Research Article

Efficient Time-Frequency Localization of a Signal

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A new representation of the Fourier transform in terms of time and scale localization is discussed that uses a newly coined A-wavelet transform (Grigoryan 2005). The A-wavelet transform uses cosine- and sine-wavelet type functions, which employ, respectively, cosine and sine signals of length 2π . For a given frequency ω , the cosine- and sine-wavelet type functions are evaluated at time points separated by $2\pi/\omega$ on the time-axis. This is a two-parameter representation of a signal in terms of time and scale (frequency), and can find out frequency contents present in the signal at any time point using less computation. In this paper, we extend this work to provide further signal information in a better way and name it as A^* -wavelet transform. In our proposed work, we use cosine and sine signals defined over the time intervals, each of length $2\pi m/(2^n \omega)$, $m \leq 2^n$, m and n are nonnegative integers, to develop cosine- and sine-type wavelets. Using smaller time intervals provides sharper frequency localization in the time-frequency plane as the frequency is inversely proportional to the time. It further reduces the computation for evaluating the Fourier transform at a given frequency. The A-wavelet transform can be derived as a special case of the A^* -wavelet transform.

1. Introduction

There are two domains for representing a signal: time and frequency domains. Depending upon the information required, either representation can be used. Fourier analysis has been the main technique for transforming a signal from one representation into another. In spite of the fact that the Fourier analysis is an ideal solution for deterministic and stationary signals, it is hardly of any use for time-varying signals or nonstationary signals, because analysis of these types of signals compromises between their transition and long term behaviors. For these types of signals, a transform is desired to represent the signal in a two-parameter form. The very first such transform in literature is the short time Fourier transform (STFT) [1]. This transform uses a time-window function to decompose the signal into segments and then the Fourier analysis is carried out on individual segments. The STFT provides local features that are present in the signal in a limited form because it uses the time-window function of fixed width for all frequency contents and thus it is unable to extract the required information in any given signal. So, a transform that can represent the signal in two-parameter form and uses the time-window function of different lengths

is needed. Application of the short time Fourier transform multiple times by using different time-window functions can be a possible solution, but it is a cumbersome process and hence may not be practically feasible. So, a transform that can represent the signal in two-parameter form and uses a time-window function of varying length is required. Such functions/transforms exist in literature since long time. Prior to a decade and so, their applications had hardly been explored in signal processing. These transforms are called wavelet transforms [2–8]. In the last two decades and so, large numbers of articles have appeared on the wavelet transforms in literature [9–13]. A wavelet transform employs a set of variable length time-window functions derived from a single time-window function, called the mother wavelet. The advantage of this type of representation of a signal is that the signal information can be obtained in different frequency bands.

There have been some studies to represent the Fourier transform in such a way that one does not need to know the function on the entire time-axis in order to find its Fourier transform in a given frequency range, possibly at a given frequency. One such study has been reported in [14]. In that study, the cosine and sine signals truncated over a period of

2π have been considered to develop the cosine- and sine-type wavelets, which have been used to represent the Fourier transform in two-parameter form unlike the traditional form. This representation is important because one does not need to compute the values of cosine- and sine-type wavelets at all time points. Their values are computed at certain time points, which are multiple of $2\pi/\omega$. In other words, this provides the signal information along the curves that are separated out by $2\pi/\omega$ in the time-frequency plane, thus requiring less computation. Besides, it provides multiresolution signal processing. This work has motivated to develop a new representation of the Fourier transform that can provide the signal information in time-frequency plane along the curves separated out by less than $2\pi/\omega$ distance. The proposed representation, named as A^* -wavelet transform, further reduces the computations and provides sharper frequency localization. In this paper, the cosine- and sine-type wavelets are derived from the cosine and sine signals. The length of the time intervals over which the cosine and sine signals are defined can be adapted as per the application requirement. Thus, the Fourier transform represented in terms of cosine- and sine-type wavelet transform can provide the required signal information by adjusting the time interval length of the cosine and sine signals. For example, the wavelets derived from the (co)sinusoidal signals with smaller period give sharper frequency localization and vice versa.

The rest of the paper is organized as follows. Section 2 reviews A -wavelet transform. In Section 3, the A^* -wavelet transform is proposed. Section 4 provides results and discussions and Section 5 elucidates the theory by using examples. Lastly, the paper is concluded in Section 6.

