Research Article

Existence and Global Attractivity of Positive Periodic Solutions for The Neutral Multidelay Logarithmic Population Model with Impulse

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Received 28 January 2013; Accepted 15 April 2013

Academic Editor: Shouming Zhong

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Sufficient and realistic conditions are established in this paper for the existence and global attractivity of a positive periodic solution to the neutral multidelay logarithmic population model with impulse by using the theory of abstract continuous theorem of k -set contractive operator and some inequality techniques. The results improve and generalize the known ones in Li 1999, Lu and Ge 2004, Y. Luo and Z. G. Luo 2010, and Wang et al. 2009. As an application, we also give an example to illustrate the feasibility of our main results.

1. Introduction

In this paper, we investigate the existence and uniqueness of the positive periodic solution of the following neutral population system with multiple delays and impulse:

$$\begin{aligned} \frac{dN(t)}{dt} = N(t) & \left[r(t) - a(t) \ln N(t) \right. \\ & - \sum_{i=1}^n b_i(t) \ln N(t - \sigma_i(t)) \\ & - \sum_{j=1}^m c_j(t) \int_{-\infty}^t K_j(t-s) \ln N(s) ds \\ & \left. - \sum_{l=1}^p d_l(t) \frac{d \ln N(t - \tau_l(t))}{dt} \right], \quad t \neq t_k, \end{aligned}$$

$$N(t_k^+) = e^{(1+\theta_k)} N(t_k), \quad k = 1, 2, \dots, \quad (1)$$

with the following initial conditions:

$$\begin{aligned} N(\xi) &= \varphi(\xi), \quad N'(\xi) = \varphi'(\xi), \quad \xi \in [-\tau, 0], \quad \varphi(0) > 0, \\ \varphi &\in C([-\tau, 0), R^+) \cap C^1([-\tau, 0)), R^+, \end{aligned} \quad (2)$$

where $r(t)$, $a(t)$, $b_i(t)$, $c_j(t)$, $d_l(t)$, $\sigma_i(t)$, $\tau_l(t)$ are positive continuous ω -periodic functions with $\sigma_i(t) \geq 0$, $\tau_l(t) \geq 0$, $t \in [0, \omega]$, $\tau = \max\{\sigma_i(t), \tau_l(t)\}$, $\sigma_i'(t) < 1$, $\tau_l'(t) < 1$, for all $i = \{1, 2, \dots, n\}$, for all $l = \{1, 2, \dots, p\}$. Furthermore, $d_l(t) \in C^1(R, R)$, $\tau_l(t) \in C^2(R, R)$, $\int_0^\infty K_j(s) ds = 1$, $\int_0^\infty s K_j(s) ds < +\infty$, for all $j = \{1, 2, \dots, m\}$, for all $l = \{1, 2, \dots, p\}$. For the ecological justification of (1) and similar types refer to [1–7]. In recent years, Gopalsamy [1] and Kirlinger [2] had proposed the following single species logarithmic model:

$$N'(t) = N(t) [a - b \ln N(t) - c \ln N(t - \tau)]. \quad (3)$$

In [3], Li considered the following nonautonomous single species logarithmic model:

$$N'(t) = N(t) [a(t) - b(t) \ln N(t) - c(t) \ln N(t - \tau(t))]. \quad (4)$$

He used the continuation theorem of the coincidence degree theory to establish sufficient conditions for the existence and attractivity of positive periodic solutions of the system (4).

For more works on the periodic solution of the neutral type logistic model or the Lotka-Volterra model, see [8–12] for details. Only little scholars considered the neutral logarithmic model (see [4–7]). Li [4] had studied the following single species neutral logarithmic model:

$$N'(t) = N(t) \left[r(t) - a(t) \ln N(t - \sigma) - b(t) (\ln N(t - \eta))' \right]. \quad (5)$$

Lu and Ge [5] and Y. Luo and Z. G. Luo [6] employed an abstract continuous theorem of k -set contractive operator to investigate the following equation:

$$N'(t) = N(t) \left[r(t) - \sum_{j=1}^n a_j(t) \ln N(t - \sigma_j(t)) - \sum_{i=1}^m b_i(t) (\ln N(t - \tau_i(t)))' \right]. \quad (6)$$

They established some criteria to guarantee the existence of positive periodic solutions of the system (6), respectively.

In [7], Wang et al. had investigated the existence and uniqueness of the positive periodic solution of the following neutral multispecies logarithmic population model:

$$N'(t) = N(t) \left[r(t) - a(t) \ln N(t) - \sum_{j=1}^n b_j(t) \ln N(t - \tau_j(t)) - \sum_{j=1}^n c_j(t) \int_{-\infty}^t k_j(t-s) \ln N(s) ds - \sum_{j=1}^n d_j(t) (\ln N(t - \eta_j(t)))' \right]. \quad (7)$$

By using an abstract continuous theorem of a k -set contractive operator, the criteria are established for the existence and global attractivity of positive periodic solutions for model (7).

On the other hand, there are some other perturbations in the real world such as fires and floods, which are not suitable to be considered continually. These perturbations bring sudden changes to the system. Systems with such sudden perturbations involving impulsive differential equations have attracted the interest of many researchers in the past twenty years [13–19], since they provide a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Such processes are often investigated in various fields of science and technology such as physics, population dynamics, ecology, biological systems, and optimal control. For details, see [20–22].

However, to this day, no scholars had done works on the existence, uniqueness, and global stability of the positive periodic solution of (1). One could easily see that the systems (5)–(7) are all special cases of the system (1). Therefore, we propose and study the system (1) in this paper.

