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Research Article

An Optimal Control for a Two-Dimensional Spatiotemporal SEIR Epidemic Model

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In this paper, we present an application of optimal control theory on a two-dimensional spatial-temporal SEIR (susceptible, exposed, infected, and restored) epidemic model, in the form of a partial differential equation. Our goal is to minimize the number of susceptible and infected individuals and to maximize recovered individuals by reducing the cost of vaccination. In addition, the existence of the optimal control and solution of the state system is proven. The characterization of the control is given in terms of state function and adjoint. Numerical results are provided to illustrate the effectiveness of our adopted approach.

1. Introduction

In the literature, there are numerous books and articles [1–7] that deal with epidemic mathematical models. It is well established that human mobility plays an important role in the spread of an epidemic [8–13]. Mathematical modelling of the spread of infectious diseases has an important influence on disease management and control [14–16]. In general, after the initial infection, a host remains in a latency period before becoming infectious, so the population can be divided into four categories: susceptible (S), exposed (E), infected (I), and recovered (R). In this contribution, we treat a model of epidemic type SEIR in which the model takes into account the total population size as a refrain for the transmission of the disease, and it is assumed that it is constant over time. The approach used is based on the work of El Alami Laaroussi et al. [17, 18], which was applied on a SIR model. So, our goal is to characterize optimal control in the form of a vaccination program, maximizing the number of people reestablished and minimizing the number of susceptible, infected people and the cost associated with vaccination over an infinite space and in a time domain. The theory of semigroups and optimal control makes it possible to show the existence of state system solutions and optimal control and to obtain the optimal characterization of this control in terms of state functions and adjoint functions. To illustrate the solutions, based on the numerical results, we find that the use of the vaccine control strategy in the spatial region helps to fight the spread of the epidemic in this region over a period of 60 days. The structure of this article is as follows. Section 2 is devoted to the basic mathematical model and the associated optimal control problem. In Section 3, we prove the existence of a strong global solution for our system. The existence of the optimal solution is proved in Section 4. The necessary optimality conditions are defined in Section 5. As an application, the numerical results associated with our control problem are given in Section 6. Finally, we conclude the paper in Section 7.

2. The Basic Mathematical Model

2.1. The Model. In this paper, we consider the following SEIR epidemic model (susceptible (S), exposed (E), infected (I), and recovered (R)):

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S - \mu S + \omega R - \beta \dfrac{SI}{N} + \mu N (1 - v(x,t)), \\ \dfrac{\partial E}{\partial t} = d_E \Delta E + \beta \dfrac{SI}{N} - (\mu + \sigma) E, \\ \dfrac{\partial I}{\partial t} = d_I \Delta I - (\mu + \gamma) I + \sigma E, \\ \dfrac{\partial R}{\partial t} = d_R \Delta R - (\mu + \omega) R + \gamma I + \mu N v(x,t), \\ (t,x) \in Q = [0,T] \times \Omega, \end{cases} \tag{1}$$

where $\beta(SI/N)$ is the total number of infection per unit of time, N is the total population $(N(t) = S(t) + E(t) + I(t) + R(t) = N(0) = N)$, μ is the rate of deaths from causes unrelated to the infection, incidence rate, ω is the rate of losing immunity, β is the transmission constant, and σ^{-1} and γ^{-1} are, respectively, the average duration of latent and infective periods. The positive constants d_S, d_E, d_I, and d_R denote the corresponding diffusion rate for susceptible, exposed, infectious, and recovered individuals. We denote by Ω a fixed and bounded domain in IR^2 with smooth boundary $\partial\Omega$ and η is the outward unit normal vector on the boundary. The initial conditions and no-flux boundary conditions are given by

$$\frac{\partial S}{\partial \eta} = \frac{\partial E}{\partial \eta} = \frac{\partial I}{\partial \eta} = \frac{\partial R}{\partial \eta} = 0, \quad (t,x) \in \Sigma = [0,T] \times \partial\Omega, \tag{2}$$

$$\begin{gathered} S(0,x) = S_0 \geq 0, \\ E(0,x) = E_0 \geq 0, \\ I(0,x) = I_0 \geq 0, \\ R(0,x) = R_0 \geq 0. \end{gathered} \tag{3}$$

2.2. The Optimal Vaccination. Eligible controls are contained in the ensemble

$$U_{\text{ad}} = \left\{ \frac{v \in L^{\infty}(Q)}{0 \leq v \leq v^{\max} \leq 1} \right\}, \tag{4}$$

where $v(x,t)$ represents the vaccination rate at time and position x. We seek to minimize the functional objective

$$J(v) = \int_0^T \int_\Omega (K_1 S(t,x) + K_2 I(t,x) - K_3 R(t,x)) \mathrm{d}x\, \mathrm{d}t + \frac{\alpha}{2} \|v\|_{L^2(Q)}^2, \tag{5}$$

for some positive constant $v^{\max}$.

K_1, K_2, and K_3 are constant weights. The cost of vaccination is a nonlinear function of v, and we choose a quadratic function indicating the additional costs associated with high vaccination rates.

The parameter $(\alpha/2)$, with the units (Population/km^2)/vaccin^2, balances the cost squared of the vaccine with the cost associated with the infected population. Our objective is to find control functions such that

$$J(v^*) = \min\{J(v), v \in U_{\text{ad}}\}. \tag{6}$$

(i) We put $H(\Omega) = (L^2(\Omega))^4$; we denote by $W^{1,2}([0,T], H(\Omega))$ the space of all absolutely continuous functions $y : [0,T] \longrightarrow H(\Omega)$ having the property that $(\partial y/\partial t) \in L^2([0,T], H(\Omega))$.

(ii) $\mathscr{L}(T,\Omega) = L^2([0,T], H^2(\Omega)) \cap L^\infty([0,T], H^1(\Omega))$.

3. Existence of Solution

We study in this section the existence of a global strong solution, positivity, and boundedness of solutions of problem for (1)–(3). Let $y = (y_1, y_2, y_3, y_4) = (S, E, I, R)$ the solution of system (1)–(3) with $y^0 = (y_1^0, y_2^0, y_3^0, y_4^0) = (S^0, E^0, I^0, R^0)$. A denotes the linear operator defined as follows:

$$\begin{aligned} A &: D(A) \subset H(\Omega) \longrightarrow H(\Omega), \\ &Ay = (d_S \Delta y_1, d_E \Delta y_2, d_1 \Delta y_3, d_R \Delta y_4) \in D(A), \\ &\forall y = (y_1, y_2, y_3, y_4) \in D(A), \end{aligned} \tag{7}$$

where the domain of A is defined by

$$D(A) = \left\{ y \in \left(H^2(\Omega)\right)^4, \frac{\partial y_1}{\partial \eta} = \frac{\partial y_2}{\partial \eta} = \frac{\partial y_3}{\partial \eta} = \frac{\partial y_4}{\partial \eta} = 0, \quad a.\text{ex} \in \partial\Omega \right\}. \tag{8}$$

Theorem 1. *Let Ω be a bounded domain from $\mathbb{R}^2$; with the boundary smooth enough, $y_i^0 \geq 0$ on Ω(with $i = 1, 2, 3, 4$), the problem (1)–(3) has a unique (global) strong solution $y \in W^{1,2}([0,T] : H(\Omega))$such that$y_i \in \mathscr{L}(T,\Omega) \cap L^\infty(Q)$ with $i = 1, 2, 3, 4$. In addition y_1, y_2, y_3, and y_4 are nonnegative. Furthermore, there exists $C > 0$ (independent of (v)) for all $t \in [0,T]$:*

$$\left\| \frac{\partial y_i}{\partial t} \right\|_{L^2(Q)} + \|y_i\|_{L^2(0,T;H^2(\Omega))} + \|y_i\|_{H^1(\Omega)} + \|y_i\|_{L^\infty(Q)} \leq C, \quad \text{for } i = 1, 2, 3, 4. \tag{9}$$

Proof. To prove the existence of a (global) strong solution for system (1)–(3), now we write system (1)–(3) as shown in (7) (see Appendix). Let

$$\begin{cases} g_1(y(t)) = -\mu y_1 + \omega y_4 - \beta \dfrac{y_1 y_3}{N} + \mu N(1 - v(x,t)), \\ g_2(y(t)) = \beta \dfrac{y_1 y_3}{N} - (\mu+\sigma) y_2, \quad t \in [0,T], \\ g_3(y(t)) = -(\mu+\gamma) y_3 + \sigma y_2, \\ g_4(y(t)) = -(\mu+\omega) y_4 + \gamma y_3 + \mu N v(x,t). \end{cases} \tag{10}$$

System (10) represents the nonlinear term of (1) and we consider the function $g(y(t)) = (g_1(y(t)), g_2(y(t)), g_3(y(t)), g_4(y(t)))$; then, we can be rewrite system (1)–(3) in the space $H(\Omega)$ as follows:

$$\begin{cases} \dfrac{\partial y}{\partial t} = Ay + g(y(t)), \quad t \in [0,T], \\ y(0) = y^0. \end{cases} \tag{11}$$

It is clear that function g is Lipschitz continuous in $y = (y_1, y_2, y_3, y_4)$ uniformly with respect to $t \in [0,T]$.