2. A-Wavelets

The Fourier analysis describes the global nature of the signals. It is beneficial if the signal does not have time-varying nature; that is, it is stationary. There are many time-varying signals, which have practical applications such as speech signals, audio signals, and video signals. For such applications, the Fourier analysis is less beneficial as these signals contain local characteristics different from global ones, which are required for further analysis. The short-time Fourier transform (STFT) provides solutions for such types of applications by using a real and symmetric time-window function having unit norm in $L_2(R)$ and nonzero only in the region of interest. Here $L_2(R)$ refers to the space of square-integrable functions. The STFT of a signal $f(t)$, denoted by $F(t, \omega)$, using the window function $g(t)$ and reconstruction of the original function $f(t)$ from its STFT are given by

$$F(t, \omega) = \int_{-\infty}^{\infty} f(\tau) g(\tau - t) e^{-j\omega\tau} d\tau \quad \text{for } \omega, t \in (-\infty, \infty)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau, \omega) g(t - \tau) e^{j\omega\tau} d\omega d\tau. \quad (1)$$

The time-window function $g(t)$ is of fixed size, which limits the STFT to determine the locations of occurrence of many frequency contents. This problem has been alleviated by

applications of the wavelet transforms. A wavelet transform employs a varying size time-window function, called wavelet function, by dilating and translating in its time parameter. Mathematically, the wavelet transform, denoted by $W(a, b)$, of an arbitrary function $f(t)$ using the wavelet function $\psi(t)$ is given by

$$W(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \psi\left(\frac{t-b}{a}\right) dt \quad (2)$$

for $a > 0, \quad b \in (-\infty, \infty)$,

where a and b are called dilation and translation parameters, respectively.

In [14], A -wavelet transform has been discussed that represents the Fourier transform in terms of cosine- and sine-type wavelet transforms. The cosine- and sine-type wavelet transforms are based on the truncated cosinusoidal and sinusoidal signals, denoted by $\psi(t)$ and $\varphi(t)$, respectively, and they are defined over a time interval of length 2π ; that is,

$$\psi(t) = \begin{cases} \cos(t) & t \in [-\pi, \pi] \\ 0 & \text{otherwise,} \end{cases} \quad (3a)$$

$$\varphi(t) = \begin{cases} \sin(t) & t \in [-\pi, \pi] \\ 0 & \text{otherwise.} \end{cases} \quad (3b)$$

Denote $\psi_{\omega,b}(t) = \psi(\omega[t - b])$ and $\varphi_{\omega,b}(t) = \varphi(\omega[t - b])$ for $t \in (-\infty, \infty)$ and define $T_{\psi}(\omega, b_n)$ and $T_{\varphi}(\omega, b_n)$ using $\psi_{\omega,b}(t)$ and $\varphi_{\omega,b}(t)$; that is,

$$T_{\psi}(\omega, b) = \int_{-\infty}^{\infty} f(t) \psi_{\omega,b}(t) dt, \quad (4)$$

$$T_{\varphi}(\omega, b) = \int_{-\infty}^{\infty} f(t) \varphi_{\omega,b}(t) dt.$$

The functions $T_{\psi}(\omega, b_n)$ and $T_{\varphi}(\omega, b_n)$ are called cosine-wavelet (C -wavelet) and sine-wavelet (S -wavelet), respectively. The A -wavelet transform representation of the Fourier transform $F(\omega)$ is given by

$$F(\omega) = \sum_{n=-\infty}^{\infty} T_{\psi}(\omega, b_n) - j \sum_{n=-\infty}^{\infty} T_{\varphi}(\omega, b_n), \quad (5)$$

where j is an imaginary quantity.

This representation (i.e., C -wavelet and S -wavelet) reduces the computations because, for a given frequency ω , the functions $T_{\psi}(\omega, b_n)$ and $T_{\varphi}(\omega, b_n)$ are evaluated at the centers of the prespecified time intervals of length $2\pi/\omega$ in time parameter. The frequency and time parameters are related by the relation $\omega t = 2\pi$. The number of time intervals on time-axis is given by $2N(\omega) + 1$ (i.e., $-N(\omega), -N(\omega) + 1, \dots, -1, 0, 1, \dots, N(\omega), N(\omega) + 1$), where $N(\omega) = 0$ for $\omega \in [0, 1/3]$, and for $\omega > 1/3$, $N(\omega) = \lfloor (3\omega - 1)/2 \rfloor$. For details on $N(\omega)$ refer to [14]. The A -wavelet transform provides the signal information in the time-frequency plane along the curves $\omega t = 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$. In the next section, we discuss A^* -wavelet transform, an extension of A -wavelet transform, that provides the signal information along the curves separated by less 2π in the time-frequency plane.

