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Research Article

Second-Order Elliptic Equation with Singularities

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On the compact Riemannian manifold of dimension $n \geq 5$, we study the existence and regularity of nontrivial solutions for nonlinear second-order elliptic equation with singularities. At the end, we give a geometric application of the above singular equation.

1. Introduction

Let (M, g) be an $(n \geq 3)$-dimensional compact Riemannian manifold, and let $a \in L^p(M)$, where $p > n/2$, and f be a positive $C^\infty(M)$ function on M. In this paper, we are interested in studying on, (M, g), the following nonlinear singular elliptic equation:

$$\Delta_g u + au = f|u|^{N-2}u, \tag{1}$$

where $\Delta_g u = -\nabla^i \nabla_i u$ is the Laplacian–Beltrami operator and $N = 2n/(n-2)$ is the critical Sobolev exponent. Equation (1) is one of the nonlinear second-order equations involving the singular term a and with critical Sobolev growth. Such problem arises from various fields of geometry and physics. There are many results for second-order elliptic equations, but most of them are focused on bounded domains Ω of $\mathbb{R}^n$ or on compact Riemannian manifold (M, g), see [1–16] for a survey. A variety of techniques have been used to solve second-order equations, and variational methods are the most suitable. Certainly, if the singular term a is replaced by $(n-2)/4(n-1)S_g$, where S_g is the scalar curvature and $f = 1$, then equation (1) becomes the famous prescribed constant scalar curvature equation which is very known in the literature as the Yamabe problem. To solve this problem, Yamabe has used the variational method, and the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. The problem is now solved, but it took a very long time to find the good approach. If f is not a constant, the problem is known as the prescribed scalar curvature problem. For more details, we refer the reader to [12, 13] and the references therein.

A famous result concerning the equation of type (1) has been obtained in [17], and it consists of the classification of positive solutions of the equation

$$\Delta u - \frac{\lambda}{|x|^2}u = u^{n+2/n-2}, \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where $0 < \lambda < (n-4)^2/4$, into the family of functions

$$u_\lambda(x) = c_\lambda \left(\frac{|x|^{a-1}}{1 + |x|^{2a}} \right)^{(n/2)-1}, \tag{3}$$

where $c_\lambda = \sqrt{1 - (4\lambda/(n-2)^2)}$.

The singular term a was introduced as follows: in [11], Madani studied equation (1) with f is a constant, $a = (n-2)/4(n-1)S_g$, and such that the metric g admits a finite number of points with singularities and is smooth outside these points. This problem can be seen as the Yamabe problem with singularities. More precisely, let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$; we denote by T^*M the cotangent space of M. The metric $g \in H_2^p(M, T^*M \otimes T^*M)$ is the space of all sections g (2-covariant tensors) such that, in normal coordinates, the components g_{ij} of g are in $H_2^p(M)$, where $H_2^p(M)$ is the completion of the space $C_0^\infty(M)$ with respect to the norm $\|u\|^2_{H_2^p(M)} = \int_M |\nabla_g^2 u|^p + |\nabla_g u|^p + |u|^p dv_g$. By Sobolev's embedding, we get that, for all $p > (n/2)$, $H_2^p(M, T^*M \otimes$

$T^*M) \subset C^{1-[n/p]}(M, T^*M \otimes T^*M)$, where $[n/p]$ denotes the entire part of n/p; then, the Christoffel symbols belong to $H_1^p(M)$, and the components of the Riemannian curvature tensor Rm_g, Ricci tensor Ric_g, and the scalar curvature S_g are in $L^p(M)$. Solving the singular Yamabe problem is equivalent to finding a positive solution $u \in H_2^p(M)$ of the equation

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = k|u|^{N-2}u, \qquad (4)$$

where k is a real constant, and in this case, the latter equation is the singular Yamabe equation. Under these assumptions on the metric g, the author in [11] proved the existence of a metric $\overline{g} = u^{N-2}g$ conformal to g such that $u \in H_2^p(M)$, $u > 0$, and the scalar curvature $S_{\overline{g}}$ of $\overline{g}$ is constant if (M, g) is not conformal to the round sphere. Moreover, we define the Yamabe invariant as follows:

$$\mu(M, g) = \inf_{u \in H_1^2(M), u \neq 0} E(u), \quad \text{where } E(u) = \frac{\int_M u P_g u \, dv_g}{\left(\int_M |u|^N dv_g\right)^{2/N}}. \qquad (5)$$

If $\mu(M, g) > 0$ and let $a = ((n-2)/4(n-1))S_g$, the singular Yamabe operator $P_g = \Delta_g + a$ is weakly conformally invariant, coercive, and invertible.

In [1] Azaiz et al. studied some singular second-order elliptic equations. They focused on the following equation:

$$\Delta_g u + au = \lambda |u|^{q-2}u + f|u|^{N-2}u, \qquad (6)$$

where $1 < q < 2$ and λ is a positive real parameter. In particular, in Theorem 1.1, under additional assumptions, they proved the existence of $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, the equation has a nontrivial weak solution. For details, see [1] and the references therein.

2. Notations and Preliminaries

In this section, we introduce some notations and materials necessary in our study. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$; we will work on the Sobolev space $H_1^2(M)$ which is the space of functions u such that $u, |\nabla_g u| \in L^2(M)$ and equipped with the usual norm

$$\|u\|_{H_1^2(M)} = \left(\int_M |\nabla_g u|^2 + u^2 dv_g\right)^{1/2}. \qquad (7)$$

By Sobolev's embedding (see [12]), $H_1^2(M) \subset L^q(M)$, where $1 < q \leq N$, and this embedding is compact when $q < N$.

The number $N = 2n/(n-2)$ is known as the critical exponent of the Sobolev embedding.

Let $K_0(n, 1)$ denote the best constant in Sobolev inequality that asserts that there exists a constant $B > 0$ such that, for any $u \in H_1^2(M)$,

$$\left(\int_M |u^N| dv_g\right)^{2/N} \leq K_0^2(n, 1) \int_M |\nabla_g u|^2 dv_g + B\|u\|_{L^2(M)}^2. \qquad (8)$$

Notice that

$$K_0^2(n, 1) = \frac{4}{n(n-2)\omega_n^{2/n}}, \qquad (9)$$

where ω_n is the volume of S^n. Denote by P_g the operator defined in the weak sense on $H_1^2(M)$ by

$$P_g(u) = \Delta_g u + au, \qquad (10)$$

P_g is an elliptic operator self-adjoint and is called coercive if there exits $C > 0$ such that, for any $u \in H_1^2(M)$,

$$\int_M u P_g(u) dv_g \geq C\|u\|_{H_1^2(M)}^2, \qquad (11)$$

where

$$\int_M u P_g(u) dv_g = \int_M \left(|\nabla_g u|^2 + au^2\right) dv_g. \qquad (12)$$

Let F be the functional defined on $H_1^2(M)$ by

$$F(u) = \int_M \left(|\nabla_g u|^2 + au^2\right) dv_g, \qquad (13)$$

and let E be the Sobolev quotient. Then, for any $u \in H_1^2(M) - \{0\}$,

$$E(u) = \frac{F(u)}{\left(\int_M f|u|^N dv_g\right)^{2/N}}. \qquad (14)$$

Traditionally, to obtain solutions of equation (1), we will use, when necessary, ideas developed in [9, 11, 18], and we will use classical variational techniques by minimizing the functional F. However, serious difficulties appear compared with the smooth case. In order, we define the quantity

$$\lambda(M, g) = \inf_{u \in A} F(u), \qquad (15)$$

where

$$A = \left\{u \in H_1^2(M) \text{ such that} \int_M f|u|^N dv_g = \left(1 + \|a\|_p\right)^{N/2}\right\}. \qquad (16)$$

Clearly, the functional E is well defined in $H_1^2(M)$ and is of class C^1, and the identity $\partial E(u) = 0$ being the equation [1], where $\partial E(u)$ is the differential of the functional E at u. So, for all $v \in H_1^2(M)$, we have

$$\partial E(u)v = \left.\frac{d}{dt}\right|_{t=0} E(u + tv). \qquad (17)$$

Note that if $f = 1$ and $a = ((n-2)/4(n-1))S_g$, it is easy to see that $\lambda(M, g)$ is not the Yamabe invariant. Indeed, $\lambda(M, g)$ is the infimum over the set A, and as above, the Yamabe invariant $\mu(M, g)$ satisfies

$$\mu(M, g) = \frac{\lambda(M, g)}{1 + \|a\|_p}. \qquad (18)$$

Throughout the paper, we will denote by $B(P,\delta)$ a geodesic ball of center P and of radius δ with $0<\delta<(r_g(M)/2)$, where $r_g(M)$ is the injectivity radius, and let η be a smooth function on M such that

$$\eta(x)=\begin{cases}1, & \text{on } B(P,\delta),\\ 0, & \text{on } M-B(P,2\delta).\end{cases} \tag{19}$$

We also let $S_g(P)$ be the scalar curvature of M at P. Now, we state our main results.

Theorem 1. *Let (M,g) be a Riemannian compact manifold of dimension $n\geq 5$. Assume that $a\in L^p(M)$, where $p>(n/2)$, f is a positive $C^\infty(M)$ function on M, and $P\in M$ such that $f(P)=sup_{x\in M}f(x)$. If*

$$\frac{\Delta f(P)}{f(P)}<\frac{S_g(P)}{3}\left(\frac{n+2}{n-4}-\|a\|_p-1\right)\left(\|a\|_p+1\right)^{-1}, \tag{20}$$

then

$$\lambda(M,g)<\frac{1+\|a\|_p}{\left(K_0^2(n,1)\right)\left(\sup_{x\in M}f(x)\right)^{2/N}}, \tag{21}$$

is satisfied, and (1) has a nontrivial positive weak solution such that $F(u)=\lambda(M,g)$ and $u\in A$. Moreover, $u\in C^{1-[n/p]}(M)$ and $u>0$.