As the operator A defined in (7) and (8) is dissipating, self-adjoint and generates a C_0-semi-group of contractions on $H(\Omega)$ [19]. Therefore, Theorem A.1 (see Appendix) assures problem (1)–(3) which admits a unique strong solution $y \in W^{1,2}([0,T], H(\Omega))$ with

$$y_1,\ y_2,\ y_3\ y_4 \in L^2\big([0,T], H^2(\Omega)\big). \tag{12}$$

In order to show that $y_i \in L^\infty(Q)$ for $i = 1, 2, 3, 4$, we denote $M = \max\left\{\|g_1\|_{L^\infty(Q)}, \|y_1^0\|_{L^\infty(\Omega)}\right\}$ and $\{S(t), t \geq 0\}$ is the C_0-semigroup generated by the operator $B : D(B) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$, where $By_1 = d_1 \Delta y_1$ and $D(B) = \{y_1 \in H^2(\Omega), (\partial y_1/\partial \eta) = 0, \text{ a.e } \partial\Omega\}$. It is clear that the function $U_1(t,x) = y_1 - Mt - \|y_1^0\|_{L^\infty(\Omega)}$ satisfies the system:

$$\begin{cases} \dfrac{\partial U_1}{\partial t}(t,x) = d_S \triangle U_1 + g_1(t, y(t)) - M, \quad t \in [0,T], \\ U_1(0,x) = y_1^0 - \left\|y_1^0\right\|_{L^\infty(\Omega)}. \end{cases} \tag{13}$$

Note that this system has a solution given by

$$U_1(t) = S(t)\big(y_1^0 - \|y_1^0\|_{L^\infty(\Omega)}\big) + \int_0^t S(t-s)\big(g_1(s, y(s)) - M\big) ds. \tag{14}$$

As $y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \leq 0$ and $g_1(s, y(s)) - M \leq 0$, we have $U_1(t,x) \leq 0$, $\forall (t,x) \in Q$. Similarly, the function $U_2(t,x) = y_1 + Mt + \|y_1^0\|_{L^\infty(\Omega)}$ satisfies $U_2(t,x) \geq 0$, $\forall (t,x) \in Q$.

Then,

$$\left|y_1(t,x)\right| \leq Mt + \left\|y_1^0\right\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q, \tag{15}$$

and analogously, we have

$$\left|y_i(t,x)\right| \leq Mt + \left\|y_i^0\right\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q, \quad \text{for } i = 2,3,4. \tag{16}$$

Thus, we have proved that

$$y_i \in L^\infty(Q) \forall (t,x) \in Q, \quad \text{for } i = 1,2,3,4. \tag{17}$$

By the first equation of (1), we obtain

$$\begin{aligned} &\int_0^t \int_\Omega \left|\frac{\partial y_1}{\partial s}\right|^2 ds\, dx + d_S^2 \int_0^t \int_\Omega \left|\Delta y_1\right|^2 ds\, dx \\ &- 2 d_S \int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \Delta y_1 ds\, dx \\ &= \int_0^t \int_\Omega \left(-\mu y_1 + \omega y_4 - \beta \frac{y_1 y_3}{N} + \mu N(1 - v(x,t))\right)^2 ds\, dx. \end{aligned} \tag{18}$$

Using the regularity of y_1 and Green's formula, we can write

$$\begin{aligned} 2\int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \Delta y_1 dx &= -\int_0^t \frac{\partial}{\partial s}\left(\int_\Omega \left|\nabla y_1^k\right|^2 dx\right) ds = -\int_\Omega \left|\nabla y_1\right|^2 dx \\ &+ \int_\Omega \left|\nabla y_1^0\right|^2 dx. \end{aligned} \tag{19}$$

Then,

$$\begin{aligned} &\int_0^t \int_\Omega \left|\frac{\partial y_1}{\partial s}\right|^2 ds\, dx + d_S^2 \int_0^t \int_\Omega \left|\Delta y_1\right|^2 ds\, dx + d_S \int_\Omega \left|\nabla y_1\right|^2 dx \\ &- d_S \int_\Omega \left|\nabla y_1^0\right|^2 dx \\ &= \int_0^t \int_\Omega \left(-\mu y_1 + \omega y_4 - \beta \frac{y_1 y_3}{N} + \mu N(1 - v(x,t))\right)^2 ds\, dx. \end{aligned} \tag{20}$$

Since $\|y_i\|_{L^\infty(Q)}$ for $i = 1, 2, 3, 4$ are bounded independently of v and $y_1^0 \in H^2(\Omega)$, we deduce that

$$y_1 \in L^\infty\big([0,T], H^1(\Omega)\big). \tag{21}$$

We make use of (12), (13), and (21) in order to get

$$y_1 \in \mathscr{L}(T,\Omega) \cap L^\infty(Q), \tag{22}$$

and conclude that the inequality in (9) holds for $i = 1$ similarly for y_2, y_3 and y_4.

In order to show the positiveness of y_i for $i = 1, 2, 3, 4$, we write system (1) in the form:

$$\begin{cases} \dfrac{\partial y_1}{\partial t} = d_1 \Delta y_1 + H_1(y_1, y_2, y_3, y_4), \\ \dfrac{\partial y_2}{\partial t} = d_2 \Delta y_1 + H_2(y_1, y_2, y_3, y_4), \\ \dfrac{\partial y_3}{\partial t} = d_3 \Delta y_3 + H_3(y_1, y_2, y_3, y_4), \\ \dfrac{\partial y_4}{\partial t} = d_4 \Delta y_3 + H_4(y_1, y_2, y_3, y_4), \\ (t,x) \in Q. \end{cases} \quad (23)$$

It is easy to see that the functions $H_1(y_1, y_2, y_3, y_4)$, $H_2(y_1, y_2, y_3, y_4)$, $H_3(y_1, y_2, y_3, y_4)$, and $H_4(y_1, y_2, y_3, y_4)$ are continuously differentiable satisfying $H_1(0, y_2, y_3, y_4) = \omega y_4 + \mu N(1 - v(x,t)) \geq 0$, $H_2(y_1, 0, y_3, y_4) = \beta(y_1 y_3 / N) \geq 0$, $H_3(y_1, y_2, 0, y_4) = \sigma y_2 \geq 0$, $H_4(y_1, y_2, y_3, 0) = \gamma y_3 + \mu N v(x,t) \geq 0$, for all $y_1, y_2, y_3, y_4 \geq 0$ (see [20]). This completes the proof. □

4. The Existence of the Optimal Solution

In this section, we will prove the existence of an optimal control for problem (5) subject to reaction diffusion system (1)–(3) and $(v) \in U_{\text{ad}}$. The main result of this section is the following theorem.

Theorem 2. *Under the hypotheses of Theorem 1, the optimal control problem (1)–(5) admits an optimal solution* $(y^*, (v^*))$.

Proof. From Theorem 1, we know that, for every $v \in U_{\text{ad}}$, there exists a unique solution y to system (1)–(3). Assume that

$$\inf_{v \in U_{\text{ad}}} J((v)) > -\infty. \quad (24)$$

Let $\{(v^n)\} \subset U_{\text{ad}}$ be a minimizing sequence such that

$$\lim_{n \longrightarrow \infty} J(v^n) = \inf_{v \in U_{\text{ad}}} J(v), \quad (25)$$

where $(y_1^n, y_2^n, y_3^n, y_4^n)$ is the solution of system (1)–(3) corresponding to the control (v^n) for $n = 1, 2, \ldots$. That is,

$$\begin{cases} \dfrac{\partial y_1^n}{\partial t} = d_1 \Delta y_1^n - \mu y_1^n + \omega y_4^n - \beta \dfrac{y_1^n y_3^n}{(y_1^n + y_2^n + y_3^n + y_4^n)} + \mu (y_1^n + y_2^n + y_3^n + y_4^n)(1 - v^n), \\ \dfrac{\partial y_2^n}{\partial t} = d_2 \Delta y_2^n + \beta \dfrac{y_1^n y_3^n}{(y_1^n + y_2^n + y_3^n + y_4^n)} - (\mu + \sigma) y_2^n, \\ \dfrac{\partial y_3^n}{\partial t} = d_3 \Delta y_3^n - (\mu + \gamma) y_3^n + \sigma y_2^n, \quad (t,x) \in Q, \\ \dfrac{\partial y_4^n}{\partial t} = d_4 \Delta y_4^n - (\mu + \omega) y_4^n + \gamma y_3^n + \mu (y_1^n + y_2^n + y_3^n + y_4^n) v^n, \end{cases} \quad (26)$$

$$\frac{\partial y_1^n}{\partial \eta} = \frac{\partial y_2^n}{\partial \eta} = \frac{\partial y_3^n}{\partial \eta} = \frac{\partial y_4^n}{\partial t} = 0, \quad (t,x) \in \Sigma (t,x) \in \Sigma, \quad (27)$$

$$y_i^n(0,x) = y_i^0 \quad \text{for} i = 1,2,3,4\, x \in \Omega. \quad (28)$$

By Theorem 1, using the estimate (9) of the solution y_i^n, there exists a constant $C > 0$ such that for all $n \geq 1, t \in [0,T]$,

$$\left\| \frac{\partial y_i^n}{\partial t} \right\|_{L^2(Q)} \leq C, \left\| y_i^n \right\|_{L^2(0,T;H^2(\Omega))} \leq C, \left\| y_i^n \right\|_{H^1(\Omega)} \leq C, \quad i = 1,2,3,4. \quad (29)$$

$H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, so we deduce that $y_1^n(t)$ is compact in $L^2(\Omega)$.