3. A^* -Wavelet Transform

The Fourier transform of a function in terms of A -wavelet transform is expressed in terms of C - and S -type wavelet transforms and these transforms have been derived from the cosine and sine functions over a time interval of length 2π (refer to (3a) and (3b)). The A -wavelet transform provides the signal information along the curves $\omega t = 2n\pi$, where n is an integer, that is, all such curves that are separated by 2π in the time-frequency plane. We call the curves $\omega t = 2n\pi$, $n = 0, \pm 1, \pm 2, \dots$, as primary curves and the curves between the primary curves as the secondary curves. The signal information along the primary curves can be obtained using the A -wavelet transform, but the A -wavelet transform cannot provide the signal information along the secondary curves. We want to get the signal information along the secondary curves in the current paper. For doing this, we develop C - and S -type wavelets using the cosine and sine signals defined over a time interval of length less than 2π . The interval length is parameterized by introducing two new integer variables n and m . Define the functions $\psi_{m,n}(t)$ and $\varphi_{m,n}(t)$ as follows:

$$\psi_{m,n}(t) = \begin{cases} \cos(t) & t \in \left[-\frac{\pi m}{2^n}, \frac{\pi m}{2^n}\right] \\ 0 & \text{otherwise,} \end{cases} \quad (6a)$$

$$\varphi_{m,n}(t) = \begin{cases} \sin(t) & t \in \left[-\frac{\pi m}{2^n}, \frac{\pi m}{2^n}\right] \\ 0 & \text{otherwise.} \end{cases} \quad (6b)$$

We look at different values of m and n . For $m = 0$, $\psi_{m,n}(t)$ is the Dirac Delta function and $\varphi_{m,n}(t)$ is identically zero and the outcome is simply a sampled signal. For $m = 2^n$, the functions $\psi_{m,n}(t)$ and $\varphi_{m,n}(t)$ are exactly the same as the $\psi(t)$ and $\varphi(t)$ functions, respectively, which are given in (3a) and (3b). For $m > 2^n$, the functions $\psi_{m,n}(t)$ and $\varphi_{m,n}(t)$ have nonzero values over the time duration, that is, more than one time interval. We are interested to have these signals as nonzero over a single time interval, which gives a condition $m \leq 2^n$. Without loss of generality, we may assume that m and n assume nonnegative integer values. For negative m , the same discussion will hold as that for positive m , but on the negative time-axis. Negative n will increase the number of time intervals over which the functions $\psi_{m,n}(t)$ and $\varphi_{m,n}(t)$ have nonzero values. The condition $m < 2^n$ makes the time interval length smaller over which the functions $\psi_{m,n}(t)$ and $\varphi_{m,n}(t)$ are nonzero and this is the requirement to develop the current work. By doing this, we intend to get the signal information along the secondary curves.

We construct a family of functions $\{\psi_{m,n,\omega,k}(t), \varphi_{m,n,\omega,k}(t)\}$ from $\psi_{m,n}(t)$ and $\varphi_{m,n}(t)$ by using the time-scaling and shift transformation; that is, $t \rightarrow \omega t$ and $t \rightarrow (t - b_k)$; that is,

$$\begin{aligned} \psi_{m,n,\omega,k}(t) &= \psi_{m,n}(\omega(t - b_k)), \\ \varphi_{m,n,\omega,k}(t) &= \varphi_{m,n}(\omega(t - b_k)), \end{aligned} \quad (7)$$

for $t \in (-\infty, \infty)$,

where the frequency variable ω assumes real values and $b_k = 2\pi k m / (2^n \omega)$, $k = 0, \pm 1, \pm 2, \dots, \pm[(3\omega - 1)2^n / 2m]$. In general,

k is infinite; however, for finite support signals, its value is finite.

Consider a function $f(t)$ that satisfies all the conditions for existence of Fourier transform. Its Fourier transform $F(\omega)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt. \quad (8)$$

For $\omega = 0$, the Fourier transform is a real quantity that is given by integrating the function. This is a trivial case and needs no further discussion. Our further discussion is meant for $\omega \neq 0$ frequencies. We write $F(\omega)$ that is defined in (8) as follows:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt. \quad (9)$$

We try to write the integrals in (9) in terms of $\psi_{m,n,\omega,k}(t)$ and $\varphi_{m,n,\omega,k}(t)$ functions that are defined in (7). For this purpose, we uniformly divide the time-axis into disjoint intervals of length $2\pi m / (2^n \omega)$ that are centered at $2\pi m k / (2^n \omega)$, $k = 0, \pm 1, \pm 2, \dots$. Denote k th interval by $I_{m,n,k}$, which can be defined as follows:

$$I_{m,n,k} = \left[\frac{(2k-1)m\pi}{2^n \omega}, \frac{(2k+1)m\pi}{2^n \omega} \right), \quad k = 0, \pm 1, \pm 2, \dots \quad (10)$$