Notice that this theorem is regarded as combined results between Theorem 3–Theorem 5.

Theorem 2. *Let (M,g) be a Riemannian compact manifold of dimension $n\geq 5$, $P\in M$ such that $f(P)=sup_{x\in M}f(x)$, a be smooth function, and r denote the distance function. If*

$$\begin{gathered}\lambda_2(M,g)K_0^2(n,1)(f(P))^{2/N}<\left(1+\left\|\frac{a}{r^2}\right\|_p\right)K^2(n,2,-2),\\ 1+a(P)K^2(n,2,-2)>0.\end{gathered} \tag{22}$$

Then, there exists $u\in H_1^2(M)$ nontrivial solution to the following equation:

$$\Delta_g u+\frac{a(x)}{r^2}u=f|u|^{N-2}u, \tag{23}$$

where r, $\lambda_2(M,g)$, and $K^2(n,2,-2)$ are given in Section 5.

Our paper is organized as follows: in Sections 1 and 2, we introduce some notations and preliminaries. In Section 3, we establish the existence and regularity result to equation (1). Section 4 is devoted to test functions which verify geometric assumptions and by the same way complete the proofs of our main theorems (Theorem 1 and Theorem 2). Section 5 deals with applications to particular equations which could arise from conformal geometry, and in Section 6, we consider the critical case $\alpha=2$.

The classical reference for conformal geometry is a survey by Lee and Parker [13].

3. Existence and Regularity of the Solution

In this section, we establish the existence and regularity result to equation (1). An elementary result we wish to briefly discuss here is the following.

Proposition 1. *If P_g is coercive, the following norm is equivalent to the usual norm on $H_1^2(M)$:*

$$\|u\|=\left(\int_M\left(\left|\nabla_g u\right|^2+au^2\right)dv_g\right)^{1/2}=\left(\int_M uP_g(u)dv_g\right)^{1/2}. \tag{24}$$

Proof. If P_g is coercive, one finds a constant $c>0$ such that, for any $u\in H_1^2(M)$,

$$\int_M uP_g(u)dv_g\geq c\|u\|^2_{H_1^2(M)}. \tag{25}$$

Since $a\in L^p(M)$, where $p>(n/2)$, and by the embedding $L^p(M)\subset L^{n/2}(M)$, it follows that there exists a constant $c_0>0$ such that $\|a\|_{n/2}\leq c_0\|a\|_p$; then, from Hölder's inequality, we get that

$$\begin{aligned}\int_M uP_g(u)dv_g&=\int_M\left(\left|\nabla_g u\right|^2+au^2\right)dv_g\\ &\leq\int_M\left|\nabla_g u\right|^2dv_g+\left(\int_M a^{n/2}dv_g\right)^{2/n}\left(\int_M u^{2n/(n-2)}dv_g\right)^{(n-2)/n}\\ &\leq\int_M\left|\nabla_g u\right|^2dv_g+\|a\|_{n/2}\|u\|_N^2\leq\int_M\left|\nabla_g u\right|^2dv_g+c_0\|a\|_p\|u\|_N^2.\end{aligned} \tag{26}$$

Now, from (8), we get that

$$\|u\|_N^2 \le \max\left(K_0^2(n,1), B\right)\|u\|_{H_1^2(M)}^2, \tag{27}$$

so we obtain

$$\begin{aligned}\int_M uP_g(u)\mathrm{d}v_g &\le \|u\|_{H_1^2(M)}^2 + c_0\|a\|_p \max\left(K_0^2(n,1),B\right)\|u\|_{H_1^2(M)}^2 \\ &\le \max\left(1, c_0\|a\|_p \max\left(K_0^2(n,1),B\right)\right)\|u\|_{H_1^2(M)}^2 \\ &\le c_1\|u\|_{H_1^2(M)}^2,\end{aligned} \tag{28}$$

where $c_1 = \max(1, c_0\|a\|_p \max(K_0^2(n,1),B)) > 0$. □

Theorem 3. *Assume that*

$$\lambda(M,g) < \frac{1+\|a\|_p}{\left(K_0^2(n,1)\right)\left(\sup_{x\in M} f(x)\right)^{2/N}}. \tag{29}$$

Then, (1) has a nontrivial positive weak solution such that $F(u) = \lambda(M,g)$ and $u \in A$.

In particular, if $f = 1$ and $a = ((n-2)/4(n-1))S_g$, condition (29) becomes the famous inequality

$$\mu(M,g) < \frac{1}{K_0^2(n,1)}. \tag{30}$$

Proof. First, we show that $\lambda(M,g)$ is finite. For any $u \in H_1^2(M)$ and by Hölder's inequality, one has

$$\begin{aligned}\int_M |a|u^2\mathrm{d}v_g &\le \left(\int_M |a|^{n/2}\mathrm{d}v_g\right)^{2/n}\left(\int_M u^{2n/(n-2)}\mathrm{d}v_g\right)^{(n-2)/n} \\ &\le \|a\|_{n/2}\|u\|_N^2 \\ &\le c_0\|a\|_p\|u\|_N^2,\end{aligned} \tag{31}$$

from

$$\int_M f\left|u^N\right|\mathrm{d}v_g = \left(1+\|a\|_p\right)^{N/2}, \tag{32}$$

we get

$$\frac{1+\|a\|_p}{\left(\inf_{x\in M} f(x)\right)^{2/N}} \ge \|u\|_N^2, \tag{33}$$

and then,

$$\int_M |a|u^2\mathrm{d}v_g \le c_0\|a\|_p \frac{1+\|a\|_p}{\left(\inf_{x\in M} f(x)\right)^{2/N}}. \tag{34}$$

Therefore,

$$\begin{aligned}F(u) &= \int_M\left(\left|\nabla_g u\right|^2 + au^2\right)\mathrm{d}v_g \ge \int_M au^2\mathrm{d}v_g \\ &\ge -c_0\|a\|_p \frac{1+\|a\|_p}{\left(\inf_{x\in M} f(x)\right)^{2/N}}.\end{aligned} \tag{35}$$

Consequently, $\lambda(M,g)$ is finite.

Now, let $(u_m)_m$ be a minimizing sequence in A of the functional F; the sequence u_m is such that

$$\begin{aligned}\lambda(M,g) &= \lim_{m\to+\infty}\int_M\left(\left|\nabla_g u_m\right|^2 + au_m^2\right)\mathrm{d}v_g, \\ \int_M f\left|u_m^N\right|\mathrm{d}v_g &= \left(1+\|a\|_p\right)^{N/2}.\end{aligned} \tag{36}$$

It is easy to see that $(|u_m|)_m$ is also a minimizing sequence. Hence, we can assume that $u_m \ge 0$; then, for m large enough, we get

$$\int_M\left(\left|\nabla_g u_m\right|^2 + au_m^2\right)\mathrm{d}v_g \le \lambda(M,g) + 1. \tag{37}$$

If P_g is coercive, by Proposition 1, $(u_m)_m$ is bounded. If not, we proceed as follows:

$$\begin{aligned}\int_M\left|\nabla_g u_m\right|^2\mathrm{d}v_g &\le \lambda(M,g) + 1 - \int_M au_m^2\mathrm{d}v_g \\ &\le \lambda(M,g) + 1 + \int_M |a|u_m^2 dv_g.\end{aligned} \tag{38}$$

By using (34), we still get

$$\int_M\left|\nabla_g u_m\right|^2\mathrm{d}v_g \le \lambda(M,g) + 1 + c_0\|a\|_p \frac{1+\|a\|_p}{\left(\inf_{x\in M} f(x)\right)^{N/2}}. \tag{39}$$

On the contrary, by the embedding $L^N(M) \subset L^2(M)$ and by inequality (33), we get that there exists $c > 0$ such that

$$\int_M u_m^2\mathrm{d}v_g \le c\left\|u_m\right\|_N^2 \le c\frac{1+\|a\|_p}{\left(\inf_{x\in M} f(x)\right)^{N/2}}. \tag{40}$$

This implies in turn that $(u_m)_m$ is bounded in $H_1^2(M)$, and after restriction to a subsequence still labeled $(u_m)_m$, we may assume that there exists $u \in H_1^2(M)$, $u \ge 0$, such that

(i) $u_m \longrightarrow u$ weakly in $H_1^2(M)$

(ii) $u_m \longrightarrow u$ strongly in $L^q(M)$ for all $q < N$ and almost everywhere on M

Furthermore, by Brezis–Lieb lemma applying to $(u_m)_m$, we get that

$$\begin{aligned}\int_M\left|\nabla_g(u_m)\right|^2\mathrm{d}v_g &= \int_M\left|\nabla_g(u)\right|^2\mathrm{d}v_g \\ &\quad + \int_M\left|\nabla_g(u_m-u)\right|^2\mathrm{d}v_g + o(1).\end{aligned} \tag{41}$$

Putting $\varphi_m = u_m - u$, then $\varphi_m \longrightarrow 0$ weakly in $H_1^2(M)$ and strongly in $L^q(M)$ for all $q < N$; therefore,

$$\begin{aligned}\int_M\left|\nabla_g u_m\right|^2\mathrm{d}v_g &= \int_M\left|\nabla_g\varphi_m\right|^2\mathrm{d}v_g + \int_M\left|\nabla_g u\right|^2\mathrm{d}v_g \\ &\quad + 2\int_M \nabla_g\varphi_m.\nabla_g u\mathrm{d}v_g.\end{aligned} \tag{42}$$