Let us show that $\{y_1^n(t), n \geq 1\}$ is equicontinuous in $C([0,T] : L^2(\Omega))$. As $(\partial y_1^n / \partial t)$ is bounded in $L^2(Q)$, this implies that for all $s, t \in [0,T]$,

$$\left| \int_\Omega (y_1^n)^2 (t,x) \mathrm{d}x - \int_\Omega (y_1^n)^2 (s,x) \mathrm{d}x \right| \leq K|t - s|. \quad (30)$$

The Ascoli–Arzela theorem (see [21]) implies that y_1^n is compact in $C([0,T] : L^2(\Omega))$. Hence, selecting further sequences, if necessary, we have $y_1^n \longrightarrow y_1^*$ in $L^2(\Omega)$, uniformly with respect to t and analogously, we have for $y_i^n \longrightarrow y_i^*$ in $L^2(\Omega)$ for $i = 2, 3\ 4$, uniformly in relation to t.

From the boundedness of Δy_i^n in $L^2(Q)$, which implies it is weakly convergent in $L^2(Q)$ on a subsequence denoted again by Δy_i^n, for all distribution φ,

$$\int_Q \varphi \Delta y_i^n = \int_Q y_i^n \Delta \varphi \longrightarrow \int_Q y_i^* \Delta \varphi = \int_Q \varphi \Delta y_i^*, \quad (31)$$

which implies that $\Delta y_i^n \longrightarrow \Delta y_i^*$ weakly in $L^2(Q), i = 1,2,3,4$. In addition, the estimates (29) lead to $(\partial y_i^n / \partial t) \longrightarrow (\partial y_i^* / \partial t)$ weakly in $L^2(Q)$,

$i = 1, 2, 3, 4$, $y_i^n \longrightarrow y_i^*$ weakly in $L^2(0, T; H^2(\Omega))$, $i = 1, 2, 3, 4$, $y_i^n \longrightarrow y_i^*$ weakly in $L^\infty(0, T; H^1(\Omega))$, $i = 1, 2, 3, 4$.

We put $N(y) = (\beta/(y_1 + y_2 + y_3 + y_4))$; we now show that $y_1^n y_3^n \longmapsto y_1^* y_3^*$ and $N(y^n) y_1^n y_3^n \longmapsto N(y^*) y_1^* y_3^*$ strongly in $L^2(Q)$, and we write

$$\begin{aligned}
y_1^n y_3^n - y_1^* y_3^* &= (y_1^n - y_1^*) y_3^n + (y_3^n - y_3^*) y_1^*, \\
N(y^n) &= \frac{\beta}{y_1^n + y_2^n + y_3^n + y_4^n}, \\
N(y^*) &= \frac{\beta}{y_1^* + y_2^* + y_3^* + y_4^*}, \\
N(y^n) y_1^n y_3^n - N(y^*) y_1^* y_3^* &= N(y^n)(y_1^n y_3^n - y_1^* y_3^*) \\
&\quad + y_1^* y_3^* (N(y^n) - N(y^*)), \\
N(y^n) - N(y^*) &= \frac{\beta}{y_1^n + y_2^n + y_3^n + y_4^n} \\
&\quad - \frac{\beta}{y_1^* + y_2^* + y_3^* + y_4^*},
\end{aligned} \tag{32}$$

and we make use of the convergences $y_i^n \longrightarrow y_i^*$ strongly in $L^2(Q)$, $i = 1, 3$, and of the boundedness of y_1^n, y_3^n in $L^\infty(Q)$, and then $y_1^n y_3^n \longrightarrow y_1^* y_3^*$ and $N(y^n) y_1^n y_3^n \longmapsto N(y^*) y_1^* y_3^*$ strongly in $L^2(Q)$.

Since v^n is bounded, we can assume that $v^n \longrightarrow v^*$ weakly in $L^2(Q)$ on a subsequence denoted again by v^n. Since U_{ad} is a closed and convex set in $L^2(Q)$, it is weakly closed, so $v^* \in U_{\text{ad}}$.

We now show that

$$v^n (y_1^n + y_2^n + y_3^n + y_4^n) \longrightarrow v^* (y_1^* + y_2^* + y_3^* + y_4^*) \text{ weakly in } L^2(Q). \tag{33}$$

Writing with $i = 1, 2, 3, 4$,

$$v^n y_i^n - v^* y_i^* = (y_i^n - y_i^*) v^n + (v^n - v^*) y_i^*, \tag{34}$$

and making use of the convergences $y_i^n \longrightarrow y_i^*$ strongly in $L^2(Q)$ and $v^n \longrightarrow v^*$ weakly in $L^2(Q)$, for $i = 1, 2, 3, 4$, one obtains that $v^n (y_1^n + y_2^n + y_3^n + y_4^n) \longrightarrow v^* (y_1^* + y_2^* + y_3^* + y_4^*)$ weakly in $L^2(Q)$.

By taking $n \longrightarrow \infty$ in (26)–(28), we obtain that y^* is a solution of (1)–(3) corresponding to $(v^*) \in U_{\text{ad}}$. Therefore,

$$\begin{aligned}
J(y^*, v^*) &= K_1 \int_0^T \int_\Omega y_1^*(t, x) \mathrm{d}x \, \mathrm{d}t + K_2 \int_0^T \int_\Omega y_3^*(t, x) \mathrm{d}x \, \mathrm{d}t - K_3 \int_0^T \int_\Omega y_4^*(t, x) \mathrm{d}x \, \mathrm{d}t + \frac{\alpha}{2} \|v^*\|_{L^2(Q)}^2 \\
&\leq \lim_{n \longrightarrow \propto} \inf \left(K_1 \int_0^T \int_\Omega y_1^n(t, x) \mathrm{d}x \, \mathrm{d}t + K_2 \int_0^T \int_\Omega y_3^n(t, x) \mathrm{d}x \, \mathrm{d}t - K_3 \int_0^T \int_\Omega y_4^n(t, x) \mathrm{d}x \, \mathrm{d}t + \frac{\alpha}{2} \|v^n\|_{L^2(Q)}^2 \right) \\
&= \lim_{n \longrightarrow \propto} \left(K_1 \int_0^T \int_\Omega y_1^n(t, x) \mathrm{d}x \, \mathrm{d}t + K_2 \int_0^T \int_\Omega y_3^n(t, x) \mathrm{d}x \, \mathrm{d}t - K_3 \int_0^T \int_\Omega y_4^n(t, x) \mathrm{d}x \, \mathrm{d}t + \frac{\alpha}{2} \|v^n\|_{L^2(Q)}^2 \right) \\
&= \inf_{(v) \in U_{ad}} J((y, v)).
\end{aligned} \tag{35}$$

This shows that J attains its minimum at (y^*, v^*), and we deduce that (y^*, v^*) verifies problem (1)–(3) and minimizes the objective functional (5). The proof is complete. □

5. Necessary Optimality Conditions

Let $v \in U_{\text{ad}}$ and $v^\varepsilon = v^* + \varepsilon v \in U_{\text{ad}}$; in this section, we show the optimality conditions to problem (1)–(3), and we find the

characterization of optimal control. First, we need the Gateaux differentiability of the mapping $v \longrightarrow y(v)$. For this reason, denoting by $y^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon) = (y_1, y_2, y_3, y_4)(v^\varepsilon)$ and $y^* = (y_1^*, y_2^*, y_3^*, y_4^*) = (y_1, y_2, y_3, y_4)(v^*)$ the solution of (1)–(3) corresponding to v^ε and v^* respectively.

$$H = \begin{pmatrix} -\dfrac{\beta y_3^*(y_2^*+y_3^*+y_4^*)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} - \mu v^* & \mu - \mu v^* & -\dfrac{\beta y_1^*(y_1^*+y_2^*+y_4^*)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} + \mu - \mu v^* & \mu - \mu v^* + \omega \\ \dfrac{\beta y_3^*(y_2^*+y_3^*+y_4^*)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} & -\mu - \sigma & \dfrac{\beta y_1^*(y_1^*+y_2^*+y_4^*)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} & 0 \\ 0 & \sigma & -\mu - \gamma & 0 \\ \mu v^* & \mu v^* & \mu v^* + \gamma & -\mu - \omega + \mu v^* \end{pmatrix}, \tag{36}$$

$$L = \begin{pmatrix} -\mu(y_1^*+y_2^*+y_3^*+y_4^*) \\ 0 \\ 0 \\ \mu(y_1^*+y_2^*+y_3^*+y_4^*) \end{pmatrix}.$$

Proposition 1. *The mapping $y : U_{\mathrm{ad}} \longrightarrow W^{1,2}([0,T]\ ; H(\Omega))$ with $y_i \in \mathscr{L}(T,\Omega)$ for $i = 1, 2, 3, 4$ is Gateaux differentiable with respect to v^*. For all direction $v \in U_{\mathrm{ad}}, y'(v^*)v = Y$ is the unique solution in $W^{1,2}([0,T]; H(\Omega))$ with $Y_i \in \mathscr{L}(T,\Omega)$ of the following equation:*

$$\begin{cases} \dfrac{\partial Y}{\partial t} = AY + HY + Lv, \quad t \in [0,T], \\ Y(0) = 0. \end{cases} \tag{37}$$

Proof. Put $Y_i^\varepsilon = y_i^\varepsilon - y_i^*/\varepsilon$ for $i = 1,2,3,4$. $F(y_1, y_2, y_3, y_4) = (\beta y_1 y_3/(y_1+y_2+y_3+y_4))$, $M_1^\varepsilon = (F(y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon) - F(y_1^*, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon))/y_1^\varepsilon - y_1^*$, and $M_3^\varepsilon = (F(y_1^*, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon) - F(y_1^*, y_2^*, y_3^*, y_4^*))/y_3^\varepsilon - y_3^*$.