In (10), k assumes all integer values in general; however, its maximum value is determined by the frequencies, denoting them by $\Theta_{k,\omega}$, that need be analyzed. For example, $k = 1$ and $\omega = 3\pi$ gives the frequency range $\Theta_{1,3\pi} = [m/(3 \cdot 2^n), 3m/(3 \cdot 2^n)]$. The secondary curves are given by the equation $\omega t = 2\pi m k / 2^n$, ($m \neq 2^n$), where m is a positive integer and n is nonnegative integer. We can write the Fourier transform $F(\omega)$ defined in (9) over the disjoint intervals $I_{m,n,k}$ as follows:

$$\begin{aligned} F(\omega) &= \sum_{k=-\infty}^{\infty} \int_{I_{m,n,k}} f(t) \cos(\omega t) dt \\ &\quad - j \sum_{k=-\infty}^{\infty} \int_{I_{m,n,k}} f(t) \sin(\omega t) dt. \end{aligned} \quad (11)$$

Changing the limits of integration, we have

$$\begin{aligned} F(\omega) &= \sum_{k=-\infty}^{\infty} \int_{I_{m,n,0}} f\left(t + \frac{2\pi m k}{2^n \omega}\right) \cos\left(\omega\left(t + \frac{2\pi m k}{2^n \omega}\right)\right) dt \\ &\quad - j \sum_{k=-\infty}^{\infty} \int_{I_{m,n,0}} f\left(t + \frac{2\pi m k}{2^n \omega}\right) \\ &\quad \times \sin\left(\omega\left(t + \frac{2\pi m k}{2^n \omega}\right)\right) dt. \end{aligned} \quad (12)$$

On simplifying, it gives

$$F(\omega) = \sum_{k=-\infty}^{\infty} \left(\int_{I_{m,n,0}} f\left(t + \frac{2\pi mk}{2^n \omega}\right) \cos(\omega t) dt - j \int_{I_{m,n,0}} f\left(t + \frac{2\pi mk}{2^n \omega}\right) \sin(\omega t) dt \right) \times e^{-j2\pi mk/2^n}, \quad (13)$$

which can further be written as follows:

$$F(\omega) = \sum_{k=-\infty}^{\infty} \left(\int_{I_{m,n,k}} f(t) \cos\left(\omega\left(t - \frac{2\pi mk}{2^n \omega}\right)\right) dt - j \int_{I_{m,n,k}} f(t) \sin\left(\omega\left(t - \frac{2\pi mk}{2^n \omega}\right)\right) dt \right) \times e^{-j2\pi mk/2^n}. \quad (14)$$

In (14) the summation contains the term $e^{-j2\pi mk/2^n}$. We can write $2^n/m = sm + r$, where s is a nonnegative integer and r is an integer such that $0 \leq r < m$. Thus, we have $e^{-j2\pi mk/2^n} = e^{-j2\pi k/(ms+r)}$, $r = 0, 1, 2, \dots, m-1$, $k = 0, \pm 1, \pm 2, \dots$. Since k assumes all integer values including zero, the expression $e^{-j2\pi k/(ms+r)}$ will have $(sm+r)$ distinct values for all values of k . For positive and negative values of k , we will have same results. For positive k , we will have intervals along positive axis and for negative k , we will have intervals along negative axis; henceforth we will confine k as nonnegative integer. For fixed m , increasing n will increase the value of s , but r will have only m distinct values. Increasing s will help in analyzing the high frequencies. Increasing m will increase the values of r ($r \neq 0$) and that in turn will increase the secondary curves. It may be noticed that the curves for $r = 0$ correspond to the primary curves. Increasing n and m will increase both primary and secondary curves in the time-frequency plane. The primary curves depend upon the time interval in which we want to analyze the desired frequency contents and the secondary curves provide further details. In limiting case, the distance between two adjacent curves can be made arbitrarily small by increasing m . If the signal is periodic, then the primary curves are sufficient enough to provide the desired information present in the signal; otherwise the secondary curves are also needed. The information corresponding to the primary curves is the same as that provided by the Fourier series for the periodic signals. The number of secondary curves in a given time interval is decided by the frequencies to be analyzed in the given signal.

We write (14) as follows:

$$F(\omega) = \sum_{k=-\infty}^{\infty} \left(\int_{I_{m,n,k}} f(t) \cos\left(\omega\left(t - \frac{2\pi k}{(sm+r)\omega}\right)\right) dt - j \int_{I_{m,n,k}} f(t) \sin\left(\omega\left(t - \frac{2\pi k}{(sm+r)\omega}\right)\right) dt \right) \times e^{-j2\pi k/(sm+r)}. \quad (15)$$

$F(\omega)$ can be written as a sum of $(sm+r)$ number of terms; that is,

$$F(\omega) = \sum_{q=0}^{sm+r-1} F_q(\omega) e^{-j(2\pi q/(sm+r))}. \quad (16)$$