We deduce that

$$F(u_m) = F(u) + \left\|\nabla_g\varphi_m\right\|_2^2 + o(1). \tag{43}$$

By definition of $\lambda(M,g)$, we obtain $F(u) \ge \lambda(M,g)(\int_M f|u^N|dv_g)^{2/N}$, and by definition of the sequence $(u_m)_m$, we obtain $F(u_m) = \lambda(M,g) + o(1)$ which implies that

$$\lambda(M,g)\left(\int_M f\left|u^N\right|dv_g\right)^{2/N}+\left\|\nabla_g\varphi_m\right\|_2^2\leq\lambda(M,g)+o(1). \tag{44}$$

Again with Brezis–Lieb lemma applying to $(u_m)_m$, we get that

$$\begin{aligned}\left(1+\|a\|_p\right)^{N/2}&=\int_M f\left|u_m\right|^N \mathrm{d}v_g=\int_M f|u|^N\mathrm{d}v_g\\&\quad+\int_M f\left|\varphi_m\right|^N\mathrm{d}v_g+o(1).\end{aligned} \tag{45}$$

Then,

$$1+\|a\|_p\leq\left(\int_M f|u|^N\mathrm{d}v_g\right)^{2/N}+\left(\int_M f\left|\varphi_m\right|^N\mathrm{d}v_g\right)^{2/N}+o(1), \tag{46}$$

and inequality (44) will be written as

$$\left\|\nabla_g\varphi_m\right\|_2^2\leq\lambda(M,g)\left[1-\left(\int_M f\left|u^N\right|\mathrm{d}v_g\right)^{2/N}\right]+o(1), \tag{47}$$

and multiplying this inequality by $1+\|a\|_p$, then we get

$$\begin{aligned}\left(1+\|a\|_p\right)\left\|\nabla_g\varphi_m\right\|_2^2&\leq\lambda(M,g)\left[\left(1+\|a\|_p\right)-\left(1+\|a\|_p\right)\left(\int_M f\left|u^N\right|dv_g\right)^{2/N}\right]+o(1)\\&\leq\lambda(M,g)\left[\left(\int_M f|u|^N\mathrm{d}v_g\right)^{2/N}+\left(\int_M f\left|\varphi_m\right|^N\mathrm{d}v_g\right)^{2/N}-\left(1+\|a\|_p\right)\left(\int_M f\left|u^N\right|\mathrm{d}v_g\right)^{2/N}\right]+o(1)\\&\leq\lambda(M,g)\left[\int_M f\left|\varphi_m\right|^N\mathrm{d}v_g)^{2/N}-\|a\|_p\left(\int_M f\left|u^N\right|\mathrm{d}v_g\right)^{2/N}\right]+o(1)\\&\leq\lambda(M,g)\left(\int_M f\left|\varphi_m\right|^N\mathrm{d}v_g\right)^{2/N}+o(1).\end{aligned} \tag{48}$$

Using Sobolev's inequality, we still get

$$\begin{aligned}\left(1+\|a\|_p\right)\left\|\nabla_g\varphi_m\right\|_2^2&\leq\lambda(M,g)\left(\sup_{x\in M}f(x)\right)^{2/N}\\&\quad\cdot K_0^2(n,1)\left\|\nabla_g\varphi_m\right\|_2^2+o(1).\end{aligned} \tag{49}$$

Thus,

$$\left(1+\|a\|_p-\lambda(M,g)\left(\sup_{x\in M}f(x)\right)^{2/N}K_0^2(n,1)\right)\left\|\nabla_g\varphi_m\right\|_2^2\leq o(1). \tag{50}$$

Now, if we assume

$$\lambda(M,g)<\frac{1+\|a\|_p}{\left(K_0^2(n,1)\right)\left(\sup_{x\in M}f(x)\right)^{2/N}}, \tag{51}$$

we find

$$\left\|\nabla_g\varphi_m\right\|_2^2=o(1). \tag{52}$$

Hence, φ_m converges strongly to 0 in $H_1^2(M)$, and then u_m converges strongly to u in $H_1^2(M)$ and in $L^N(M)$. It follows that

$$\lim\int_M f\left|u_m-u\right|^N\mathrm{d}v_g=0, \tag{53}$$

which leads to

$$\int_M f\left|u^N\right|\mathrm{d}v_g=\left(1+\|a\|_p\right)^{N/2}, \tag{54}$$

and we can conclude that $u\in A$, and u is a nontrivial positive weak solution of (1).

Concerning the regularity of solutions of equations (1), Madani in [11] proved through the classical techniques a regularity result with f a constant function and $a=((n-2)/4(n-1)S_g$. By following the same procedure, though the presence of the nonconstant function f adds further technical difficulties, we can prove the regularity of solutions of equations (1). This result is formulated in the following theorem. □

Theorem 4. *Let (M,g) be a Riemannian compact manifold of dimension $n\geq 3$, and let $a\in L^p(M)$, where $p>(n/2)$, and f be a positive $C^\infty(M)$ function on M. If $u\in H_1^2(M)$ is a nontrivial positive weak solution of*

$$\Delta_g u+au=fu^{N-1}, \tag{55}$$

then $u\in H_2^p(M)\subset C^{1-[n/p]}(M)$ and $u>0$.

Proof. The proof of this theorem is reduced to show that $u\in L^{N+\epsilon}$ for some $\epsilon>0$. Indeed, u verifies the equation

$$\Delta_g u+\left(a-fu^{N-2}\right)u=0, \tag{56}$$

and if $u\in L^{N+\epsilon}$, it follows that $a-fu^{N-2}\in L^r(M)$, where $r=\min(p,(n/2)+\epsilon)>(n/2)$ (see [10] for some details); hence, one has $\Delta_g u\in L^p(M)$, and by the regularity theorem, we deduce that $u\in H_2^p(M)$. Let $l>0$ be a real number and H,F be two continuous functions on $\mathbb{R}^+$ given by

$$H(x)=\begin{cases}t^{\gamma}, & \text{if } 0\le t\le l,\\ l^{q-1}\left(ql^{q-1}t-(q-1)l^{q}\right), & \text{if } t>l,\end{cases}$$
$$F(x)=\begin{cases}t^{q}, & \text{if } 0\le t\le l,\\ ql^{q-1}t-(q-1)l^{q}, & \text{if } t>l,\end{cases} \tag{57}$$

where $\gamma=2q-1$ and $1<q<(n(p-1)/n(p-2))$. Since $u\ge 0$ and $u\in H_1^2(M)$, then it follows that $H^{\circ}u$ and $F^{\circ}u$ are both in $H_1^2(M)$,

$$\begin{aligned}qH(t)&=F(t)F'(t),\\ \left(F'(t)\right)^2&\le qH'(t),\\ F^2(t)&\ge tH(t).\end{aligned} \tag{58}$$

Let u be a weak solution of (55); then, for all $v\in H_1^2(M)$, one has

$$\int_M \nabla_g u\nabla_g v\,dv_g+\int_M auv\,dv_g=\int_M fu^{N-1}v\,dv_g. \tag{59}$$

Now, as in Section 2, we define a cutoff function $\eta\in C^1(M)$ such that

$$\eta(x)=\begin{cases}1, & \text{on } B(P,\delta),\\ 0, & \text{on } M-B(P,2\delta).\end{cases} \tag{60}$$

Chosen $v=\eta^2H^{\circ}u$ and plugging this function in (59), we get

$$\begin{aligned}&\int_M \eta^2H'\circ u\left|\nabla_g u\right|^2dv_g+2\int_M \eta H\circ u\nabla_g u\nabla_g\eta\,dv_g\\ &=\int_M fu^{N-1}\eta^2H\circ u\,dv_g-\int_M au\eta^2H\circ u\,dv_g.\end{aligned} \tag{61}$$

We put $h=F\circ u$. Now, let us evaluate each of the above integrals by using h and (58). We have $\nabla_g h=F'\circ u\nabla_g u$; thus, by applying the second relationship of (58), this implies

$$\left|\nabla_g h\right|^2=\left(F'\circ u\right)^2\left|\nabla_g u\right|^2\le qH'\circ u\left|\nabla_g u\right|^2. \tag{62}$$

We deduce that the first integral of (61) is bounded; then,

$$\frac{1}{q}\left\|\eta\nabla_g h\right\|_2^2\le\int_M \eta^2H'\circ u\left|\nabla_g u\right|^2dv_g. \tag{63}$$

The first relationship of (58) and the Cauchy–Schwarz inequality imply that the second integral of (61) is bounded; hence,

$$\begin{aligned}2\int_M \eta H\circ u\nabla_g u\nabla_g\eta\,dv_g&=\frac{2}{q}\int_M \eta h\nabla_g h\nabla_g\eta\,dv_g\\ &\ge\frac{-2}{q}\left\|h\nabla_g\eta\right\|_2\left\|\eta\nabla_g h\right\|_2.\end{aligned} \tag{64}$$

By using the latter relationship of (58), we obtain $uH^{\circ}u\le h^2$. In the same vain, the two integrals of the right-hand side member in (61) are bounded; thus,

$$\begin{aligned}&\left|\int_M fu^{N-1}\eta^2H\circ u\,dv_g-\int_M au\eta^2H\circ u\,dv_g\right|\\ &\le\left(\sup_{x\in M}f(x)\right)\|u\|_{N,2\delta}^{4/(n-2)}\|\eta h\|_N^2+\|a\|_p\|\eta h\|_{2p/(p-1)}^2,\end{aligned} \tag{65}$$

where $\|u\|_{N,r}^N=\int_{B(P,r)}u^N dv_g$. If we group these estimates together, equality (61) becomes

$$\begin{aligned}&\left\|\eta\nabla_g h\right\|_2^2-2\left\|h\nabla_g\eta\right\|_2\left\|\eta\nabla_g\overline{f}\right\|_2\\ &\le q\left[\left(\sup_{x\in M}f(x)\right)\|u\|_{N,2\delta}^{4/(n-2)}\|\eta h\|_N^2+\|a\|_p\|\eta h\|_{2p/(p-1)}^2\right].\end{aligned} \tag{66}$$

