Research Article

Basicity of Systems of Sines with Linear Phase in Weighted Sobolev Spaces

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The perturbed systems of sines, which appear when solving some partial differential equations by the Fourier method, are considered in this paper. Basis properties of these systems in weighted Sobolev spaces of functions are studied.

1. Introduction

When solving many problems in mathematical physics by Fourier method (see e.g., [1–4]), there appear perturbed systems of sines and cosines of the following form:

$$\{\sin(nt + \alpha(t))\}_{n=1}^{\infty}, \quad (1)$$

$$1 \cup \{\cos(nt + \alpha(t))\}_{n=1}^{\infty}, \quad (2)$$

where $\alpha(t) = \beta t + \gamma$, $\beta, \gamma \in \mathbb{R}$ are real parameters. Using Fourier method requires the study of basis properties of the above systems in Lebesgue and Sobolev spaces of functions. Relevant investigations date back to the well-known works by Paley and Wiener [5] and Levinson [6]. For $\gamma = 0$, basis properties of these systems in spaces L_p , $1 < p < +\infty$, are completely studied in [7–12]. The case of weighted L_p was considered by E. I. Moiseev in [13, 14]. Basis properties of some perturbed systems of exponents in Sobolev spaces are studied in [15–19]. Further references include [20–23].

Our paper is devoted to the study of basis properties of these systems in weighted Sobolev spaces. Unlike previous works, we offer a different method of investigation.

2. Auxiliary Facts

Let $L_{p,\rho}(0, \pi)$ and $W_{p,\rho}^1(0, \pi)$ be weighted Lebesgue and Sobolev spaces with the following norms:

$$\|u\|_{L_{p,\rho}}^p = \int_0^\pi |u(\theta)|^p \rho(\theta) d\theta, \quad \|u\|_{W_{p,\rho}^1} = \|u\|_{L_{p,\rho}} + \|u'\|_{L_{p,\rho}}, \quad (3)$$

where $\rho(\theta) = \theta^a(\pi - \theta)^b|\theta - \theta_0|^c$, $\theta_0 \in (0, \pi)$, $a, b, c \in (-1, +\infty)$. Denote by $\mathcal{L}_{p,\rho}$ the following direct sum:

$$\mathcal{L}_{p,\rho} = L_{p,\rho} \oplus C, \quad (4)$$

where C is a complex plane. The norm in this space is defined as follows: $\|\hat{u}\|_{\mathcal{L}_{p,\rho}} = \|u\|_{L_{p,\rho}} + |\lambda|$, where $\hat{u} = (u, \lambda) \in \mathcal{L}_{p,\rho}$. The following easily provable lemmas play an important role in obtaining our main result.

Lemma 1. Let $p \in (1, +\infty)$, $a, b, c \in (-1, p-1)$. Then the operator

$$A(u; \lambda) = \lambda + \int_0^t u(\theta) d\theta \quad (5)$$

performs an isomorphism between the spaces $\mathcal{L}_{p,\rho}$ and $W_{p,\rho}^1(0, \pi)$; that is, the spaces $\mathcal{L}_{p,\rho}$ and $W_{p,\rho}^1$ are isomorphic.

Proof. First we show the boundedness of this operator. We have the following:

$$\begin{aligned}
& \|A\hat{u}\|_{W_{p,\rho}} \\
&= \left\| \lambda + \int_0^t u(\theta) d\theta \right\|_{L_{p,\rho}} + \|u\|_{L_{p,\rho}} \\
&\leq \|\lambda\|_{L_{p,\rho}} + \left\| \int_0^t u(\theta) d\theta \right\|_{L_{p,\rho}} + \|u\|_{L_{p,\rho}} \\
&= \left(\int_0^\pi |\lambda|^p \rho(\theta) d\theta \right)^{1/p} + \left(\int_0^\pi \left| \int_0^t u(\theta) d\theta \right|^p \rho(\tau) d\tau \right)^{1/p} \\
&\quad + \|u\|_{L_{p,\rho}} \\
&\leq |\lambda| \|\rho\|_{L_1}^{1/p} + \left(\int_0^\pi \left(\int_0^t |u(\theta)| d\theta \right)^p \rho(\tau) d\tau \right)^{1/p} + \|u\|_{L_{p,\rho}} \\
&\leq |\lambda| \|\rho\|_{L_1}^{1/p} + \left(\int_0^\pi \left(\int_0^\pi |u(\theta)| d\theta \right)^p \rho(\tau) d\tau \right)^{1/p} \\
&\quad + \|u\|_{L_{p,\rho}} \\
&= |\lambda| \|\rho\|_{L_1}^{1/p} + \int_0^\pi |u(\theta)| d\theta \|\rho\|_{L_1}^{1/p} + \|u\|_{L_{p,\rho}}.
\end{aligned} \tag{6}$$