We can write $k = (sm+r)p + q$, $0 \leq q < (sm+r)$. From (15), $F_q(\omega)$ is given by

$$\begin{aligned} F_q(\omega) &= \sum_{p=-\infty}^{\infty} \left(\int_{I_{m,n,(sm+r)p+q}} f(t) \times \cos\left(\omega\left(t - \frac{2\pi((sm+r)p+q)}{(sm+r)\omega}\right)\right) dt - j \int_{I_{m,n,(sm+r)p+q}} f(t) \times \sin\left(\omega\left(t - \frac{2\pi((sm+r)p+q)}{(sm+r)\omega}\right)\right) dt \right). \end{aligned} \quad (17)$$

Using the cosine and sine functions in terms of $\psi_{m,\omega,b_n}(t)$ and $\varphi_{m,\omega,b_n}(t)$ (refer to (6a), (6b), and (7)), we can write (17) as follows:

$$F_q(\omega) = \sum_{p=-\infty}^{\infty} \left(\int_{I_{m,n,(sm+r)p+q}} f(t) \psi_{m,n,\omega,b_{(sm+r)p+q}}(t) dt - j \int_{I_{m,n,(sm+r)p+q}} f(t) \varphi_{m,n,\omega,b_{(sm+r)p+q}}(t) dt \right), \quad (18)$$

where

$$b_{(sm+r)p+q} = \frac{2\pi((sm+r)p+q)}{(sm+r)\omega}. \quad (19)$$

The integrals in (18), for a fixed value of q , are defined over $((sm+r)p+q)$ th interval of length $2\pi m/(2^n \omega) = 2\pi/(sm+r)\omega$ that is centered at $2\pi((sm+r)p+q)/((sm+r)\omega)$, $p = 0, \pm 1, \pm 2, \dots \lfloor (3(sm+r)\omega - 1)/2 \rfloor$. Denote these integrals as follows:

$$T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q}) = \int_{I_{m,n,(sm+r)p+q}} f(t) \psi_{m,n,\omega,b_{(sm+r)p+q}}(t) dt, \quad (20a)$$

$$T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q}) = \int_{I_{m,n,(sm+r)p+q}} f(t) \varphi_{m,n,\omega,b_{(sm+r)p+q}}(t) dt. \quad (20b)$$

At the point $(0,0)$, $T_{m,n,\psi,q}(0,0) = \int_{-\infty}^{\infty} f(t) dt$ and $T_{m,n,\varphi,q}(0,0) = 0$.

Writing expressions given in (18) in terms of $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q})$, we get

$$F_q(\omega) = \sum_{p=-\infty}^{\infty} \left(T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q}) - jT_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q}) \right). \quad (21)$$

Using $F_q(\omega)$ from (21) in (16) gives

$$\begin{aligned} F(\omega) &= \sum_{q=0}^{(sm+r)-1} \left(\cos\left(\frac{2\pi q}{(sm+r)}\right) \right. \\ &\quad \times \sum_{p=-\infty}^{\infty} T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q}) \\ &\quad - \sin\left(\frac{2\pi q}{(sm+r)}\right) \\ &\quad \times \sum_{p=-\infty}^{\infty} T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q}) \Bigg) \\ &\quad - j \sum_{q=0}^{(sm+r)-1} \left(\sin\left(\frac{2\pi q}{(sm+r)}\right) \right. \\ &\quad \times \sum_{p=-\infty}^{\infty} T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q}) \\ &\quad + \cos\left(\frac{2\pi q}{(sm+r)}\right) \\ &\quad \times \sum_{p=-\infty}^{\infty} T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q}) \Bigg). \end{aligned} \quad (22)$$

The terms in (22) are the weighted sums of different cosine- and sine- wavelet transforms and it is named as A^* -wavelet transform, an extension of A -wavelet transform. The relation (22) is the main outcome of this paper.

4. Results and Discussions

The Fourier transform in (16) has been decomposed into different components and each component that has been represented in terms of time and scale localization can be analyzed explicitly and independently. The work [14] contains only one component; whereas the current work contains multiple components. Thus, it provides much more information about the desired signal. In fact, the signal information obtained in [14] can be given by the first component $F_0(\omega)$ in a better localized form using the current work. The signals $F_i(\omega)$ ($i \neq 0$) give the information about the signal along the curves $\omega b_{m,n} = 2\pi/(i+1)$, $i = 1, 2, 3, \dots, (sm+r-1)$ in the time-frequency plane. For $i = 0$, it gives information along the primary curves. The equation $\omega b_{m,n} = 2\pi/i$ indeed represents a family of hyperbolic curves of second kind, one for each value of i , in the time-frequency plane. Maximum value of i or the number of hyperbolic curves is determined by the values of m and n . The set of functions $\{\psi_{m,n,\omega,k}(t), \varphi_{m,n,\omega,k}(t)\}$ constitutes a basis to