Now, let a_1, b_1, c_1, and d_1 be four real numbers; if $a_1^2-2a_1b\le c_1^2+d_1^2$, we easily obtain that $a_1\le 2b_1+c_1+d_1$. Then, (66) becomes

$$\begin{aligned}\left\|\eta\nabla_g h\right\|_2&\le\sqrt{q\sup_{x\in M}f(x)}\|u\|_{N,2\delta}^{2/(n-2)}\|\eta h\|_N\\ &\quad+\sqrt{q\|a\|_p}\|\eta h\|_{2p/(p-1)}+2\left\|h\nabla_g\eta\right\|_2.\end{aligned} \tag{67}$$

By Sobolev's embedding, we then get that there exists a constant $c>0$ depending only on n such that

$$\|\eta h\|_N\le c\left(\left\|\eta\nabla_g h\right\|_2+\left\|h\nabla_g\eta\right\|_2+\|h\eta\|_2\right). \tag{68}$$

Since $q<N$ and after using (67), we obtain

$$\begin{aligned}&\left(1-c\sqrt{N\sup_{x\in M}f(x)}\|u\|_{N,2\delta}^{2/(n-2)}\right)\|\eta h\|_N\\ &\le c\left(\sqrt{N\|a\|_p}\|\eta h\|_{2p/(p-1)}+3\left\|h\nabla_g\eta\right\|_2+\|h\eta\|_2\right).\end{aligned} \tag{69}$$

For δ sufficiently small, one has

$$\|u\|_{N,2\delta}^{2/(n-2)}\le\frac{1}{2c\sqrt{N\sup_{x\in M}f(x)}}. \tag{70}$$

When l goes to $+\infty$, we then get that there exists a constant $C>0$ depending only on $n,\delta,\|\eta\|_\infty,\|\nabla_g\eta\|_\infty\|a\|_p$, and f such that

$$\left\|u^q\right\|_{N,2\delta}\le C\left(\left\|u^q\right\|_2+\left\|u^q\right\|_{2p/(p-1)}\right). \tag{71}$$

Now, from the boundedness of u in $L^N(M)$ and as $(2p/(p-1))q<N$, we still get

$$\left\|u^q\right\|_{qN,2\delta}\le C. \tag{72}$$

Since M is compact, it can be covered by a finite number of balls $\{B(P_i,\delta)\}_{i\in I}$, and let $(\eta_i)_{i\in I}$ be a partition of unity subordinated to the covering; then,

$$\|u\|_{qN}^{qN}=\sum_{i\in I}\left\|\eta_i u\right\|_{qN,\delta_i}^{qN}\le C. \tag{73}$$

It follows that $u\in L^{qN}(M)$ with $qN>N$. □

4. Test Functions

The purpose of this section is to find conditions such that (29) will be true. Consider a normal geodesic coordinate system centered at a point P. Denote by $S(r)$ the geodesic sphere centered at P and of radius r with $r<r_g(M)$, where $r_g(M)$ is the injectivity radius. Let $d\Omega$ be the volume

element of the $n-1$-dimensional Euclidean unit sphere S^{n-1}, and put

$$G(r) = \frac{1}{\omega_{n-1}} \int_{S(r)} \sqrt{|g|} d\Omega, \quad (74)$$

where ω_{n-1} is the volume of S^{n-1} and $|g|$ is the determinant of the Riemannian metric g. The formula of Taylor's expansion of $G(r)$ in a neighborhood of P is given by

$$G(r) = 1 - \frac{S_g(P)}{6n} r^2 + o(r^2), \quad (75)$$

where $S_g(P)$ is the scalar curvature of M at P. As in Section 2, let η be a smooth function on M such that

$$\eta(x) = \begin{cases} 1, & \text{on } B(P,\delta), \\ 0, & \text{on } M - B(P,2\delta). \end{cases} \quad (76)$$

For $\epsilon > 0$, we define the radial function u_ϵ as follows:

$$u_\epsilon = \eta(r)(r^2 + \epsilon^2)^{-((n-2)/2)}, \quad (77)$$

where $r = d(P,x)$ is the distance from P to x and $f(P) = \max_{x\in M} f(x)$. For further computations, we need the following integrals; then, for any real positive numbers p, q such that $p - q > 1$, we put

$$I_p^q = \int_0^{+\infty} (1+t)^{-p} t^q dt. \quad (78)$$

Furthermore, it can be easily seen that

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q, \qquad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q. \quad (79)$$

Theorem 5. *Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. Assume that $a \in L^p(M)$, where $p > (n/2)$, f is a positive $C^\infty(M)$ function on M, and $P \in M$ such that $f(P) = sup_{x\in M} f(x)$. If*

$$\frac{\Delta f(P)}{f(P)} < \frac{S_g(P)}{3} \left(\frac{n+2}{n-4} - \|a\|_p - 1 \right) \left(\|a\|_p + 1 \right)^{-1}, \quad (80)$$

then, (29) is true.

Proof. To proof this theorem, it suffices to show that

$$E(u_\varepsilon) < \frac{1 + \|a\|_p}{(K_0^2(n,1)) (\sup_{x\in M} f(x))^{2/N}}. \quad (81)$$

The aim of the following is to compute expansions of these integrals:

$$J_1 = \int_M f |u_\varepsilon|^N dv_g, \qquad J_2 = \int_M a u_\varepsilon^2 dv_g, \qquad J_3 = \int_M |\nabla_g u_\varepsilon|^2 dv_g, \quad (82)$$

on the geodesic ball $B(P,\delta)$. To compute the first term, we need the following limited development of f at P:

$$f(x) = f(P) + \frac{1}{2} \nabla_{i,j} f(P) y^i y^j + o(r^2). \quad (83)$$

We have

$$J_1 = \int_M f |u_\varepsilon|^N dv_g = \int_0^\delta |u_\varepsilon|^N \left(\int_{S(r)} f \sqrt{|g|}\, d\Omega \right) r^{n-1} dr, \quad (84)$$

where

$$\begin{aligned} \int_{S(r)} f \sqrt{|g|} d\Omega &= \int_{S(r)} \left(f(P) + \frac{1}{2} \nabla_{i,j} f(P) y^i y^j \right) \\ &\quad \cdot \left(1 - \frac{1}{6} R_{i,j} \right) y^i y^j \Big) d\Omega + o(r^2) \\ &= \omega_{n-1} \left(f(P) - \left(\frac{\Delta f(P)}{2n} + \frac{f(P) S_g(P)}{6n} \right) r^2 + o(r^2) \right). \end{aligned} \quad (85)$$

Put

$$L = \frac{\Delta f(P)}{2n} + \frac{f(P) S_g(P)}{6n}. \quad (86)$$

Then,

$$\begin{aligned} J_1 &= \omega_{n-1} \int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^n} (f(P) - L r^2) dr + o(r^2) \\ &= \omega_{n-1} \left(f(\mathrm{P}) \int_0^\delta \frac{r^{n-1}}{(r^2+\epsilon^2)^n} dr - L \int_0^\delta \frac{r^{n+1}}{(r^2+\epsilon^2)^n} dr \right) + o(r^{n+1}). \end{aligned} \quad (87)$$

Now, we set

$$t = \frac{r^2}{\epsilon^2}, \qquad dr = \frac{\epsilon dt}{2dt}, \qquad r = \epsilon \sqrt{t}. \quad (88)$$

By changing the variable as above, it follows that

$$\begin{aligned} J_1 &= \omega_{n-1} \left(\mathrm{f(P)} \int_0^{(\delta/\epsilon)^2} \frac{t^{n/2}}{2\epsilon^{n-2}(1+t)^n} dt \right. \\ &\quad \left. - L \int_0^{(\delta/\epsilon)^2} \frac{t^{(n/2)}}{2\epsilon^{n-2}(1+t)^n} dt \right) + o(\epsilon^{n+1}) \\ &= \frac{\omega_{n-1}}{2\epsilon^n} \left(f(P) I_n^{(n/2)-1} - L\epsilon^2 I_n^{(n/2)} \right) + o(\epsilon^2). \end{aligned} \quad (89)$$

From

$$I_n^{n/2} = \frac{n}{n-2} I_n^{(n/2)-1}, \qquad \omega_n = 2^{n-1} \omega_{n-1} I_n^{(n/2)-1}, \quad (90)$$

we get that

$$J_1 = \frac{\omega_{n-1}}{2\epsilon^n} I_n^{(n/2)-1}\left(f(P) - \left(\frac{\Delta f(P)}{2(n-2)} + \frac{f(P)S_g(P)}{6(n-2)} \right)\epsilon^2 \right) + o\left(\epsilon^2\right)$$
$$= \frac{\omega_{n-1}}{2\epsilon^n} I_n^{(n/2)-1} f(P)\left(1 - \left(\frac{\Delta f(P)}{2(n-2)f(P)} + \frac{S_g(P)}{6(n-2)} \right)\epsilon^2 \right) + o\left(\epsilon^2\right). \quad (91)$$

Therefore,

$$J_1^{-(2/N)} = J_1^{-((n-2)/n)}$$
$$= \left(\frac{\omega_{n-1}}{2\epsilon^n} I_n^{(n/2)-1} f(P) \right)^{-((n-2)/n)} \left(1 + \frac{n-2}{n}\left(\frac{\Delta f(P)}{2(n-2)f(P)} + \frac{S_g(P)}{6(n-2)} \right)\epsilon^2 \right) + o\left(\epsilon^2\right)$$
$$= \frac{2^{((n-2)/n)}\epsilon^{n-2}}{\left(\omega_{n-1} I_n^{(n-2)-1} f(P)\right)^{((n-2)/n)}} \left(1 + \left(\frac{\Delta f(P)}{2nf(P)} + \frac{S_g(P)}{6n} \right)\epsilon^2 \right) + o\left(\epsilon^2\right). \quad (92)$$