We denote S^ε system (1) corresponding to v^ε and S^* system (1) corresponding to v^*, subtracting system S^ε from S^*; so, we have:

$$\begin{cases} \dfrac{\partial Y_1^\varepsilon}{\partial t} = d_S \Delta Y_1^\varepsilon + (-M_1^\varepsilon - \mu v^\varepsilon) Y_1^\varepsilon + (\mu - \mu v^\varepsilon) Y_2^\varepsilon + (-M_3^\varepsilon + \mu - \mu v^\varepsilon) Y_3^\varepsilon + (\mu - \mu v^\varepsilon + \omega) Y_4^\varepsilon - \mu v (y_1^* + y_2^* + y_3^* + y_4^*), \\ \dfrac{\partial Y_2^\varepsilon}{\partial t} = d_E \Delta Y_2^\varepsilon + M_1^\varepsilon Y_1^\varepsilon - (\mu + \sigma) Y_2^\varepsilon + M_3^\varepsilon Y_3^\varepsilon, \quad (x,t) \in Q, \\ \dfrac{\partial Y_3^\varepsilon}{\partial t} = d_I \Delta Y_3^\varepsilon + \sigma Y_2^\varepsilon - (\mu + \gamma) Y_3^\varepsilon, \\ \dfrac{\partial Y_4^\varepsilon}{\partial t} = d_R \Delta Y_4^\varepsilon + \mu v^\varepsilon Y_1^\varepsilon + \mu v^\varepsilon Y_2^\varepsilon + (\mu v^\varepsilon + \gamma) Y_3^\varepsilon + (-\mu - \omega + \mu v^\varepsilon) Y_4^\varepsilon + \mu v (y_1^* + y_2^* + y_3^* + y_4^*), \end{cases} \tag{38}$$

with the homogeneous Neumann boundary conditions:

$$\frac{\partial Y_1^\varepsilon}{\partial \eta} = \frac{\partial Y_2^\varepsilon}{\partial \eta} = \frac{\partial Y_3^\varepsilon}{\partial \eta} = \frac{\partial Y_4^\varepsilon}{\partial \eta} = 0, \quad (x,t) \in \Sigma, \tag{39}$$

$$Y_i^\varepsilon(0,x) = 0, \quad x \in \Omega, \ \text{for } i = 1,2,3,4. \tag{40}$$

We prove that Y_i^ε are bounded in $L^2(Q)$ uniformly with respect to ε. For this end, denoting by $Y^\varepsilon = (Y_1^\varepsilon, Y_2^\varepsilon, Y_3^\varepsilon, Y_4^\varepsilon)$,

$$H^{\varepsilon} = \begin{pmatrix} -M_1^{\varepsilon} - \mu v^{\varepsilon} & \mu - \mu v^{\varepsilon} & -M_3^{\varepsilon} + \mu - \mu v^{\varepsilon} & \mu - \mu v^{\varepsilon} + \omega \\ M_1^{\varepsilon} & -\mu - \sigma & M_3^{\varepsilon} & 0 \\ 0 & \sigma & -\mu - \gamma & 0 \\ \mu v^{\varepsilon} & \mu v^{\varepsilon} & \mu v^{\varepsilon} + \gamma & -\mu - \omega + \mu v^{\varepsilon} \end{pmatrix},$$

$$L = \begin{pmatrix} -\mu\left(y_1^* + y_2^* + y_3^* + y_4^*\right) \\ 0 \\ 0 \\ \mu\left(y_1^* + y_2^* + y_3^* + y_4^*\right) \end{pmatrix}. \tag{41}$$

Then, (38) is rewritten as

$$\begin{cases} \dfrac{\partial Y^{\varepsilon}}{\partial t} = AY^{\varepsilon} + H^{\varepsilon} Y^{\varepsilon} + Lv, \quad t \in [0, T], \\ Y^{\varepsilon}(0) = 0. \end{cases} \tag{42}$$

Let $(S(t), t \geq 0)$ be the semigroup generated by A; then, the solution of (42) can be expressed as

$$Y^{\varepsilon}(t) = \int_0^t S(t-s) H^{\varepsilon}(s) Y^{\varepsilon}(s) ds + \int_0^t S(t-s) L v(s) ds. \tag{43}$$

On the other hand, the coefficients of the matrix H^{ε} are bounded uniformly with respect to ε; using Gronwall's inequality, we have

$$\left\| Y_i^{\varepsilon} \right\|_{L^2(Q)} \leq \Gamma, \tag{44}$$

where $\Gamma > 0\,(i = 1, 2, 3, 4)$. Then,

$$\left\| y_i^{\varepsilon} - y_i^* \right\|_{L^2(Q)} = \varepsilon \left\| Y_i^{\varepsilon} \right\|_{L^2(Q)}. \tag{45}$$

Hence, $y_i^{\varepsilon} \longrightarrow y_i^*$ in $L^2(Q)$, $i = 1, 2, 3, 4$. Denote

$$H = \begin{pmatrix} -M_1^* - \mu v^* & \mu - \mu v^* & -M_3^* + \mu - \mu v^* & \mu - \mu v^* + \omega \\ M_1^* & -\mu - \sigma & M_3^* & 0 \\ 0 & \sigma & -\mu - \gamma & 0 \\ \mu v^* & \mu v^* & \mu v^* + \gamma & -\mu - \omega + \mu v^* \end{pmatrix}, \tag{46}$$

where $M_1^* = (\partial F(y_1^*, y_2^*, y_3^*, y_4^*))/\partial y_1$, $M_3^* = (\partial F(y_1^*, y_2^*, y_3^*, y_4^*))/\partial y_3$, and $Y = (Y_1, Y_2, Y_3, Y_4)$.

Hence, system (38)–(40) can be written in the form

$$\begin{cases} \dfrac{\partial Y}{\partial t} = AY + HY + Lv, \quad t \in [0, T], \\ Y(0) = 0, \end{cases} \tag{47}$$

and its solution can be expressed as

$$Y(t) = \int_0^t S(t-s) H(s) Y(s) ds + \int_0^t S(t-s) L v(s) ds. \tag{48}$$

By (43) and (48), we deduce that

$$Y^{\varepsilon}(t) - Y(t) = \int_0^t S(t-s)\left[H^{\varepsilon}(s)\left(Y^{\varepsilon} - Y\right) + Y(s)\left(H^{\varepsilon}(s) - H(s)\right)\right] ds. \tag{49}$$

Thus, all the coefficients of the matrix H^{ε} tend to the corresponding coefficients of the matrix H in $L^2(Q)$. An application of Gronwall's inequality yields to $Y_i^{\varepsilon} \longrightarrow Y_i$ in $L^2(Q)$ as $\varepsilon \longrightarrow 0$, for $i = 1, 2, 3, 4$.

Let v^* be an optimal control of (1)–(4), $y^* = (y_1^*, y_2^*, y_3^*, y_4^*)$ be the optimal state, Z be the matrix defined by $Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $K = (K_1, K_2, 0, K_3)$, Z^* be the adjoint matrix associated to Z, H^* be the adjoint matrix associated to H, and $p = (p_1, p_2, p_3, p_4)$ be the adjoint variable; we can write the dual system associated to system (1)–(4):

$$\begin{cases} -\dfrac{\partial p}{\partial t} - Ap - H^* p = Z^* Z K, \quad t \in [0, T], \\ p(T, x) = 0. \end{cases} \tag{50}$$

Lemma 1. *Under hypotheses of Theorem 1, if $(y^*, (v^*))$ is an optimal pair, then there exists a unique strong solution $p \in W^{1,2}([0, T]; H(\Omega))$ to system (50) with $p_i \in \mathscr{L}(T, \Omega)$ for $i = 1, 2, 3, 4$.*

Proof. Similar to Theorem 1, by making the change of variable $s = T - t$ and the change of functions $q_i(s, x) = p_i(T - s, x) = p_i(t, x)$, $(t, x) \in Q$, $i = 1, 2, 3, 4$, we can easily prove the existence of the solution to this lemma.