represent the Fourier transform of a given signal in terms of scale and time localization. The Fourier transform is given by the functions $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q})$ (these functions include the signal whose Fourier transform is determined). For a given frequency, these functions are evaluated at various points, each lying on a different curve. In this work, we have discretized the frequency-time plane in different curves. The value of n used in (6a) and (6b) determines the number of curves in the time-frequency plane. The maximum distance between two consecutive curves can be 2π , the coarsest discretization that has been considered in [14]. The minimum distance (in limiting case) can be zero, which corresponds to the continuous domain and in that case the number of curves is infinite. In conventional Fourier transform, we implicitly consider infinite number of curves and, in the Fourier series, finite number of curves, each separated by 2π . That is why infinite frequencies (or time) are needed to represent the Fourier transform at a given point in time (or frequency).

The results derived in this paper help finding the values of $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q})$ over the time interval $[-\pi m/2^n, \pi m/2^n] = [-\pi/(sm+r), \pi/(sm+r)]$. This can be analogous to a stack of rectangular blocks, each having time and scale (frequency) as its two sides. The zeroth block ($s = 0$) has the time parameter length equal to the signal support and the frequency length is decided by the frequencies to be analyzed. If the desired frequencies are not resolved in the zeroth block, then the first block ($s = 1$) is considered whose time parameter length is half of that of the zeroth block. If the first block fails to resolve the desired frequencies, then the second block ($s = 2$) is considered whose time parameter length is half of that of the first block or one-fourth of the signal support, and so on. It may be noted that the segment in time parameter of a block is appropriately chosen from the support of the signal maintaining the length of sth block as $t/2^s$, where t is the signal support. The zeroth block is used to analyze the smallest frequencies starting from zero. The parameter m specifies distinct (type) curves in a block and r specifies a particular (type) curve in that block. A block may contain several curves of the same type. Theoretically, a signal contains infinite frequencies, which means there are infinite numbers of blocks. For analyzing high frequencies, we need higher indexed blocks. For taking s sufficiently large, we can decrease the distance between two curves as small as we please. Practical signals are generally both band-limited and time-limited. Thus, most of the time, we have both the number of blocks and number of time samples (curves) as finite. The work [14] considers only primary curves for representing the signal information and the work [16] considers both primary and secondary curves for representing the signal information, but only one block that corresponds to the first block of our proposed work that consists of multiple blocks, each containing both primary as well as secondary curves. Figures 1(a)–1(d) show frequency contents in different blocks for a signal with support for $t \in (4, 36)$. We want to determine the frequency contents in the vicinity of the time point 4.5. Figure 1(d) shows that the frequency contents up to three decimal points can be

FIGURE 1: Frequency resolution in various blocks: (a) $s = 0$, (b) $s = 2$, (c) $s = 4$, and (d) $s = 6$.

resolved. If we want to resolve further closer frequencies, we need to increase the number of blocks.

The representation (22) has $(sm + r)$ components $F_i(\omega)$, $i = 0, 1, \dots, (sm + r - 1)$. Depending upon the domain under consideration in the time-frequency plane, the points on the curves are considered. For example, evaluating the frequencies $\omega \in [0, 1/12]$ in the time range $[-5, 5]$, only straight line passing through the origin and parallel to the frequency-axis is needed and for $\omega = 1.0$, four curves (i.e., four samples) are needed. Thus, we need to have the values of $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\phi,q}(\omega, b_{(sm+r)p+q})$ at the points $(\omega, 2\pi((sm+r)p+q)/((sm+r)\omega))$, where m is a fixed integer; $r = 0, 1, 2, \dots, (m-1)$; $q = 0, 1, 2, \dots, (ms+r-1)$; and $p = 0, \pm 1, \pm 2, \dots$. The set of points at which the signals $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\phi,q}(\omega, b_{(sm+r)p+q})$ are computed is given by

$$\mathfrak{R} = \left\{ \left(\omega, \frac{(2\pi((sm+r)p+q))}{((sm+r)\omega)} \right) : \omega \in (-\infty, \infty), \right. \\ m \text{ is a fixed integer and} \\ r = 0, 1, 2, \dots, (m-1), p = 0, \pm 1, \pm 2, \dots, \\ \left. \pm \left\lfloor \frac{(3(ms+r)\omega - 1)}{2} \right\rfloor, q = 0, 1, \dots, ms+r-1 \right\}. \quad (23)$$

FIGURE 2: Loci of points in time-frequency plane of S -wavelet transform for $m = 4$.