Let us compute the third integral. First, we have

$$\left|\nabla_g u_\varepsilon\right| = \left|\frac{\partial u_\varepsilon}{\partial r}\right| = (n-2)\frac{r}{\left(r^2+\epsilon^2\right)^{n/2}}. \quad (93)$$

Then, in a similar way, we get

$$J_3 = \omega_{n-1}\int_0^\delta \frac{(n-2)^2 r^2}{\left(r^2+\epsilon^2\right)^n}\left(1 - \frac{S_g(P)}{6n}r^2 + o\left(r^2\right) \right) r^{n-1}\mathrm{d}r$$
$$= \frac{(n-2)^2\omega_{n-1}}{\epsilon^{n-2}}\int_0^{(\delta/\epsilon)^2} \frac{t^{n/2}\mathrm{d}t}{2(1+t)^n} - \int_0^{(\delta/\epsilon)^2} \frac{S_g(P)\epsilon^2 t^{(n/2)+1}\mathrm{d}t}{12n(1+t)^n} + o\left(\epsilon^2\right)$$
$$= \frac{(n-2)^2\omega_{n-1}}{\epsilon^{n-2}}\left(\frac{n}{2(n-2)} I_n^{(n/2)-1} - \frac{S_g(P)\epsilon^2 n(n+2)}{12n(n-4)(n-2)} I_n^{(n/2)-1} + o\left(\epsilon^2\right) \right)$$
$$= \frac{(n-2)^2}{\epsilon^{n-2}}\omega_{n-1} I_n^{(n/2)-1}\left(\frac{n}{2(n-2)} - \frac{S_g(P)\epsilon^2 n(n+2)}{12n(n-4)(n-2)} + o\left(\epsilon^2\right) \right)$$
$$= \frac{(n-2)}{\epsilon^{n-2}}\omega_{n-1} I_n^{(n/2)-1}\left(\frac{n}{2} - \frac{S_g(P)\epsilon^2 n(n+2)}{12n(n-4)} + o\left(\epsilon^2\right) \right). \quad (94)$$

That is,

$$J_3 = \frac{(n-2)}{\epsilon^{n-2}}\omega_{n-1} I_n^{(n/2)-1}\frac{n}{2}\left(1 - \frac{S_g(P)\epsilon^2 (n+2)}{6n(n-4)} + o\left(\epsilon^2\right) \right). \quad (95)$$

Now, compute the second integral J_2. By using Hölder's inequality, we get

$$J_2 = \int_M a u_\varepsilon^2 \mathrm{d}v_g$$
$$\leq \left(\int_M a^p \mathrm{d}v_g \right)^{1/p} \left(\int_M u^{2p/(p-1)} \mathrm{d}v_g \right)^{(p-1)/p}$$
$$\leq \|a\|_p \left\|u_\varepsilon\right\|_{2p/(p-1)}^2. \quad (96)$$

Then, a direct computation shows that

$$\begin{aligned}
\|u_\varepsilon\|^2_{2p/(p-1)} &= \left(\int_M u_\varepsilon^{2p/(p-1)} \mathrm{d}v_g\right)^{(p-1)/p} \\
&= (\omega_{n-1})^{(p-1)/p}\left(\int_0^\delta\left(\frac{r^{n-1}}{(r^2+\epsilon^2)^{(n-2)p/(p-1)}} - \frac{S_g(P)}{6n}\frac{r^{n+1}}{(r^2+\epsilon^2)^{(n-2)p/(p-1)}} + o\left(r^{n+1}\right)\right)\mathrm{d}r\right)^{(p-1)/p} \\
&= \left(\frac{1}{2}\right)^{(p-1)/p}(\omega_{n-1})^{(p-1)/p}\epsilon^{-n+2+2-(n/p)}\left(I^{(n/2)-1}_{(n-2)p/(p-1)} - \frac{S_g(P)}{3n}\epsilon^2 I^{(n/2)}_{(n-2)p/(p-1)} + o\left(\epsilon^2\right)\right)^{(p-1)/p} \\
&= \left(\frac{1}{2}\right)^{(p-1)/p}(\omega_{n-1})^{(p-1)/p}\epsilon^{-n+2+2-(n/p)}\left(I^{(n/2)-1}_{(n-2)p/(p-1)} - \frac{S_g(P)(n+2)(p-1)}{3n(pn-8p+4-n)}\epsilon^2 I^{(n/2)-1}_{(n-2)p/(p-1)} + o\left(\epsilon^2\right)\right)^{(p-1)/p} \\
&= \left(\frac{1}{2}\right)^{(p-1)/p}(\omega_{n-1})^{(p-1)/p}\epsilon^{-n+2+2-(n/p)}\left(I^{(n/2)-1}_{(n-2)p/(p-1)}\right)^{(p-1)/p}\left(1 - \frac{S_g(P)(n+2)(p-1)}{3n(pn-8p+4-n)}\epsilon^2 + o\left(\epsilon^2\right)\right)^{(p-1)/p} \\
&= \left(\frac{1}{2}\right)^{(p-1)/p}(\omega_{n-1})^{(p-1)/p}\epsilon^{-n+2+2-(n/p)}\left(I^{(n/2)-1}_{(n-2)p/(p-1)}\right)^{(p-1)/p}\left(1 - \beta\epsilon^2 + o\left(\epsilon^2\right)\right),
\end{aligned} \tag{97}$$

where

$$\beta = \frac{S_g(P)(n+2)(p-1)^2}{3np(pn-8p+4-n)}. \tag{98}$$

It follows that

$$\begin{aligned}
\|u_\varepsilon\|^2_{2p/(p-1)} &= \left(\frac{1}{2}\right)^{(p-1)/p}(\omega_{n-1})^{(p-1)/p}\epsilon^{-n+2+2-(n/p)}\left(I^{(n/2)-1}_{(n-2)p/(p-1)}\right)^{(p-1)/p}\left(1 - \beta\epsilon^2 + o\left(\epsilon^2\right)\right), \\
\int_M a u_\varepsilon^2 \mathrm{d}v_g &\le \left(\frac{1}{2}\right)^{p-1/p}(\omega_{n-1})^{p-1/p}\epsilon^{-n+2+2-(n/p)}\|a\|_p\left(I^{(n/2)-1}_{(n-2)p/p-1}\right)^{p-1/p}\left(1 - \beta\epsilon^2 + o\left(\epsilon^2\right)\right).
\end{aligned} \tag{99}$$

Independently, we can easily show that, for $\varphi = |\nabla_g u_\varepsilon|^2$ or $\varphi = f|u_\varepsilon|^N$ or $\varphi = a u_\varepsilon^2$, we get

$$\int_{B(P,2\delta)-B(P,\delta)} \varphi \mathrm{d}v_g \longrightarrow 0. \tag{100}$$

Let us derive estimates for $E(u_\varepsilon)$. Using expansions of J_1, J_2, and J_3, we get that

$$E(u_\varepsilon) = \left(\int_M \left|\nabla_g u_\varepsilon\right|^2 + a u_\varepsilon^2 \mathrm{d}v_g\right)\left(\int_M f\left|u_\varepsilon\right|^N \mathrm{d}v_g\right)^{-(2/N)}, \tag{101}$$

which yields

$$\begin{aligned}
E(u_\varepsilon) \le &\left[\left(\frac{1}{2}\right)^{(p-1)/p}(\omega_{n-1})^{(p-1)/p}\epsilon^{-n+2+2-(n/p)}\|a\|_p\left(I^{(n/2)-1}_{(n-2)p/p-1}\right)^{(p-1)/p}\left(1 - \beta\epsilon^2 + o\left(\epsilon^2\right)\right)\right. \\
&\left. + \frac{(n-2)}{\epsilon^{n-2}}\omega_{n-1} I_n^{(n/2)-1}\frac{n}{2}\left(1 - \frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)} + o\left(\epsilon^2\right)\right)\right] \\
&\times\left[\frac{2^{n-2/n}\epsilon^{n-2}}{\left(\omega_{n-1} I_n^{(n/2)-1} f(P)\right)^{n-2/n}}\left(1 + \left(\frac{\Delta f(P)}{2nf(P)} + \frac{S_g(P)}{6n}\right)\epsilon^2\right) + o\left(\epsilon^2\right)\right].
\end{aligned} \tag{102}$$

Next,

$$E(u_\varepsilon) \le \left[\left(\frac{1}{2}\right)^{(p-1)/p} (\omega_{n-1})^{(p-1)/p} \epsilon^{2-(n/p)} \|a\|_p \left(I_{(n-2)p/p-1}^{(n/2)-1}\right)^{(p-1)/p} \left(1 - \beta\epsilon^2 + o\left(\epsilon^2\right)\right) + (n-2)\omega_{n-1} I_n^{(n/2)-1} \frac{n}{2} \left(1 - \frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)} + o\left(\epsilon^2\right)\right) \right] \times \left[\frac{2^{n-2/n}}{\left(\omega_{n-1} I_n^{(n/2)-1} f(P)\right)^{n-2/n}} \left(1 + \left(\frac{\Delta f(P)}{2nf(P)} + \frac{S_g(P)}{6n}\right)\epsilon^2\right) + o\left(\epsilon^2\right) \right]. \tag{103}$$