To obtain the necessary conditions for the optimal control problem, applying standard optimality techniques, analyzing the objective functional and utilizing relationships between the state and adjoint equations, we obtain a characterization of the control optimal. □

Theorem 3. *Let $\alpha > 0$, v^* be an optimal control of (1)–(4) and let y^* and $p \in W^{1,2}([0, T]; H(\Omega))$ with y_i^* and $p_i \in \mathscr{L}(T, \Omega)$ for $i = 1, 2, 3, 4$. p is the adjoint variable, and y^* is the optimal state.*

y^* is the solution to (1)–(4) with the control v^*. Then,

$$v^* = \min\left(v^{\max}, \max\left(0, \frac{\mu\left(y_1^* + y_2^* + y_3^* + y_4^*\right)\left(p_1 - p_4\right)}{\alpha}\right)\right). \tag{51}$$

Proof. We suppose v^* is an optimal control and $y^* = (y_1^*, y_2^*, y_3^*, y_4^*) = (y_1, y_2, y_3, y_4)(v^*)$ are the corresponding state variables. Consider $v^{\varepsilon} = v^* + \varepsilon h \in U_{\text{ad}}$ and corresponding state solution $y^{\varepsilon} = (y_1^{\varepsilon}, y_2^{\varepsilon}, y_3^{\varepsilon}, y_4^{\varepsilon}) = (y_1, y_2, y_3, y_4)(v^{\varepsilon})$; we have

$$
\begin{aligned}
J'(v^*)(h) &= \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon}\left(J(v^\varepsilon) - J(v^*)\right) \\
&= \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon}\Bigg(\int_0^T \int_\Omega K_1 (y_1^\varepsilon - y_1^*)(t,x)\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega K_2 (y_3^\varepsilon - y_3^*)(t,x)\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega K_3 (y_4^\varepsilon - y_4^*)(t,x)\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{2}\int_0^T \int_\Omega \left((v^\varepsilon)^2 - (v^*)^2\right)(t,x)\mathrm{d}x\,\mathrm{d}t\Bigg) \\
&= \lim_{\varepsilon \longrightarrow 0}\Bigg(\int_0^T \int_\Omega K_1 \left(\frac{y_1^\varepsilon - y_1^*}{\varepsilon}\right)(t,x)\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega K_2 \left(\frac{y_3^\varepsilon - y_3^*}{\varepsilon}\right)(t,x)\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega K_3 \left(\frac{y_4^\varepsilon - y_4^*}{\varepsilon}\right)(t,x)\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{2}\int_0^T \int_\Omega \left(\varepsilon h^2 + 2hv^*\right)(t,x)\mathrm{d}x\,\mathrm{d}t\Bigg) \\
&= \int_0^T \int_\Omega K_1 Y_1 (t,x)\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega K_2 Y_3 (t,x)\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega K_3 Y_4 (t,x)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \alpha \int_0^T \int_\Omega (hv^*)(t,x)\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \langle ZK, ZY\rangle_{H(\Omega)}\mathrm{d}t + \int_0^T \langle \alpha v^*, h\rangle_{L^2(\Omega)}\mathrm{d}t.
\end{aligned} \tag{52}
$$

We use (37) and (50), and we have

$$
\begin{aligned}
\int_0^T \langle ZK, ZY\rangle_{H(\Omega)} dt &= \int_0^T \left\langle Z^* ZK, Y\right\rangle_{H(\Omega)}\mathrm{d}t \\
&= \int_0^T \left\langle -\frac{\partial p}{\partial t} - Ap - H^* p, Y\right\rangle_{H(\Omega)}\mathrm{d}t \\
&= \int_0^T \left\langle p, \frac{\partial Y}{\partial t} - AY - HY\right\rangle_{H(\Omega)}\mathrm{d}t \\
&= \int_0^T \left\langle p, Lh\right\rangle_{H(\Omega)}\mathrm{d}t \\
&= \int_0^T \left\langle L^* p, h\right\rangle_{L^2(\Omega)}\mathrm{d}t.
\end{aligned} \tag{53}
$$

Since J is Gateaux differentiable at v^* and U_{ad} is convex, as the minimum of the objective functional is attained at v^*, it is seen that $J'(v^*)(u - v^*) \geq 0$ for all $u \in U_{\mathrm{ad}}$.

We take $h = u - v^*$ and we use (52) and (53); then, $J'(v^*)(u - v^*) = \int_0^T \langle L^* p + \alpha v^*, (u - v^*)\rangle_{L^2(\Omega)}\mathrm{d}t$. We conclude that $J'(v^*)(u - v^*) \geq 0$ equivalent to $\int_0^T \langle L^* p + \alpha v^*, (u - v^*)\rangle_{L^2(\Omega)}\mathrm{d}t \geq 0$ for all $u \in U_{\mathrm{ad}}$. By standard arguments varying u, we obtain

$$
\alpha v^* = -L^* p. \tag{54}
$$

Then,

$$
v^* = \frac{\mu(y_1^* + y_2^* + y_3^* + y_4^*)p_1 - \mu(y_1^* + y_2^* + y_3^* + y_4^*)p_4}{\alpha}. \tag{55}
$$

As $v^* \in U_{\mathrm{ad}}$, we have

$$
v^* = \min\left(v^{\max},\ \max\left(0, \frac{\mu(y_1^* + y_2^* + y_3^* + y_4^*)(p_1 - p_4)}{\alpha}\right)\right). \tag{56}
$$

□

6. Numerical Results

In this section, we give the results obtained by the numerical resolution of the optimality system ((1)–(3), (50), (51)) using forward-backward sweep method (FBSM) [22]. We adopt two situations for the resolution: the first is that the disease starts with the middle of domain $\Omega(1)$, and in the second situation, the disease begins with the lower corner $\Omega(2)$. A rectangular area of $30\,\mathrm{km} \times 40\,\mathrm{km}$ is considered, and the parameter values and the initial values are given in Table 1. The upper limits of the optimality condition are considered to be $v^{\max} = 1$ [23]. The constant weighting values in the objective function are $K_1 = 1$, $K_2 = 1$, $K_3 = -1$, $\alpha = 2$.

Figures 1–4 show the results without vaccination, and we can see clearly the spread of the disease throughout the domain, for both situations of the onset of the disease.

In Figure 3, in the absence of control and for the two situations at the beginning of the epidemic, it can be seen that the number of infected individuals increases by $I_0(x, y) = 0$ for $(x, y)/\notin \Omega_i$, $i = 1;\ 2$, to about 9 people infected. To validate our vaccination strategy, we consider two ways to do it.

The first is to inject vaccination after 30 days of the onset of the disease, while for the second, vaccination begins on the first day of the epidemic. In Figures 5 and 6, when injecting the

Table 1: Initial conditions and parameter values.

Notations	Value	Description (units)
$S_0(x,y)$	5 for $(x,y) \in \Omega_i \quad i=1,2$ 10 for $(x,y) \notin \Omega_i$	Initial susceptible population (people/km^2)
$E_0(x,y)$	3 for $(x,y) \in \Omega_i \quad i=1,2$ 0 for $(x,y) \notin \Omega_i$	Initial exposed population (people/km^2)
$I_0(x,y)$	3 for $(x,y) \in \Omega_i \quad i=1,2$ 0 for $(x,y) \notin \Omega_i$	Initial infected population (people/km^2)
$R_0(x,y)$	1 for $(x,y) \in \Omega$	Initial recovered population (people/km^2)
σ^{-1}	$(1/0.09)$	Average duration of latent
μ	0.01	Rate of deaths (day^{-1})
β	1	Transmission constant
γ^{-1}	$(1/0.04)$	Infective periods
ω	0.2	The rate of losing immunity (day^{-1})
d_S	0.2	Diffusion rate for susceptible
d_E	0.3	Diffusion rate for exposed
d_I	0.5	Diffusion rate for infected
d_R	0.5	Diffusion rate for recovered
t	$[1, 60]$	Time period (day)

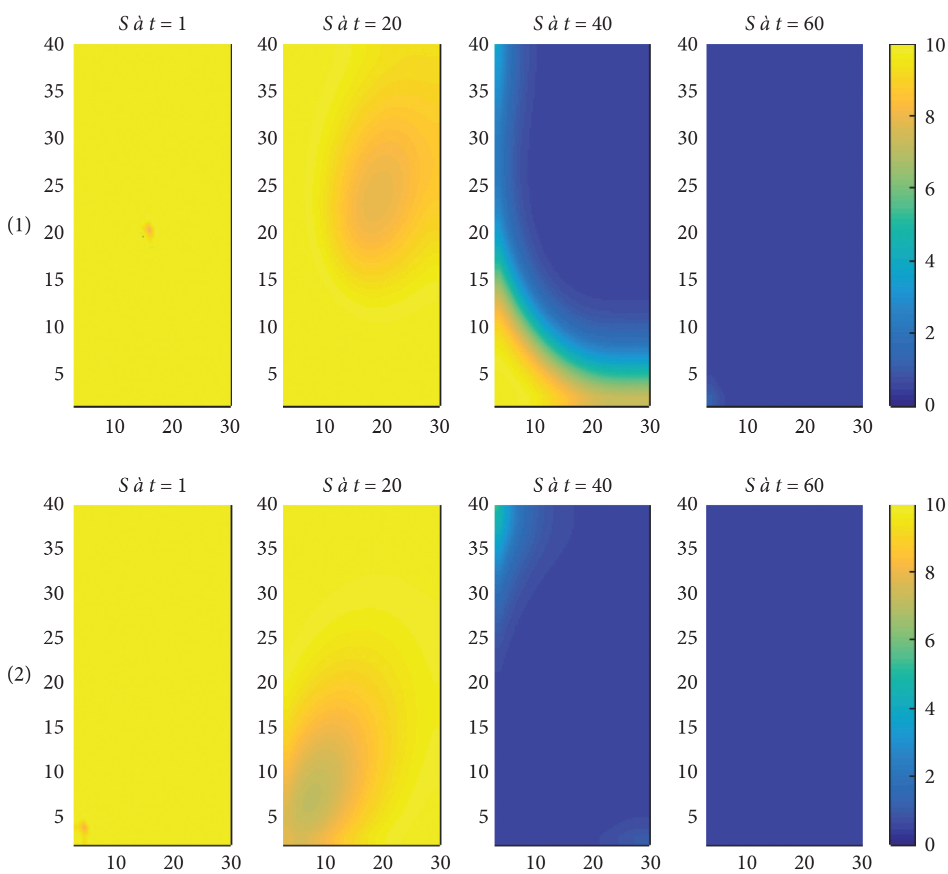

Figure 1: Susceptible behavior within Ω without control.

vaccine after 30 days, it is easy to see the effectiveness of our control strategy in slowing the spread of the epidemic, since in Figure 6, after 60 days, the number of infected individuals has decreased to about 6 infected individuals, which is a gain by comparing it with the uncontrolled case. Another benefit of our control strategy is illustrated in Figure 7 for recovered individuals, as the number of individuals has increased to reach 4 recovered individuals. In the second case, when the vaccination against the disease starts from the first day ($t = 1$), the effectiveness of our vaccination strategy to control the spread of the epidemic is clear, since the disease disappears quickly (Figures 8–10).