For $s = 0$ and $r = 1$, (22) gives

$$F(\omega) = F_0(\omega). \quad (24)$$

It is not difficult to show that the signal information given by $F_0(\omega)$ is the same as that obtained in [14]. For $m = 4$ and $n = 4$, the plot of the equation $\omega b_{m,n} = 2\pi m/2^n$ is shown in Figure 2. (Two consecutive curves are $\pi/4$ apart). Figure 2 helps determining the number of points in set $\mathfrak{R}$ defined in (23) in the time-frequency plane for a given frequency range. For example, for all frequencies less than $1/12$, only one point is needed; for frequencies less than $1/4$, two points are needed, and so on. In this figure, the curves drawn correspond to the loci of points in

TABLE 1: Fourier transform computation for the frequency $\omega = 1.75$ of a function defined in time duration $(-10, 10)$.

$p/\text{Component}$	$F_0(\omega)$	$F_1(\omega)$	$F_2(\omega)$	$F_3(\omega)$
$0, \pm 4, \pm 8$	Yes			
$\pm 1, \pm 5, \pm 9$		Yes		
$\pm 2, \pm 6, \pm 10$			Yes	
$\pm 3, \pm 7$				Yes

the time-frequency plane for $\omega \in (0, 5)$, $b_n \in (-5, 5)$, where b_n signifies the shift in time and $m = 4$. In this figure, there are four types of curves, each corresponding to $F_i(\omega)$, $i = 0, 1, 2, 3$. The dark black curves are the primary curves that correspond to first component $F_0(\omega)$ of (22). These are the curves that can also be identified using the *A-wavelet* transform [14]. The important point here is that though the curves are the same, yet the time intervals considered in the current paper are of smaller size and as a result this gives better frequency localization. The dashed, dash-dotted, and dotted curves correspond to second component $F_1(\omega)$, third component $F_2(\omega)$, and fourth component $F_3(\omega)$, respectively. These are the secondary curves which cannot be obtained using the *A-wavelet* transform.

5. Illustration

We will elucidate the above discussed theory with an example. The function $\lfloor (3(ms + 1)\omega - 1)/2 \rfloor$ determines the number of points at which the values of the signals $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q})$, for $q = 0, 1, 2, \dots, (sm + r - 1)$, are needed for the given frequency ω . Consider a time-limited signal defined in the time interval $(-10, 10)$ whose Fourier transform for the frequency $\omega = 1.75$ is to be evaluated. For $m = 4$, $s = 1$, and $r = 0$, the maximum value of p , denoted by $p_{\max}$, is given by $p_{\max} = \lfloor (3(sm + r)\omega - 1)/2 \rfloor = \lfloor (12 * 1.75 - 1)/2 \rfloor = 11$. Thus, the number of points at which the signals $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q})$ are to be evaluated, for $\omega = 1.75$, is 21, that is, $0, \pm 1, \pm 2, \pm 3, \dots, \pm 10$. Since $m = 4$, the Fourier transform contains four components, that is, $F_0(\omega)$, $F_1(\omega)$, $F_2(\omega)$, and $F_3(\omega)$, and since $s = 1$, we need to consider the curves corresponding to the first block. $F_0(\omega)$ needs to be computed at the center of every fourth time interval in both directions ($p = 0, \pm 4, \pm 8$) starting from the first one (assuming the index of the first interval as 0), that is, (ω, b_0) , $(\omega, \pm b_4)$, and $(\omega, \pm b_8)$, where b_i represents the time (location) parameter; $F_1(\omega)$ is computed at the center of every fourth time interval in both directions starting from the second one, that is, $(\omega, \pm b_1)$, $(\omega, \pm b_5)$, and $(\omega, \pm b_9)$; $F_2(\omega)$ is computed at the center of every fourth time interval in both directions starting from the third one, that is, $(\omega, \pm b_2)$, $(\omega, \pm b_6)$, and $(\omega, \pm b_{10})$; $F_3(\omega)$ is computed at the center of every fifth time interval in both directions starting from the fourth one, that is, $(\omega, \pm b_3)$ and $(\omega, \pm b_7)$. Apart from these computations, no values of any other function or translation of any function is needed. The above discussion has been summarized in Table 1.