Applying Hlder's inequality, we obtain the following:

$$\begin{aligned}
& \int_0^\pi |u(\theta)| d\theta \\
&= \int_0^\pi |u(\theta)| \rho^{1/p} \rho^{-(1/p)} \\
&\leq \int_0^\pi |u(\theta)|^p \rho d\theta \left(\int_0^\pi \rho^{-(q/p)}(\theta) d\theta \right)^{1/q} \\
&= \|u\|_{L_{p,\rho}} \|\rho^{1/(1-p)}\|_{L_1}^{1/q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned} \tag{7}$$

Consequently,

$$\begin{aligned}
\|A\hat{u}\|_{W_{p,\rho}} &\leq |\lambda| \|\rho\|_{L_1}^{1/p} + \|u\|_{L_{p,\rho}} \|\rho\|_{L_1}^{1/p} \|\rho^{1/(1-p)}\|_{L_1}^{1/q} + \|u\|_{L_{p,\rho}} \\
&= |\lambda| \rho_{L_1}^{1/p} + \left(\|\rho\|_{L_1}^{1/p} \|\rho^{1/(1-p)}\|_{L_1}^{1/q} + 1 \right) \|u\|_{L_{p,\rho}} \\
&\leq M \left(|\lambda| + \|u\|_{L_{p,\rho}} \right) = M \|\hat{u}\|_{L_{p,\rho}},
\end{aligned} \tag{8}$$

where $M = \max\{\|\rho\|_{L_1}^{1/p}; \|\rho\|_{L_1}^{1/p} \|\rho^{1/(1-p)}\|_{L_1}^{1/q}\}$.

Let us show that $\text{Ker } A = \{0\}$. Let $A\hat{u} = 0$; that is,

$$\lambda + \int_0^t u(\theta) d\theta = 0, \quad \forall t \in (0, \pi), \tag{9}$$

where $\lambda \in C$, $u \in L_{p,\rho}$. Differentiating this equation, we obtain $u(\theta) = 0$ a.e. on $(0, \pi)$.

It follows that $\lambda = 0$. From (9) it directly follows that $u = 0$ a.e. on $(0, \pi)$ and this implies that $\hat{u} = 0$.

Show that $\text{Im } A = W_{p,\rho}^1(0, \pi)$. Let $v \in W_{p,\rho}^1(0, \pi)$ be an arbitrary function. Assume $\hat{v} = (v', v(0))$. It is clear that $A\hat{v} = v$ and $\hat{v} \in L_{p,\rho}$. Then by Banach theorem we find that the operator A has a bounded inverse. This proves Lemma 1. $\square$

Now let us prove the following lemma.

Lemma 2. Let $p \in (1, +\infty)$ and $a, b, c \in (-1, p-1)$. Then for all $p_0 \in (1, \alpha) : L_{p,\rho} \subset L_{p_0}(0, \pi)$, where $\alpha = \min\{p; p/(a+1); p/(b+1); p/(c+1); 2\}$.

Proof. Let $f \in L_{p,\rho}(0, \pi)$, $p_0 \in (1, \alpha)$. We have the following:

$$\begin{aligned}
& \int_0^\pi |f|^{p_0} dt \\
&= \int_0^\pi |f|^{p_0} \rho^{p_0/p} \rho^{-(p_0/p)} dt \\
&\leq \left(\int_0^\pi |f|^p \rho dt \right)^{p_0/p} \left(\int_0^\pi \rho^{p_0/(p-p)} dt \right)^{(p-p_0)/p} \\
&= \|f\|_{L_{p,\rho}}^{p_0} \left(\int_0^\pi t^{ap_0/(p_0-p)} |t-t_0|^{cp_0/(p_0-p)} \right. \\
&\quad \left. (\pi-t)^{bp_0/(p_0-p)} dt \right)^{(p-p_0)/p}.
\end{aligned} \tag{10}$$

As $p_0 < p/(a+1)$ and $a > -1$, then $ap_0/(p_0-p) > -1$. Similarly, we find that $bp_0/(p_0-p) > -1$ and $cp_0/(p_0-p) > -1$. It is easy to see that $f \in L_{p_0}(0, \pi)$ and, moreover,

$$\|f\|_{p_0} \leq C \|f\|_{L_{p,\rho}}. \tag{11}$$

This proves the lemma. $\square$

From results of the paper [24] it follows the validity of the following lemma.

Lemma 3. Let $f \in L_{p,\rho}(0, \pi)$, $\rho = t^a |t-t_0|^c (\pi-t)^b$, $a, b, c \in (-1, p-1)$, and $f(t) = \sum_{n=1}^\infty c_n \sin(n+\beta/2)t$ in $L_{p,\rho}(0, \pi)$. Then the series $\sum_{n=1}^\infty (c_n/(n+\beta/2))$ converges absolutely.