The comparison of these results with those obtained when vaccination is introduced at 30 days allows to conclude the influence of the vaccination from the first days for the elimination of the epidemic.

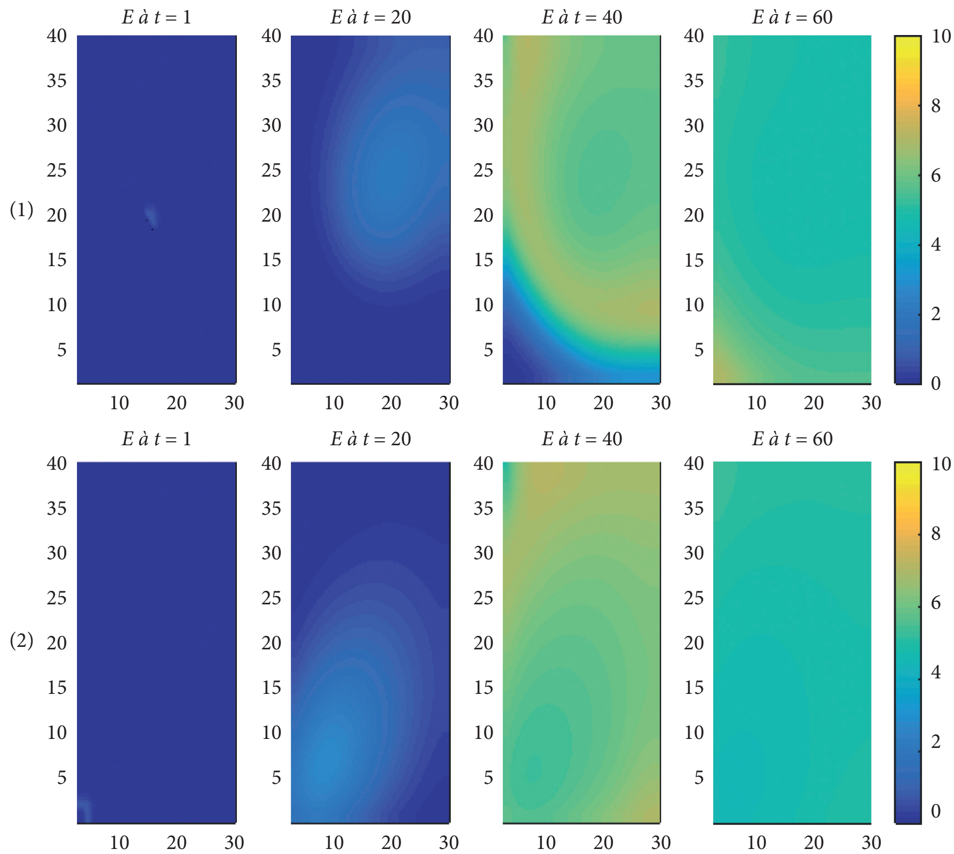

FIGURE 2: Exposed behavior within Ω without control.

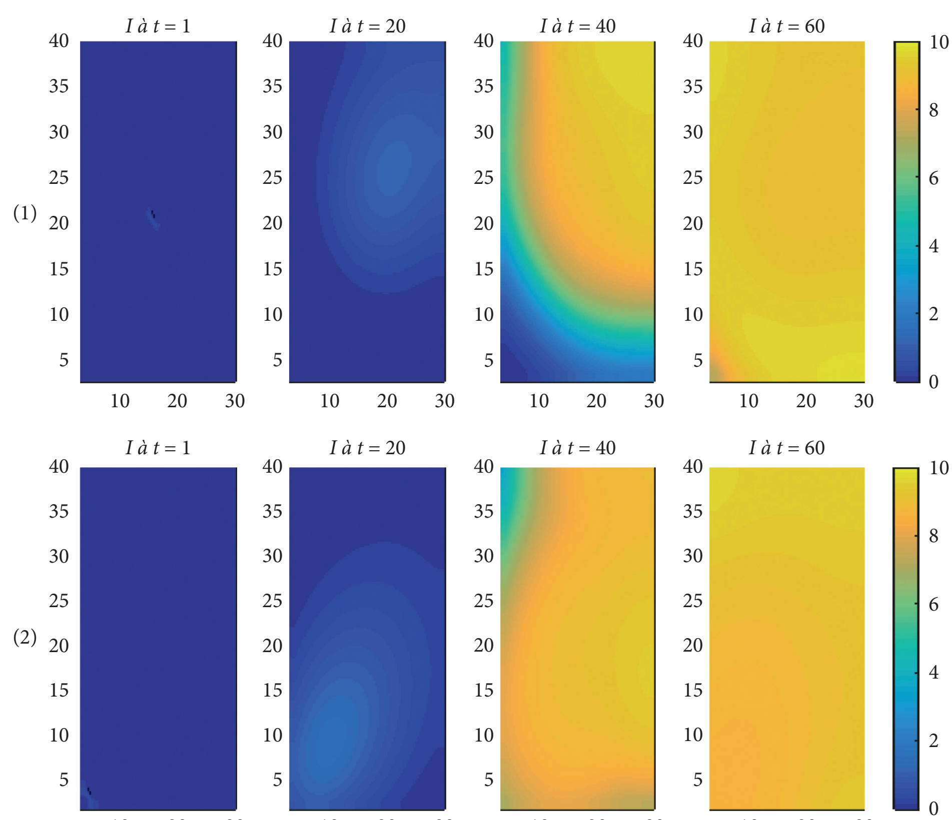

FIGURE 3: Infected behavior within Ω without control.

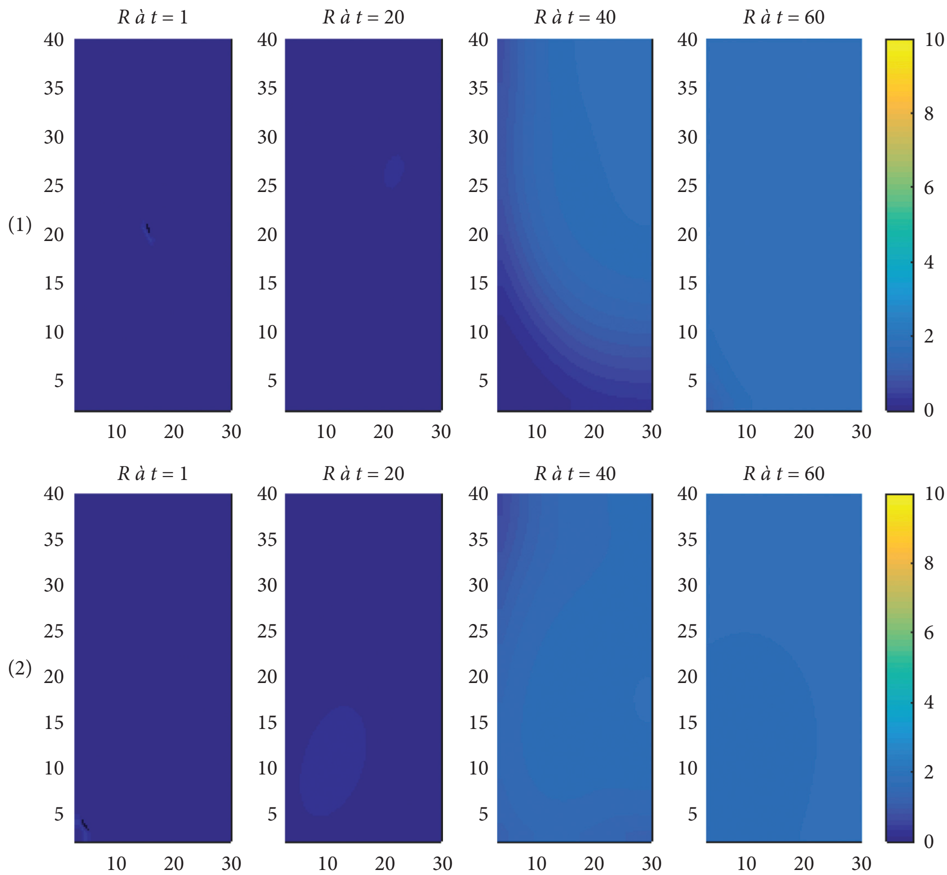

Figure 4: Recovered behavior within Ω without control.