Let ϵ be sufficiently small such that

$$\left(\frac{1}{2}\right)^{(p-1)/p} (\omega_{n-1})^{(p-1)/p} \epsilon^{2-(n/p)} \|a\|_p \left(I_{(n-2)p/p-1}^{(n/2)-1}\right)^{(p-1)/p} \cdot \left(\frac{2^{(n-2)/n}}{\left(\omega_{n-1} I_n^{(n/2)-1} f(P)\right)^{(n-2)/n}} \right) \le \|a\|_p A, \tag{104}$$

where $A = (n-2)n((\omega_{n-1} I_n^{(n/2)-1})/2)^{2/n} (f(P))^{-(2/N)}$, and as $2 - (n/p) > 0$,

$$\epsilon^{2-(n/p)} \beta\epsilon^2 = o\left(\epsilon^2\right). \tag{105}$$

Then, we get that

$$E(u_\varepsilon) \le \left[\|a\|_p A + o\left(\epsilon^2\right) + (n-2)\omega_{n-1} I_n^{(n/2)-1} \frac{n}{2} \left(\frac{2^{(n-2/n)}}{\left(\omega_{n-1} I_n^{(n/2)-1} f(P)\right)^{(n-2/n)}} \right) \left(1 - \frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)} + o\left(\epsilon^2\right)\right) \right] \times \left[\left(1 + \left(\frac{\Delta f(P)}{2nf(P)} + \frac{S_g(P)}{6n}\right)\epsilon^2\right) + o\left(\epsilon^2\right) \right]. \tag{106}$$

Therefore,

$$E(u_\varepsilon) \le \left[\|a\|_p A + o\left(\epsilon^2\right) + (n-2)n \left(\frac{\omega_{n-1} I_n^{(n/2)-1}}{2} \right)^{2/n} (f(P))^{-(2/N)} \left(1 - \frac{S_g(P)\epsilon^2(n+2)}{6n(n-4)} + o\left(\epsilon^2\right)\right) \right] \times \left[\left(1 + \left(\frac{\Delta f(P)}{2nf(P)} + \frac{S_g(P)}{6n}\right)\epsilon^2\right) + o\left(\epsilon^2\right) \right]. \tag{107}$$

Put

$$C = \frac{S_g(P)(n+2)}{6n(n-4)}, \qquad D = \frac{\Delta f(P)}{2nf(P)} + \frac{S_g(P)}{6n}, \tag{108}$$

i.e.,

$$E(u_\varepsilon) \le \left[\|a\|_p A + o\left(\epsilon^2\right) + A - AC\epsilon^2 + o\left(\epsilon^2\right) \right] \times \left[1 + D\epsilon^2 + o\left(\epsilon^2\right) \right]. \tag{109}$$

Direct calculation gives

$$E(u_\varepsilon) \le A\left(\|a\|_p + 1\right) \left[1 + \left(\|a\|_p D + D - C\right) \frac{\epsilon^2}{\|a\|_p + 1} \right] + o\left(\epsilon^2\right). \tag{110}$$

Knowing that

$$(n-2)n \left(\frac{\omega_{n-1} I_n^{(n/2)-1}}{2} \right)^{2/n} = K_0^{-2}(n, 1), \tag{111}$$

we obtain

$$E(u_\varepsilon) \le K_0^{-2}(n,1)(f(P))^{-(2/N)}\left(\|a\|_p + 1\right) \cdot \left[1 + \left(\|a\|_p D + D - C\right)\frac{\epsilon^2}{\|a\|_p + 1}\right] + o\left(\epsilon^2\right). \quad (112)$$

To ensure assumption (29),

$$\lambda(M,g) < \frac{1 + \|a\|_p}{\left(K_0^2(n,1)\right)\left(\sup_{x\in M} f(x)\right)^{2/N}}, \quad (113)$$

we must take

$$\|a\|_p D + D - C < 0. \quad (114)$$

Then, we get that

$$E(u_\varepsilon) < \left(1 + \|a\|_p\right)\left(K_0^{-2}(n,1)\right)(f(P))^{-(2/N)}. \quad (115)$$

It follows that (114) means that

$$\frac{\Delta f(P)}{f(P)} < \frac{S_g(P)}{3}\left(\frac{n+2}{n-4} - \|a\|_p - 1\right)\left(\|a\|_p + 1\right)^{-1}. \quad (116)$$

□

5. Application

Let $P \in M$; we define a function on M by

$$r_p(q) = \begin{cases} d(p,q), & \text{if } d(p,q) < \delta(M), \\ \delta(M), & \text{if } d(p,q) \ge \delta(M), \end{cases} \quad (117)$$

where $\delta(M)$ is the injectivity radius of M. For brevity, we denote this function by r. We define the weighted $L_p(M, r^\gamma)$ space as the set of measurable functions u on M such that $r^\gamma |u|^p$ are integrable, where $p \ge 1$. We endow $L^p(M, r^\gamma)$ with the norm

$$\|u\|_{p,r^\gamma}^p = \left(\int_M r^\gamma |u|^p dv_g\right). \quad (118)$$

In this section, we need the Hardy–Sobolev inequality and the Rellich–Kondrakov embedding whose proofs are given in [11].

Theorem 6 (Hardy inequality). *For any function in $u \in C_0^\infty(M)$, there exists a constant $c > 0$ such that*

$$\left\||x|^\gamma u\right\|_p \le c\left\||x|^\beta \nabla u\right\|_q, \quad (119)$$

where $1 \le q \le p \le (qn/(n - lq))$, $\gamma = \beta - l + n((1/q) - (1/p)) > -(n/p)$ and $n > lq$.

This type of inequality in one dimension was introduced by Hardy and generalized for all dimensions. For more details, see the book of V. G. Maz'ja, where we can find the proof of this theorem. In our case, we are interested when $\beta = 0$ and $l = 1$. The following theorems were proved by Madani in [11].

Theorem 7. *Let (M,g) be a Riemannian compact manifold of dimension n and p, q and γ be real numbers such that $((\gamma + n)/p) = -1 + (n/q) > 0$ and $1 \le q \le p \le (qn/(n-q))$. For any $\epsilon > 0$, there exists $A(\epsilon, q, \gamma)$ such that, for any $u \in H_1^q(M)$,*

$$\|u\|_{p,r^\gamma} \le (K(n,q,\gamma) + \epsilon)\left\|\nabla_g u\right\|_q + A(\epsilon,q,\gamma)\|u\|_q, \quad (120)$$

where the number $K(n,q,\gamma) = c$ is the best constant in Hardy inequality.

In particular, $K(n,q,0) = K(n,q)$ is the best constant in Sobolev' embedding $u \in H_1^q(M) \subset L^p(M)$.

Moreover, in the case of $q = 2$, we obtain

$$\|u\|_{p,r^\gamma}^2 \le \left(K^2(n,2,\gamma) + \epsilon\right)\left\|\nabla_g u\right\|_2^2 + A(\epsilon,2,\gamma)\|u\|_2^2, \quad (121)$$

where $(\gamma + n/p) = -1 + (n/2) > 0$ and $2 \le p \le (2n/n-2)$.

Lemma 1. *For all function $u \in H_1^q(M)$ with compact support included in the ball $B(P,\delta)$, we get*

$$\|u\|_{p,r^\gamma} \le (K(n,q,\gamma) + \epsilon)\left\|\nabla_g u\right\|_q, \quad (122)$$

and when δ tends to 0, $K_\delta(n,q,\gamma)$ goes to $K(n,q,\gamma)$.

Theorem 8. *Let (M,g) be a Riemannian compact manifold of dimension n.*

(1) If $((\gamma + n)/p) = -1 + (n/q) > 0$ and $1 \le q \le p$, then the embedding $u \in H_1^q(M) \subset L^p(M, r^\gamma)$ is continuous

(2) If $(\gamma + n/p) = -1 + (n/q) > 0$, $\gamma \le 0$ and $q \le p$, then this embedding is compact

We consider the following second-order equation:

$$\Delta_g u + \frac{a(x)}{r^\alpha}u = f|u|^{N-2}u, \quad (123)$$

where a is the smooth function, r denotes the distance function defined as above, and α will be precise later. Let

$$F_\alpha(u) = \int_M\left(\left|\nabla_g u\right|^2 + \frac{a(x)}{r^\alpha}u^2\right)dv_g, \quad (124)$$

be the energy functional, and consider the Sobolev quotient: for any $u \in H_1^2(M) - \{0\}$,

$$E_\alpha(u) = \frac{F_\alpha(u)}{\left(\int_M f|u^N|dv_g\right)^{2/N}}. \quad (125)$$

Let

$$\lambda_\alpha(M,g) = \inf_{u\in A} F_\alpha(u), \quad (126)$$

where

$$A = \left\{u \in H_1^2(M) \text{ such that} \int_M f\left|u^N\right|dv_g = \left(1 + \left\|\frac{a}{r^\alpha}\right\|_p\right)^{N/2}\right\}. \quad (127)$$

Proposition 2. *If $0 < \alpha < (n/p) < 2$ and*

$$\lambda_\alpha(M,g) < \left(1 + \|a\|_p\right)\left(K_0^{-2}(n,1)\right)(f(P))^{-(2/N)}, \quad (128)$$

then equation (123) has a positive solution $u_\alpha \in C^{1-[n/p]}(M)$ such that

$$F_\alpha(u_\alpha) = \lambda_\alpha(M, g). \tag{129}$$

Proof. Let $\overline{a} = (a(x)/r^\alpha)$; since $\alpha < (n/p) < 2$, it follows that $\overline{a} \in L^p(M)$, and then from Theorem 3 and Theorem 4, we get the result.