3. Main Result

Theorem 4. Let $p \in (1, +\infty)$, $a, b, c \in (-1, p-1)$. Then system (2) forms a basis for $W_{p,\rho}^1(0, \pi)$ if and only if system (1) forms a basis for $L_{p,\rho}(0, \pi)$, where $\alpha(t) = (\beta/2)t$, $\beta \neq -2k$, $k = 1, 2, 3, \dots$.

Proof. First let us assume that the system

$$\left\{ \sin \left(n + \frac{\beta}{2} \right) t \right\}_{n=1}^\infty \tag{12}$$

forms a basis for $L_{p,\rho}(0, \pi)$. Show that the system $\{\hat{u}_n(t)\}_{n=0}^\infty$ forms a basis for $\mathcal{L}_{p,\rho}$, where $\hat{u}_0 = (0; 1)$, $\hat{u}_n = (-(n +$

$\beta/2) \sin[(n + \beta/2)t]; 1)$, for $n \geq 1$. It suffices to show that every element $\hat{u} = (u; \lambda)$ of space $\mathcal{L}_{p,\rho}$ can be expanded in a unique way as a series of the following:

$$\hat{u} = \sum_{n=0}^{\infty} \lambda_n \hat{u}_n; \quad (13)$$

that is,

$$u(t) = - \sum_{n=1}^{\infty} \left(n + \frac{\beta}{2}\right) \lambda_n \sin \left[\left(n + \frac{\beta}{2}\right)t\right], \quad (14)$$

$$\lambda = \lambda_0 + \sum_{n=1}^{\infty} \lambda_n. \quad (15)$$

Since system (1) forms a basis for $L_{p,\rho}(0, \pi)$, we have expansion (14), and the coefficients $\{\lambda_n\}_{n=1}^{\infty}$ are determined uniquely. By Lemma 3, the series $\sum_{n=1}^{\infty} \lambda_n$ converges absolutely. Then, it is clear that the number λ_0 in (15) is determined uniquely. And this implies that the system $\{\hat{u}_n\}_{n=0}^{\infty}$ forms a basis for $\mathcal{L}_{p,\rho}$.

Consider the system $\{\nu_n\}_{n=0}^{\infty}$, where $\nu_n = A\hat{u}_n$, $n \geq 0$, $A\hat{u} = A[(u; \lambda)] = \lambda + \int_0^t u(\tau) d\tau$, and $\hat{u} = (u; \lambda) \in \mathcal{L}_{p,\rho}$. By Lemma 1, the operator A is an isomorphism in $L(W_{p,\rho}^1; \mathcal{L}_{p,\rho})$. Then the system $\{\nu_n\}_{n=0}^{\infty}$ forms a basis for $W_{p,\rho}^1(0, \pi)$. It is easy to see that

$$\nu_0 = 1, \quad \nu_n = \cos\left(n + \frac{\beta}{2}\right)t, \quad n \geq 1. \quad (16)$$

Now we prove the converse. Assume that the system $1 \cup \{\cos(n + \beta/2)t\}_{n=1}^{\infty}$ forms a basis for $W_{p,\rho}^1(0, \pi)$.

Consider the system $\hat{u}_n = A^{-1}\nu_n$, $n \geq 0$. The inverse operator A^{-1} is defined as $A^{-1}\nu = (\nu'; \nu(0))$. It is absolutely clear that the system $\{\hat{u}_n\}_{n=0}^{\infty}$ forms a basis for $\mathcal{L}_{p,\rho}$ and

$$\hat{u}_0 = (0, 1), \quad \hat{u}_n = \left(-\left(n + \frac{\beta}{2}\right) \sin\left(n + \frac{\beta}{2}\right)t; 1\right). \quad (17)$$

Consequently, $\hat{u} = (u; \lambda) \in \mathcal{L}_{p,\rho}$ has a unique expansion of the form (14), (15) in $\mathcal{L}_{p,\rho}$. As a result, we obtain that for all $u \in L_{p,\rho}$ has an expansion with respect to system (1) in $L_{p,\rho}$. Suppose there exists another expansion of u in $L_{p,\rho}$:

$$u(t) = \sum_{n=1}^{\infty} \left(n + \frac{\beta}{2}\right) \mu_n \sin\left(n + \frac{\beta}{2}\right)t. \quad (18)$$

Then Lemma 3 implies the absolute convergence of series $\sum_{n=1}^{\infty} \mu_n$.

Assume $\mu_0 = \lambda - \sum_{n=1}^{\infty} \mu_n$. It is clear that the biorthogonal coefficients of the element $\hat{u} = (u; \lambda)$ are $(\bar{\mu}; \mu_0)$, where $\bar{\mu} = \{\mu_n\}_{n=1}^{\infty}$.

Then, from basicity of system $\{\hat{u}_n\}_{n=0}^{\infty}$ in $\mathcal{L}_{p,\rho}$, we obtain that $\mu_n = \lambda_n$, for all $n \geq 0$. So we came upon a contradiction which proves the theorem. $\square$

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