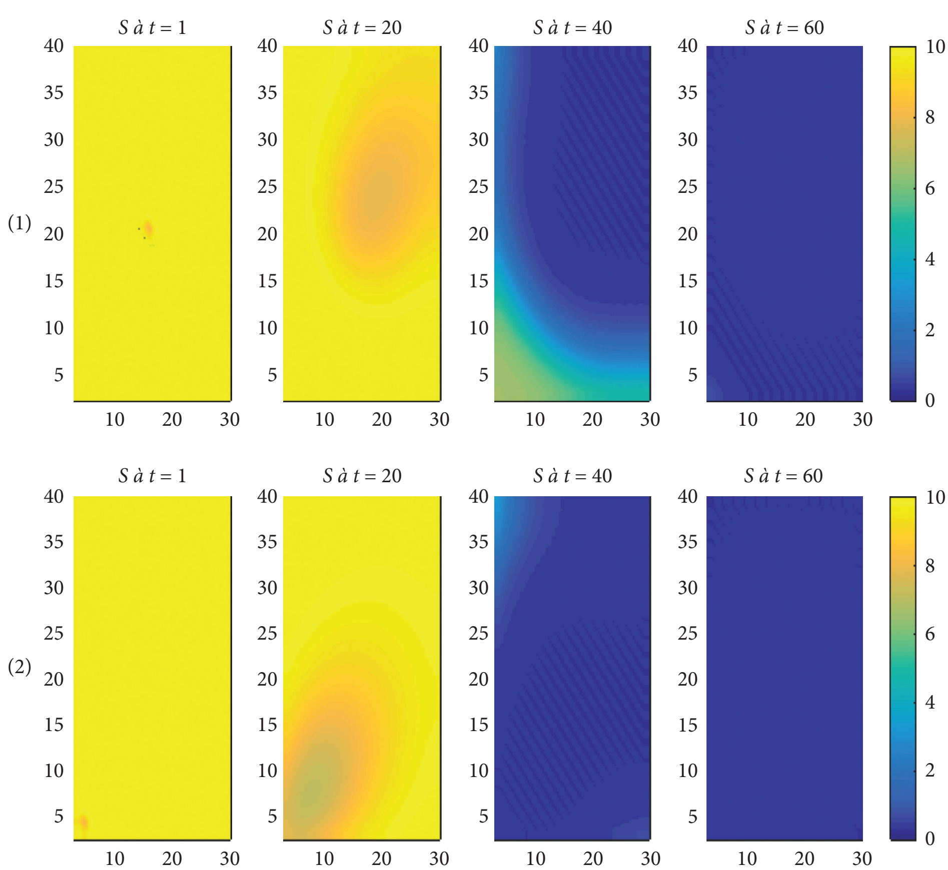

Figure 5: Susceptible behavior within Ω without control (vaccine strategy after 30 days).

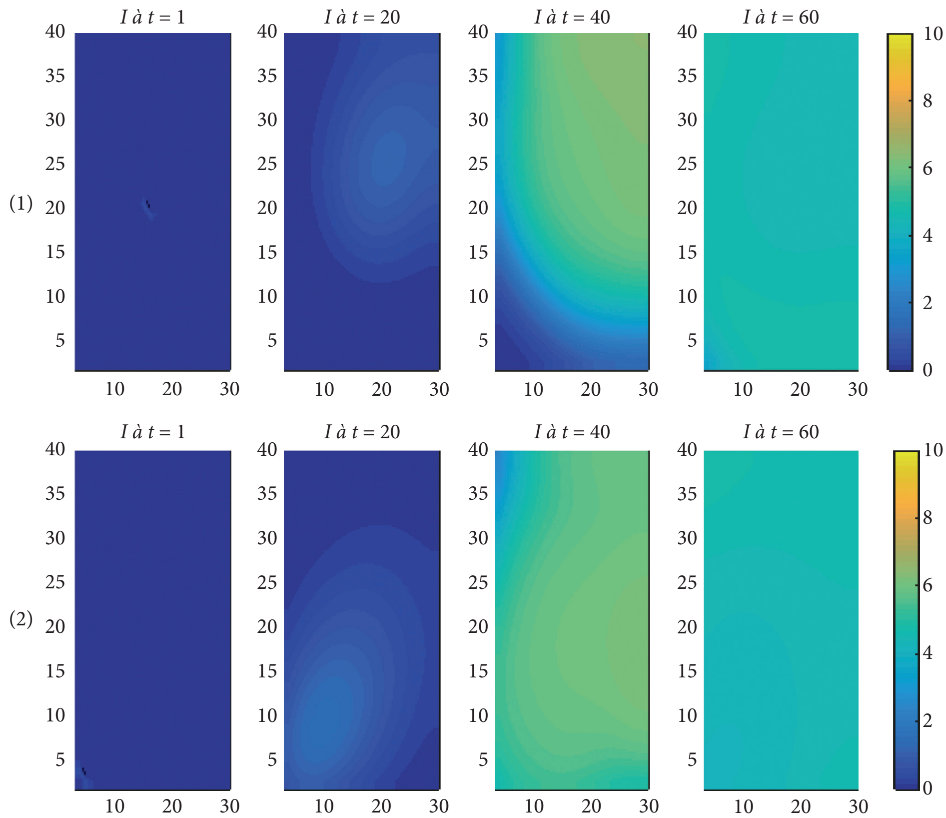

Figure 6: Infected behavior within Ω without control (vaccine strategy after 30 days).

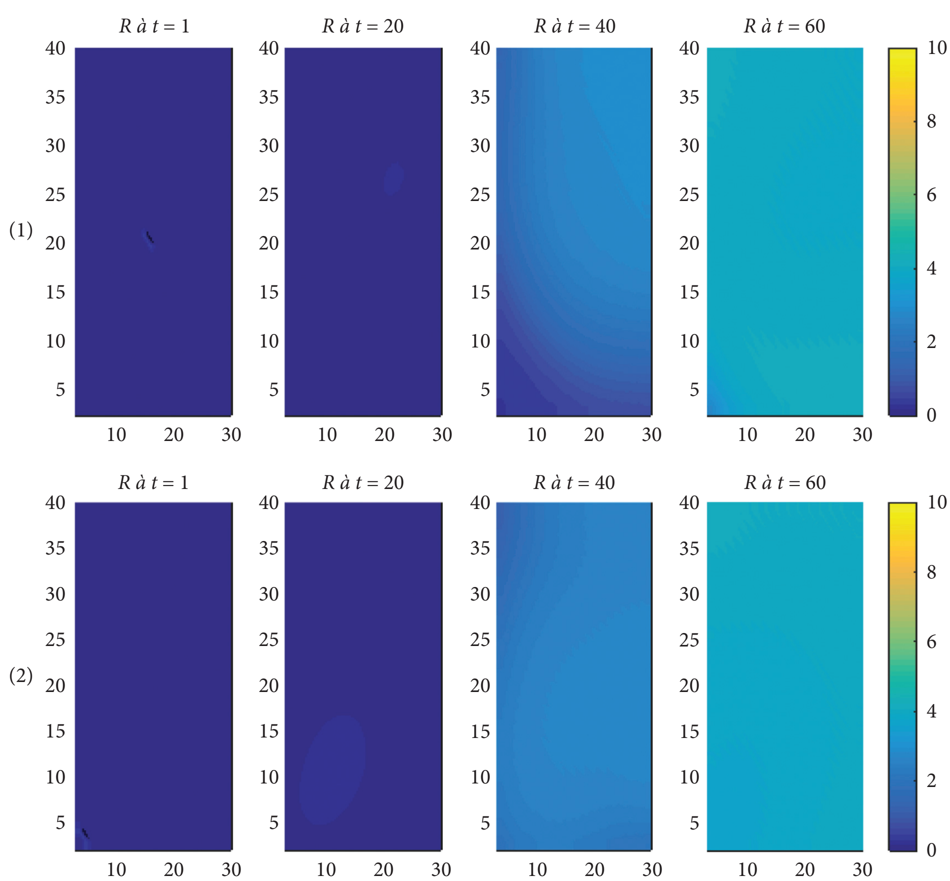

Figure 7: Recovered behavior within Ω with control (vaccine strategy after 30 days).