In the case of $\alpha = 2$, $\overline{a}$ does not necessarily belong to $L^p(M)$, and Theorem 3 is no longer valid, so we look for subcritical cases $0 < \alpha < 2$ and we tend α to 2. This can be done by adding an assumption and using Lebesgue's theorem. Let P_g be such that

$$P_{g,\alpha} = \Delta_g + \frac{a}{r^\alpha}. \tag{130}$$

□

Proposition 3. *If $P_{g,2}$ is coercive, the following norm is equivalent to the usual norm on $H_1^2(M)$:*

$$\|u\| = \left(\int_M \left(\left|\nabla_g u\right|^2 + \frac{a}{r^2}u^2\right)\mathrm{d}v_g\right)^{1/2} = \left(\int_M uP_{g,2}(u)\mathrm{d}v_g\right)^{1/2}. \tag{131}$$

Proof. If P_g is coercive, then there exits $c_0 > 0$ such that, for any $u \in H_1^2(M)$,

$$\int_M uP_{g,2}(u)\mathrm{d}v_g \geq c_0\|u\|^2_{H_1^2(M)}. \tag{132}$$

By Hardy–Sobolev inequality (121) with $p = 2$ and $\gamma = -2$, we get

$$\begin{aligned}\int_M uP_{g,2}(u)\mathrm{d}v_g &\leq \int_M \left|\nabla_g u\right|^2\mathrm{d}v_g + \|a\|_\infty \int_M \frac{u^2}{r^2}\mathrm{d}v_g \\ &\leq \int_M \left|\nabla_g u\right|^2\mathrm{d}v_g + \|a\|_\infty\left(\left(K(n,2,-2)^2 + \epsilon\right)\left\|\nabla_g u\right\|_2^2 + A(\epsilon,2,-2)\|u\|_2^2\right) \\ &\leq \max\left(1, \left(\|a\|_\infty K(n,2,-2)^2, \|a\|_\infty A(\epsilon,2,-2)\right)\right)\|u\|^2_{H_1^2(M)} \\ &\leq c_\alpha\|u\|^2_{H_1^2(M)},\end{aligned} \tag{133}$$

where

$$c_\alpha = \max\left(1, \left(\|a\|_\infty K(n,2,-2)^2, \|a\|_\infty A(\epsilon,2,-2)\right)\right) > 0. \tag{134}$$

Therefore,

$$c_0\|u\|^2_{H_1^2(M)} \leq \int_M uP_{g,2}(u)dv_g \leq c_\alpha\|u\|^2_{H_1^2(M)}. \tag{135}$$

□

Proposition 4. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$ such that the metric $g \in H_2^p(M, T^*M \otimes T^*M)$. Assume that f is a positive $C^\infty(M)$ function on M and $P \in M$ such that $f(P) = sup_{x\in M} f(x)$. If $\mu(M, g) > 0$ and*

$$\frac{\Delta f(P)}{f(P)} < \frac{S_g(P)}{3}\left(\frac{n+2}{n-4} - \|a\|_p - 1\right)\left(\|a\|_p + 1\right)^{-1}, \tag{136}$$

where $a = (n-2/4(n-1))S_g$, then there exists a metric $\overline{g} = u^{N-2}g$ conformal to g such that the scalar curvature $S_{\overline{g}} = f$.

Proof. As in Section 1, if the singular Yamabe invariant $\mu(M, g) > 0$, the singular Yamabe operator $P_g = \Delta_g + a$ is weakly conformally invariant, and by Theorem 3, Theorem 4, and Theorem 5, there exists $u \in C^{1-[n/p]}(M)$, $u > 0$, solution of the following equation:

$$\Delta_g u + \frac{n-2}{4(n-1)}S_g u = f|u|^{N-2}u. \tag{137}$$

On the contrary, by the weak conformal invariance of P_g and if $\overline{g} = u^{N-2}g$ is conformal to g, one has

$$\Delta_g u + \frac{n-2}{4(n-1)}S_g u = S_{\overline{g}}|u|^{N-2}u. \tag{138}$$

Then, we deduce that the metric $\overline{g} = u^{N-2}g$ is such that the scalar curvature $S_{\overline{g}} = f$. □

6. The Critical Case $\alpha = 2$

Keeping the notations adapted above, in this section, we are going to prove the second main theorem, Theorem 2.

Proof □

Step 1. We have $F_2(u) > -\infty$ and

$$\lim_{\alpha\longrightarrow 2}\lambda_\alpha(M, g) = \lambda_2(M, g). \tag{139}$$

Let $\delta(M)$ be the injectivity radius of M. For any $\epsilon > 0$, there exists δ such that $0 < \delta < \min(1, \delta(M))$, and if $Q \in B(P, \delta)$, we get $|a(Q) - a(P)| < \epsilon$, and also, if $u \in H_1^2(M)$ with $\int_M f|u^N|\mathrm{d}v_g = (1 + \|a/r^2\|_p)^{N/2}$, then we have

$$\int_M \frac{a}{r^2}u^2\mathrm{d}v_g \geq (a(P) - \epsilon)\int_{B(P,\delta)} r^{-2}u^2\mathrm{d}v_g - \frac{\|a\|_\infty}{\delta^2}\|u\|_2^2. \tag{140}$$

By applying Theorem 7, the following inequality holds:

$$\int_{B(P,\delta)} \frac{u^2}{r^2}\mathrm{d}v_g \leq \left(K^2(n,2,-2) + \epsilon\right)\left\|\nabla_g u\right\|_2^2 + A(\epsilon,2,-2)\|u\|_2^2. \tag{141}$$

Combining (140) and (141), we get

$$\int_M a\frac{u^2}{r^2}dv_g \geq (\min(a(P),0)-\epsilon)\left(K^2(n,2,-2)+\epsilon\right)\left\|\nabla_g u\right\|_2^2 + \left\{(\min(a(P),0)-\epsilon)A(\epsilon,2,-2)-\frac{\|a\|_\infty}{\delta^2}\right\}\|u\|_2^2. \quad (142)$$

Independently, by Hölder's inequality, one has

$$\|u\|_2^2 \leq \left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)}. \quad (143)$$

Then, we get that

$$\int_M a\frac{u^2}{r^2}dv_g \geq (\min(a(P),0)-\epsilon)\left(K^2(n,2,-2)+\epsilon\right)\left\|\nabla_g u\right\|_2^2 + \left\{(\min(a(P),0)-\epsilon)A(\epsilon,2,-2)-\frac{\|a\|_\infty}{\delta^2}\right\}\left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)}. \quad (144)$$

Hence,

$$F_2(u) \geq \left\|\nabla_g u\right\|_2^2 + (\min(a(P),0)-\epsilon)\left(K^2(n,2,-2)+\epsilon\right)\left\|\nabla_g u\right\|_2^2 + \left\{(\min(a(P),0)-\epsilon)A(\epsilon,2,-2)-\frac{\|a\|_\infty}{\delta^2}\right\}\left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)}, \quad (145)$$

which yields

$$F_2(u) \geq \left\{(1+(\min(a(P),0)-\epsilon)\left(K^2(n,2,-2)+\epsilon\right)\right\}\left\|\nabla_g u\right\|_2^2 + \left\{(\min(a(P),0)-\epsilon)A(\epsilon,2,-2)-\frac{\|a\|_\infty}{\delta^2}\right\}\left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)}. \quad (146)$$

Now, if $1+a(P)K^2(n,2,-2)>0$, then we can chose ϵ sufficiently small such that the first term of the right-hand side of the latter equality will be strictly positive; then, it follows that

$$F_2(u) > \left\{(\min(a(P),0)-\epsilon)A(\epsilon,2,-2)-\frac{\|a\|_\infty}{\delta^2}\right\} \cdot \left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)}. \quad (147)$$

Consequently,

$$F_2(u) > -\infty. \quad (148)$$

On the contrary, by Lebesgue's dominated convergence theorem, we get that

$$\int_M \frac{a}{r^\alpha}u^2 dv_g \longrightarrow \int_M \frac{a}{r^2}u^2 dv_g, \quad \text{when } \alpha \longrightarrow 2. \quad (149)$$

Hence,

$$\lim_{\alpha\longrightarrow 2} E_\alpha(u) = E_2(u), \quad (150)$$

and by passing to the infimum over u such that $\int_M f|u^N|dv_g = (1+\|a/r^\alpha\|_p)^{N/2}$, we obtain

$$\lim_{\alpha\longrightarrow 2} \lambda_\alpha(M,g) = \lambda_2(M,g), \quad (151)$$

which implies that there exists α_0 such that, for all $\alpha \in [\alpha_0, 2)$, we also have

$$\lambda_\alpha(M,g) \leq \left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(K_0^{-2}(n,1)\right)\left(f(P)\right)^{-(2/N)}. \quad (152)$$

Step 2. We claim that the sequence $(u_\alpha)_\alpha$ is bounded in $H_1^2(M)$ and converges to a weak solution.

In a similar way, by Proposition 4 and from Hölder's inequality, we get then for all $\alpha \in [\alpha_0, 2)$, the solution u_α of equation (123) satisfies

$$\left\|u_\alpha\right\|_2^2 \leq \left(1+\|a/r^\alpha\|_p\right)\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)}. \quad (153)$$

Knowing that

$$F_\alpha(u_\alpha) = \int_M \left(\left|\nabla_g u_\alpha\right|^2 + \frac{a}{r^\alpha}u^2\right)dv_g \geq \int_M \left|\nabla_g u_\alpha\right|^2 + \int_{B(P,\delta)} \frac{a}{r^\alpha}u^2 - \frac{\left\|u_\alpha\right\|_\infty}{\delta^2}\left\|u_\alpha\right\|_2^2 dv_g, \quad (154)$$

we get by the assumption made in (128) that

$$\left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(K_0^{-2}(n,1)\right)(f(P))^{-(2/N)} \geq F_\alpha(u_\alpha) \geq \left\|\nabla_g u_\alpha\right\|_2^2 + \int_{B(P,\delta)} \frac{a}{r^\alpha}u^2 - \frac{\left\|u_\alpha\right\|_\infty}{\delta^2}\int_M u_\alpha^2 dv_g, \quad (155)$$

which leads to

$$\left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(K_0^{-2}(n,1)\right)(f(P))^{-(2/N)} + \frac{\left\|u_\alpha\right\|_\infty}{\delta^2}\int_M u_\alpha^2 dv_g \geq \left\|\nabla_g u_\alpha\right\|_2^2 + \int_{B(P,\delta)} \frac{a}{r^\alpha}u^2 dv_g, \quad (156)$$

and by letting

$$A = \left(1+\left\|\frac{a}{r^2}\right\|_p\right)\left(K_0^{-2}(n,1)\right)(f(P))^{-(2/N)}, \quad (157)$$

the last inequality will be written as

$$A + \frac{\left\|u_\alpha\right\|_\infty}{\delta^2}\int_M u_\alpha^2 dv_g \geq \left\|\nabla_g u_\alpha\right\|_2^2 + \int_{B(P,\delta)} \frac{a}{r^\alpha}u_\alpha^2 dv_g. \quad (158)$$