We now illustrate the computation of the Fourier transform and compare the results with those of the *A-wavelet* transform [14] and SC- and SS-wavelet transforms [16]. We take the signal $f(t) = \cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$ and evaluate $T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q})$ and $T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q})$, $r = 0$, $s = 1$, $m = 4$, and $q = 0, 1, \dots, (sm + r - 1)$ using the current work. Substituting the values of $f(t)$ and $\psi_{m,n,\omega,k}(t)$ in (20a), we have

$$\begin{aligned}
 & T_{m,n,\psi,q}(\omega, b_{(sm+r)p+q}) \\
 &= \int_{(2q-1)\pi/((sm+r)\omega)}^{(2q+1)\pi/((sm+r)\omega)} (\cos(1.3t) + \cos(2.5t) + \cos(3.6t) \\
 &\quad + \cos(4.8t)) \\
 &\quad \times \cos(\omega(t - b_{(sm+r)p+q})) dt,
 \end{aligned} \tag{25}$$

where

$$b_{(sm+r)p+q} = \frac{2\pi((sm+r)p+q)}{(sm+r)\omega}. \tag{26}$$

The corresponding plots in 3D and 2D are shown in Figures 3(a) and 3(d), respectively, for the first block. In fact, these figures can also be obtained by using the work [16]. However, the plots shown in Figure 3(b) can only be obtained by using the current work. We have omitted the 2D graphs for the current work because Figure 3(b) is good enough to show the advantage over the works [14, 16]. The cosine wavelet transform corresponding to the *A-wavelet* transform [14] for $f(t)$ is given by

$$\begin{aligned}
 & T_{\psi}(\omega, b) \\
 &= \int_{-\pi/\omega}^{\pi/\omega} (\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)) \\
 &\quad \times \cos(\omega(t - b)) dt.
 \end{aligned} \tag{27}$$

The corresponding plots in 3D and 2D are shown in Figures 3(c) and 3(e), respectively.

As discussed in the beginning of this section, the first component $F_0(\omega)$ provides the same information as that obtained in [14] but along the primary curves and also the work [16] but in the first block along both primary and secondary curves. So, we have computed only the first component $F_0(\omega)$. The graphs in Figure 3(a) have sharper peaks in the time-frequency plane than that of Figure 3(c). Comparing 2D plots, the frequencies are better resolved in Figure 3(d) than that in Figure 3(e). In Figure 3(e), the frequencies are either not resolved at all or are poorly resolved. Figure 3(b) has sharper peaks in the time-frequency plane than that of Figure 3(a) and better resolves the frequency contents.

We compute the values of $T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q})$, sine wavelet transform, developed in the current paper for the

FIGURE 3: (a) 3D cosine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (b) 3D cosine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (c) 3D cosine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (d) 2D cosine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (e) 2D cosine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$.

same function $f(t) = \cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. Substituting the values of $f(t)$ and $\varphi_{m,n,\omega,k}(t)$ in (20b), we have

$$\begin{aligned}
 & T_{m,n,\varphi,q}(\omega, b_{(sm+r)p+q}) \\
 &= \int_{(2q-1)\pi/((sm+r)\omega)}^{(2q+1)\pi/((sm+r)\omega)} (\cos(1.3t) + \cos(2.5t) + \cos(3.6t) \\
 &\quad + \cos(4.8t)) \\
 &\quad \times \sin(\omega(t - b_{(sm+r)p+q})) dt,
 \end{aligned} \tag{28}$$

where

$$b_{(sm+r)p+q} = \frac{2\pi((sm+r)p+q)}{(sm+r)\omega}. \tag{29}$$

The sine wavelet transform corresponding to the A -wavelet transform of $f(t)$ is computed by

$$\begin{aligned}
 & T_{\varphi}(\omega, b) \\
 &= \int_{-\pi/\omega}^{\pi/\omega} (\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)) \\
 &\quad \times \sin(\omega(t - b)) dt.
 \end{aligned} \tag{30}$$

The corresponding 3D and 2D plots are shown in Figures 4(a) and 4(d), respectively. Here, also, the frequency localization is better than that of the A -wavelet transform as evident from Figures 4(a) and 4(c). In 2D plots, the frequencies are better resolved than that of the A -wavelet transform as shown in Figures 4(d) and 4(e). In Figure 4(b), the frequency

FIGURE 4: (a) 3D sine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (b) 3D sine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (c) 3D sine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (d) Projection of sine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$. (e) 2D sine wavelet transform for $\cos(1.3t) + \cos(2.5t) + \cos(3.6t) + \cos(4.8t)$.

localization is better than both Figures 4(a) and 4(c). Using the proposed work, we can resolve the very close frequencies by taking appropriate block. Important unitary transforms have been discussed in [15]. The results obtained in this paper can be applied to them too.

6. Conclusion

In this paper, we have proposed a new representation of the Fourier transform, A^* -wavelet transform, which provides

better frequency localization than that of A -wavelet transform. The A -wavelet transform is a particular case of the A^* -wavelet transform that provides the signal information along the primary curves, which are separated out by 2π in the time-frequency plane. The proposed work can provide the signal information along the secondary curves and the separation between two curves can be made arbitrarily small, which is not possible in the A -wavelet transform. In this work, we can theoretically resolve the frequency contents as small as we please. This study can be useful in many areas such as image processing and computer vision.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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