n(t), I_n(t), \) and \( R_n(t), \) noticing that \( \delta S_n(t) = \delta E_n(t) = \delta I_n(t) = \delta R_n(t) = 0, \) yields

\[
\begin{align*}
\delta S_{n+1}(t) &= \delta S_n(t) + \delta \int_0^t \lambda_1(r) \left\{ \frac{dS_n(r)}{dr} + \beta \frac{S_n(r)}{N} I_n(r) + \frac{Z S_n(r)}{N} - (\rho_I + \rho_E) \left( \frac{\rho_I}{N} + \frac{\rho_E}{N} \right) S_n(r) - \gamma N_n(r) + \mu S_n(r) \right\} dr = 0, \\
\delta E_{n+1}(t) &= \delta E_n(t) + \delta \int_0^t \lambda_2(r) \left\{ \frac{dE_n(r)}{dr} - \beta \frac{S_n(r)}{N} I_n(r) - \frac{Z S_n(r)}{N} + \left( \alpha + \left( \frac{\rho_I}{N} + \frac{\rho_E}{N} \right) + \mu + \sigma \right) E_n(r) \right\} dr = 0, \\
\delta I_{n+1}(t) &= \delta I_n(t) + \delta \int_0^t \lambda_3(r) \left\{ \frac{dI_n(r)}{dr} - \alpha E_n(r) + \left( \gamma + \frac{\rho_I}{N} + \frac{\rho_E}{N} + \mu \right) I_n(r) \right\} dr = 0, \\
\delta R_{n+1}(t) &= \delta R_n(t) + \delta \int_0^t \lambda_4(r) \left\{ \frac{dR_n(r)}{dr} - \gamma I_n(r) + \mu \tilde{R}_n(r) - \sigma \tilde{E}_n(r) \right\} dr = 0.
\end{align*}
\]

(11)

Therefore, the Lagrange multiplier can readily be identified:

\[
\begin{align*}
S_{n+1}(t) &= S_n(t) - \int_0^t \left\{ \frac{dS_n(r)}{dr} + \beta \frac{S_n(r)}{N} I_n(r) + \frac{Z S_n(r)}{N} - (\rho_I + \rho_E) \left( \frac{\rho_I}{N} + \frac{\rho_E}{N} \right) S_n(r) - \gamma N_n(r) + \mu S_n(r) \right\} dr, \\
E_{n+1}(t) &= E_n(t) - \int_0^t \left\{ \frac{dE_n(r)}{dr} - \beta \frac{S_n(r)}{N} I_n(r) - \frac{Z S_n(r)}{N} + \left( \alpha + \left( \frac{\rho_I}{N} + \frac{\rho_E}{N} \right) + \mu + \sigma \right) E_n(r) \right\} dr, \\
I_{n+1}(t) &= I_n(t) - \int_0^t \left\{ \frac{dI_n(r)}{dr} - \alpha E_n(r) + \left( \gamma + \frac{\rho_I}{N} + \frac{\rho_E}{N} + \mu \right) I_n(r) \right\} dr, \\
R_{n+1}(t) &= R_n(t) - \int_0^t \left\{ \frac{dR_n(r)}{dr} - \gamma I_n(r) + \mu \tilde{R}_n(r) - \sigma \tilde{E}_n(r) \right\} dr.
\end{align*}
\]

(12)

With initial approximations \( S(0) = 2500, E(0) = 1, \)
\( I(0) = 1, \) \( R(0) = 0, \) and \( N = 2502, \) which in turn gives successive approximations, and considering the following values for parameters (see [20]) \( \beta = 0.8, \alpha = 0.75, \)
\[ \sigma = 0.1, \gamma = 0.05, \nu = 0.009/N, \mu = 0.01, \]
\[ Z = 0.001, \rho_T = 0.15, \rho_E = 0.15, \varrho_T = 0.01, \text{and} \varrho_E = 0.03, \]

we obtain
\[ \begin{aligned}
S_1(t) &= 2500 - 25.53132774t, \\
E_1(t) &= 1 - 0.059656274t, \\
I_1(t) &= 1 + 0.689984012t, \\
R_1(t) &= 0.15t.
\end{aligned} \tag{13} \]

Similarly, we get the following system after two terms:
\[ \begin{aligned}
S_2(t) &= 2500 - 25.53132774t - 0.143869926t^2 + 0.013851166t^3 - 8.48550280810^{-5}t^4 \\
&\quad - 4.79158897410^{-7}t^5 + 4.31004739710^{-8}t^6, \\
E_2(t) &= 1 - 0.059656274t + 0.29733881t^2 - 0.09860941t^3 + 4.8381106310^{-4}t^4 \\
&\quad - 4.79158897410^{-7}t^5 + 4.31004739710^{-8}t^6, \\
I_2(t) &= 1 + 0.689984012t - 0.043076138t^2 + 0.074410709t^3 - 3.52042524910^{-4}t^4, \\
R_2(t) &= 0.15t + 0.013516786t^2 + 9.1483020810^{-3}t^3 - 4.69390033310^{-5}t^4.
\end{aligned} \tag{15} \]