By applying (153), we still get

$$A + \frac{\|a\|_\infty}{\delta^2}\left(1+\left\|\frac{a}{r^\alpha}\right\|_p\right)\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)} \geq \left\|\nabla_g u_\alpha\right\|_2^2 + \int_{B(P,\delta)} \frac{a}{r^\alpha}u_\alpha^2 dv_g, \quad (159)$$

and as

$$\left(1+\left\|\frac{a}{r^2}\right\|_p\right) \leq \left(1+\left\|\frac{a}{r^\alpha}\right\|_p\right), \quad (160)$$

by letting

$$B = \left(1+\left\|\frac{a}{r^\alpha}\right\|_p\right)\left\{\left(K_0^{-2}(n,1)\right)(f(P))^{-(2/N)} + \frac{\|a\|_\infty}{\delta^2}\left(\inf_{x\in M} f(x)\right)^{-(2/N)} V(M)^{1-(2/N)}\right\}, \quad (161)$$

we also get

$$B \geq \left\|\nabla_g u_\alpha\right\|_2^2 + \int_{B(P,\delta)} \frac{a}{r^\alpha}u_\alpha^2 dv_g. \quad (162)$$

Now, by Lemma 1 and for δ sufficiently small, we obtain

$$\int_{B(P,\delta)} \frac{a}{r^\alpha}u_\alpha^2 dv_g \geq (\min(a(P),0)-\epsilon)K^2(n,2,-2)\left\|\nabla_g u_\alpha\right\|_2^2. \quad (163)$$

Then, inequality (162) gives

$$1 + (\min(a(P),0)-\epsilon)K^2(n,2,-2)\left\|\nabla_g u_\alpha\right\|_2^2 \leq B. \quad (164)$$

Then, the sequence $(u_\alpha)_\alpha$ is bounded in $H_1^2(M)$, and then there exists a sequence $\alpha_m \in [\alpha_0, 2)$ which converges to 2 such that the sequence $(u_{\alpha_m})_m = (u_m)_m$ converges weakly in $H_1^2(M)$, and after restriction to a subsequence still labeled $(u_m)_m$, we may assume that there exists $u \in H_1^2(M)$ such that

(i) $u_m \longrightarrow u$ weakly in $H_1^2(M)$.

(ii) $u_m \longrightarrow u$ strongly in $L^p(M)$ for all $p < N$ and almost everywhere on M. Then, from Hardy–Sobolev's embedding Theorem 8, we have $u_m \longrightarrow u$ weakly in $L^2(M, r^{-2})$, and we deduce that, for any $\varphi \in L^2(M)$,

$$\int_M \frac{a}{r^2}u_m\varphi dv_g = \int_M \frac{a}{r^2}u\varphi dv_g + o(1). \quad (165)$$

In particular, for any $\varphi \in H_1^2(M)$, we have

$$\int_M \left(\Delta_g u_m + \frac{a}{r^{\alpha_m}}u_m\right)\varphi dv_g = \lambda_m(M,g)\int_M f\left|u_m\right|^{N-2}u_m\varphi dv_g. \quad (166)$$

By the weak convergence in $H_1^2(M)$, we get

$$\begin{aligned}
\int_M \varphi\Delta_g u_m dv_g &= \int_M \varphi\Delta_g u dv_g + o(1), \\
\int_M \varphi\left(\frac{a}{r^{\alpha_m}}u_m - \frac{a}{r^2}u\right)dv_g &= \int_M \varphi\left(\frac{a}{r^{\alpha_m}}u_m - \frac{a}{r^2}u_m + \frac{a}{r^2}u_m - \frac{a}{r^2}u\right)dv_g \\
&= \int_M a\varphi u_m\left(\frac{1}{r^{\alpha_m}} - \frac{1}{r^2}\right)dv_g + \int_M \frac{a}{r^2}\varphi(u_m - u)dv_g.
\end{aligned} \quad (167)$$

Again with the weak convergence in $L^2(M, r^{-2})$ and Lebesgue's dominated convergence theorem, we get that

$$\int_M a\varphi u_m\left(\frac{1}{r^{\alpha_m}} - \frac{1}{r^2}\right)\mathrm{d}v_g + \int_M \frac{a}{r^2}\varphi\left(u_m - u\right)\mathrm{d}v_g \longrightarrow 0. \tag{168}$$

On the contrary, since (u_m) is bounded in $L^N(M)$, the sequence $(|u_m|^{N-2}u_m)_m$ is bounded in $L^{N/(N-1)}(M)$; hence,

$$\lambda_m(M, g)\int_M f|u_m|^{N-2}u_m\varphi\mathrm{d}v_g \longrightarrow \lambda(M, g)\int_M f|u|^{N-2}u\varphi\mathrm{d}v_g. \tag{169}$$

Step 3. u is nonidentically null.

By Sobolev's inequality applying to $(u_m)_m$, we can easily have

$$\left(1 + \left\|\frac{a}{r^{\alpha_m}}\right\|_p\right)(f(P))^{-(2/N)} \le \|u_m\|_N^2 \le K_0^2(n, 1)\left\|\nabla_g u_m\right\|_2^2 + B\|u_m\|_2^2, \tag{170}$$

and since u_m satisfies (123), we also have

$$\left\|\nabla_g u_m\right\|_2^2 = \lambda_m(M, g) - \int_M u_m^2\frac{a}{r^{\alpha_m}}\mathrm{d}v_g. \tag{171}$$

Writing

$$\begin{aligned}\int_M a\frac{u_m^2}{r^{\alpha_m}}dv_g \ge{}& (\min(a(P), 0) - \epsilon)\left(K^2(n, 2, -\alpha) + \epsilon\right)\left\|\nabla_g u_m\right\|_2^2\\ &+ \left\{(\min(a(P), 0) - \epsilon)A(\epsilon, 2, -\alpha) - \frac{\|a\|_\infty}{\delta^2}\right\}\|u_m\|_2^2,\end{aligned} \tag{172}$$

then

$$\begin{aligned}\left\|\nabla_g u_m\right\|_2^2 \le{}& \lambda_m(M, g) - (\min(a(P), 0) - \epsilon)\left(K^2(n, 2, -\alpha) + \epsilon\right)\left\|\nabla_g u_m\right\|_2^2\\ &- \left\{(\min(a(P), 0) - \epsilon)A(\epsilon, 2, -\alpha) - \frac{\|a\|_\infty}{\delta^2}\right\}\|u_m\|_2^2.\end{aligned} \tag{173}$$

It follows that

$$\begin{aligned}&\left\{1 + (\min(a(P), 0) - \epsilon)\left(K^2(n, 2, -\alpha) + \epsilon\right)\right\}\left\|\nabla_g u_m\right\|_2^2\\ &\quad\le \lambda_m(M, g) - \left\{(\min(a(P), 0) - \epsilon)A(\epsilon, 2, -\alpha) - \frac{\|a\|_\infty}{\delta^2}\right\}\|u_m\|_2^2,\end{aligned} \tag{174}$$

which implies

$$\left\|\nabla_g u_m\right\|_2^2 \le \frac{\lambda_m(M, g) - \left\{(\min(a(P), 0) - \epsilon)A(\epsilon, 2, -\alpha) - \left(\|a\|_\infty/\delta^2\right)\right\}\|u_m\|_2^2}{1 + (\min(a(P), 0) - \epsilon)\left(K^2(n, 2, -\alpha) + \epsilon\right)}. \tag{175}$$

Plugging this in (170), we get

$$\left(1 + \left\|\frac{a}{r^{\alpha_m}}\right\|_p\right)(f(P))^{-(2/N)} \le \frac{\lambda_m(M, g) - \left\{(\min(a(P), 0) - \epsilon)A(\epsilon, 2, -\alpha) - \left(\|a\|_\infty/\delta^2\right)\right\}\|u_m\|_2^2}{1 + (\min(a(P), 0) - \epsilon)\left(K^2(n, 2, -\alpha) + \epsilon\right)}K_0^2(n, 1) + B\|u_m\|_2^2. \tag{176}$$

Hence,

$$\begin{aligned}&\left(1 + \left\|\frac{a}{r^{\alpha_m}}\right\|_p\right)(f(P))^{-(2/N)} - \frac{\lambda_m(M, g)K_0^2(n, 1)}{1 + (\min(a(P), 0) - \epsilon)\left(K^2(n, 2, -\alpha) + \epsilon\right)}\\ &\quad\le \left\{B - \frac{K_0^2(n, 1)\left\{(\min(a(P), 0) - \epsilon)A(\epsilon, 2, -\alpha) - \left(\|a\|_\infty/\delta^2\right)\right\}}{1 + (\min(a(P), 0) - \epsilon)\left(K^2(n, 2, -\alpha) + \epsilon\right)}\right\}\|u_m\|_2^2.\end{aligned} \tag{177}$$

In the end, if (22) is satisfied, as above for m large enough, we can chose ϵ sufficiently small such that

$$0<\left(1+\left\|\frac{a}{r^{\alpha_m}}\right\|_p\right)(f(P))^{-(2/N)}-\frac{\lambda_m(M,g)K_0^2(n,1)}{1+(\min(a(P),0)-\epsilon)(K^2(n,2,-\alpha)+\epsilon)}. \quad (178)$$

Then, it follows that the solution u is not trivial.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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