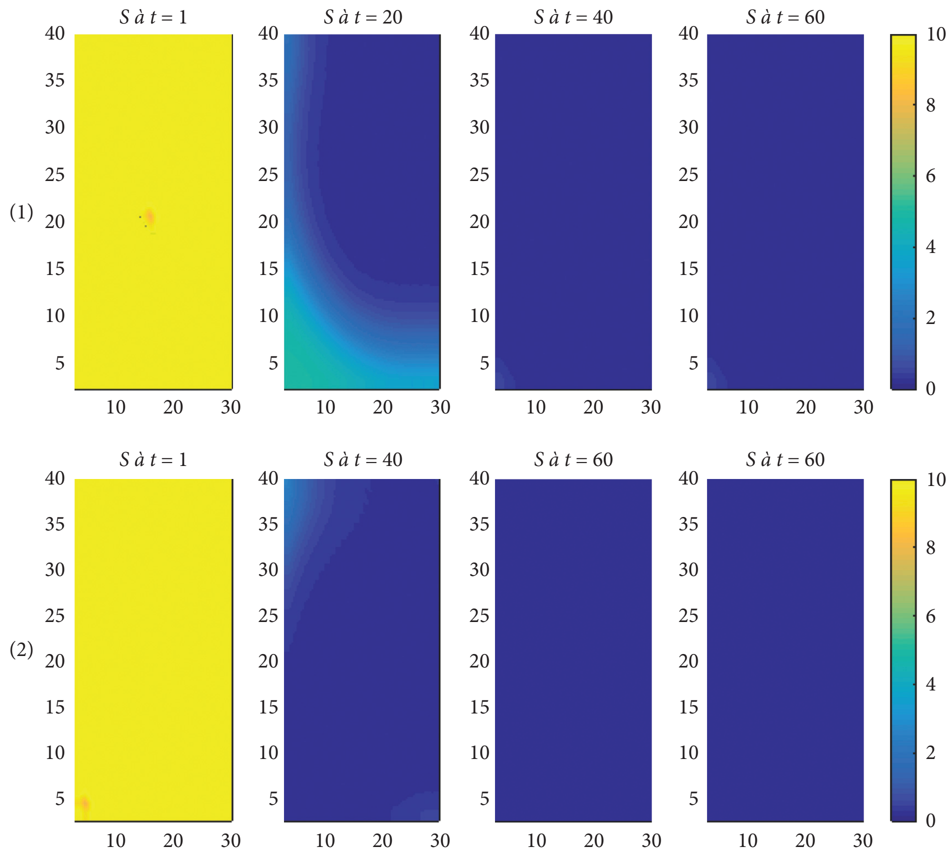

Figure 8: Susceptible behavior within Ω with control (vaccine strategy starts from the first day).

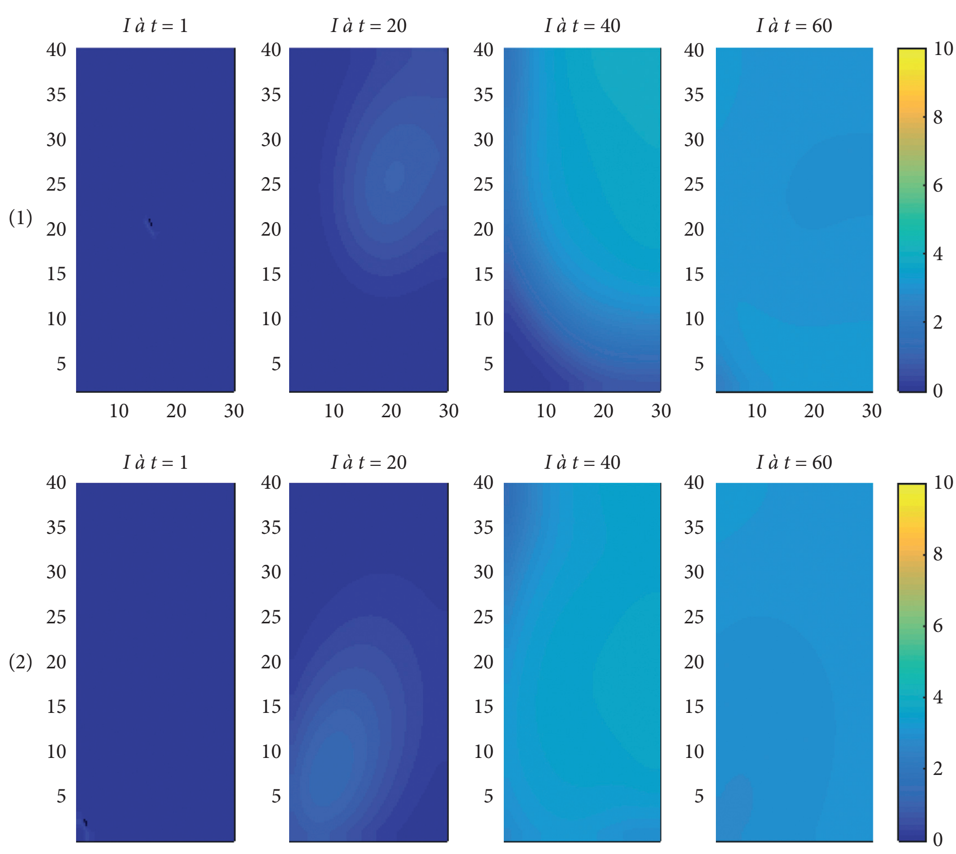

Figure 9: Infected behavior within Ω without control (vaccine strategy starts from the first day).

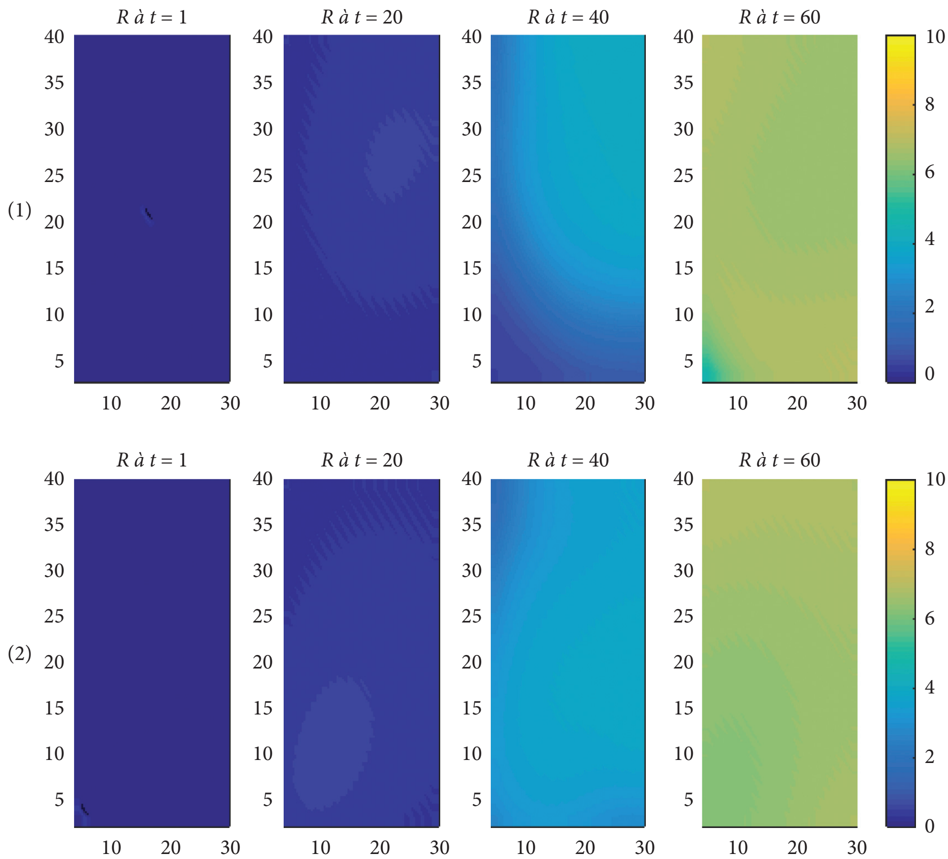

Figure 10: Recovered behavior within Ω with control (vaccine strategy starts from the first day).

7. Conclusion

In this work, we proposed an effective vaccination strategy for a two-dimensional spatiotemporal SEIR model, in order to minimize the number of susceptible and infected individuals and to maximize the number of individuals recovered. To achieve this goal, we have based our mathematical work on the use of semigroup theory and optimal control to show the existence of solutions for our state system, and these solutions are positive and related. In addition, we have proven the existence and characterization of optimal control that achieves both our goal and reduce the cost of vaccination. The characterization of the control was made in terms of state functions and adjoint functions. A numerical simulation was given to validate our control strategy.

Appendix

First, recall a general existence result which we use in the sequel (Proposition 1.2, p. 175, [24]; see also [19, 25]). Consider the initial value problem

$$\begin{cases} \dfrac{\partial z}{\partial t} = Az(t) + g(t, z(t)), \quad t \in [0, T], \\ \\ z(0) = z_0, \end{cases} \tag{A.1}$$

where A is a linear operator defined on a Banach space X, with the domain $D(A)$ and $g : [0, \mathrm{T}] \times X \longrightarrow X$ is a given function. If X is a Hilbert space endowed with the scalar product $(\cdot, \cdot)$, then the linear operator A is called dissipative if $(Az, z) \leq 0$, $(\forall z \in D(A))$.

Theorem A.1. *Let X be a real Banach space, $A : D(A) \subseteq X \longrightarrow X$ be the infinitesimal generator of a C_0–semigroup of linear contractions $S(t)$, $t \geq 0$ on X, and $g : [0, \mathrm{T}] \times X \longrightarrow X$ be a function measurable in t and Lipschitz continuous in $x \in X$, uniformly with respect to $t \in [0, T]$.*

- *(i) If $z_0 \in X$, then problem (A.1) admits a unique mild solution, i.e., a function $z \in C([0, T]; X)$ which verifies the equality $z(t) = S(t)z_0 + \int_0^t S(t-s)g(s, z(s))ds$, $(\forall t \in [0, T])$.*
- *(ii) If X is a Hilbert space, A is self-adjoint and dissipative on X, and $z_0 \in D(A)$, then the mild solution is in fact a strong solution and $z \in W^{1,2}([0, T];\ X \cap L^2(0, T;\ D(A)))$*

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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