For the solution after three terms, we can write
\[ \begin{aligned}
S_3(t) &= 2500 - 25.53132774t - 0.143869926t^2 + 0.013851166t^3 - 8.48550280810^{-5}t^4 \\
&\quad - 4.79158897410^{-7}t^5 + 4.31004739710^{-8}t^6, \\
E_3(t) &= 1 - 0.059656274t + 0.29733881t^2 - 0.09860941t^3 + 4.8381106310^{-4}t^4 \\
&\quad - 4.79158897410^{-7}t^5 + 4.31004739710^{-8}t^6, \\
I_3(t) &= 1 + 0.689984012t - 0.043076138t^2 + 0.074410709t^3 - 3.52042524910^{-4}t^4, \\
R_3(t) &= 0.15t + 0.013516786t^2 + 9.1483020810^{-3}t^3 - 4.69390033310^{-5}t^4.
\end{aligned} \tag{16} \]

4. The Differential Transformation Method

The basic definition and the fundamental theorems of the DTM and its applicability to various kinds of differential equations are given in [17, 21]. According to the operations of differential transformation given in Table 1 in [21], we have the following recurrence relation:

\[ \begin{aligned}
S(k + 1) &= \frac{1}{k + 1} \left[ \frac{\beta}{N} \sum_{m=0}^{k} \kappa (m) S(k - m) + \frac{Z}{N} S(k) + (\rho_T + \rho_E) \delta(k) + \nu N (k) \right] - \left[ \frac{\varrho_T + \varrho_E}{N} + \mu \right] S(k), \\
E(k + 1) &= \frac{1}{k + 1} \left[ \frac{\beta}{N} \sum_{m=0}^{k} \kappa (m) I(k - m) + \frac{Z}{N} S(k) \right] - \left[ \alpha + \frac{\varrho_T + \varrho_E}{N} + \mu + \sigma \right] E(k), \\
I(k + 1) &= \frac{1}{k + 1} \left[ \alpha E(k) \right] - \left[ \gamma + \frac{\varrho_T + \varrho_E}{N} + \mu \right] I(k), \\
R(k + 1) &= \frac{1}{k + 1} \left[ \gamma I(k) - \mu R(k) + \sigma E(k) \right].
\end{aligned} \tag{17} \]

The inverse differential transformation of \( S(k) \) is defined as follows: when \( t_0 \) is taken as zero, the given function \( S(t) \) is declared by a finite series, and the above equation can be written in the form
\[ S(t) = \sum_{k=0}^{\infty} S(k)t^k. \tag{18} \]

By solving the above equations for \( S(k + 1), E(k + 1), I(k + 1), \) and \( R(k + 1) \) up to order 3, we obtain the functions of \( S(k), E(k), I(k), \) and \( R(k), \) respectively:
With initial approximations, $S(0) = 2500$, $E(0) = 1$, $I(0) = 1$, $R(0) = 0$, and $N = 2502$ and parameters $\beta = 0.8$, $\alpha = 0.75$, $\sigma = 0.1$, $\gamma = 0.05$, $\nu = 0.009/N$, $\mu = 0.01$, $Z = 0.001$, $\rho_I = 0.15$, $\rho_E = 0.15$, $\varphi_I = 0.01$, and $\varphi_E = 0.03$, and applying the conditions in equations (16) and (18), we obtain the approximate solution after three terms as follows:

$$S(t) = 2500 - 25.53132774t - 0.143869688t^2 + 0.013851335t^3,$$
$$E(t) = 1 - 0.059656274t + 0.297292652t^2 - 0.099439782t^3,$$
$$I(t) = 1 + 0.689984012t - 0.043076138t^2 + 0.075184915t^3,$$
$$R(t) = 0.15t + 0.013516786t^2 + 9.1467634810^{-3}t^3. \quad (19)$$

5. Conclusions

In this paper, we have developed the SEIR model of the COVID-19 epidemic in China that incorporates key features of this pandemic. For solving this model, we used the variational iteration method (VIM) and differential transformation method (DTM). It is found that these methods are effective in providing analytic form solutions for such problems. The comparison of the results obtained by these two methods is in excellent agreement.

For further research, we propose the study of the fractional-order model using the Caputo–Fabrizio derivative [22, 23]. In addition, we propose to extend the results of the
present paper and combine them with the results in [6] (Figures 3–6).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