Throughout this paper, we make the following notations. Let $\omega > 0$ be a constant,

$C_\omega = \{x \mid x \in C(R, R), x(t + \omega) = x(t)\}$, with the norm defined by $\|x\|_0 = \max_{t \in [0, \omega]} |x(t)|$,

$C_\omega^1 = \{x \mid x \in C^1(R, R), x(t + \omega) = x(t)\}$, with the norm defined by $\|x\| = \max_{t \in [0, \omega]} \{|x|_0, |x'|_0\}$.

Then, those spaces are both Banach spaces. We also denote that

$$\bar{h} = \frac{1}{\omega} \int_0^\omega h(t) dt, \quad \text{for any } h \in PC_\omega. \quad (8)$$

For the sake of generality and convenience, we always make the following fundamental assumptions:

- (H₁) $r(t), a(t), b_i(t), c_j(t), d_l(t), \sigma_i(t), \tau_l(t)$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; l = 1, 2, \dots, p$) are all positive periodic continuous functions with period $\omega > 0$;
- (H₂) $0 < t_1 < t_2 < \dots < t_k < \dots$ are fixed impulsive points with $\lim_{k \rightarrow \infty} t_k = +\infty$;
- (H₃) $\{\theta_k\}$ is a real sequence such that $\theta_k + 1 > 0$, and $\prod_{0 < t_k < t} (1 + \theta_k)$ is an ω -periodic function.

In the following section, some definitions and some useful lemmas are listed. In the third section, by using an abstract continuous theorem of k -set contractive operator and some inequality techniques, we acquired some sufficient conditions which ensure the existence and uniqueness of the positive periodic solution of the systems (1) and (2). Finally, we give an example to show our results.

2. Preliminaries

In order to obtain the existence and uniqueness of a periodic solution for the systems (1) and (2), we first give some definitions and lemmas.

Definition 1 (see [21]). A function $N : R \rightarrow (0, +\infty)$ is said to be a positive solution of (1) and (2), if the following conditions are satisfied:

- (a) $N(t)$ is absolutely continuous on each (t_k, t_{k+1}) ;
- (b) for each $k \in Z_+$, $N(t_k^+)$ and $N(t_k^-)$ exist and $N(t_k^-) = N(t_k)$;
- (c) $N(t)$ satisfies the first equation of (1) and (2) for almost everywhere (for short a.e.) in $[0, \infty) \setminus \{t_k\}$ and satisfies $N(t_k^+) = (1 + \theta_k)N(t_k^-)$ for $t = t_k, k \in Z_+ = \{1, 2, \dots\}$.

Definition 2. The system (1) is said to be globally attractive, if there exists a positive solution $x(t)$ of (1) such that $\lim_{t \rightarrow +\infty} |x(t) - y(t)| = 0$ for any other positive solution $y(t)$ of (1).

We can easily get the following Lemma 3.

Lemma 3. The region $R = \{N(t) : N(0) > 0, t \geq 0\}$ is invariant with respect to (1).

Proof. In view of biological population, we obtain $N(0) > 0$. By the system (1), we have

$$\begin{aligned}
 N(t) &= N(0) \\
 &\times \exp \left\{ \int_0^t \left[r(\eta) - a(\eta) \ln N(\eta) \right. \right. \\
 &\quad - \sum_{i=1}^n b_i(\eta) \ln N(\eta - \sigma_i(\eta)) \\
 &\quad - \sum_{j=1}^m c_j(\eta) \int_{-\infty}^t K_j(\eta - s) \ln N(s) ds \\
 &\quad \left. \left. - \sum_{l=1}^p d_l(\eta) \frac{d \ln N(\eta - \tau_l(\eta))}{d\eta} \right] d\eta \right\} \\
 &> 0, \quad t \in [0, t_1], \\
 N(t) &= N(t_k) \\
 &\times \exp \left\{ \int_{t_k}^t \left[r(\eta) - a(\eta) \ln N(\eta) \right. \right. \\
 &\quad - \sum_{i=1}^n b_i(\eta) \ln N(\eta - \sigma_i(\eta)) \\
 &\quad - \sum_{j=1}^m c_j(\eta) \int_{-\infty}^t K_j(\eta - s) \ln N(s) ds \\
 &\quad \left. \left. - \sum_{l=1}^p d_l(\eta) \frac{d \ln N(\eta - \tau_l(\eta))}{d\eta} \right] d\eta \right\} \\
 &> 0, \quad t \in (t_k, t_{k+1}], \\
 N(t_k^+) &= e^{(1+\theta_k)} N(t_k) > 0, \quad k \in \mathbb{N}.
 \end{aligned} \tag{9}$$

Then, the solution of (1) and (2) is positive.

Under the above hypotheses $(H_1)-(H_3)$, we consider the following neutral nonimpulsive system:

$$\begin{aligned}
 \frac{dy(t)}{dt} &= y(t) \left[r(t) - A(t) \ln y(t) \right. \\
 &\quad - \sum_{i=1}^n B_i(t) \ln y(t - \sigma_i(t)) \\
 &\quad - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t - s) \ln y(s) ds \\
 &\quad \left. - \sum_{l=1}^p D_l(t) \frac{d \ln y(t - \tau_l(t))}{dt} \right],
 \end{aligned} \tag{10}$$

with the following initial conditions:

$$\begin{aligned}
 y(\xi) &= \varphi(\xi), \quad y'(\xi) = \varphi'(\xi), \quad \xi \in [-\tau, 0], \quad \varphi(0) > 0, \\
 \varphi &\in C([- \tau, 0), R^+) \cap C^1([- \tau, 0)), R^+,
 \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 A(t) &= a(t) \prod_{0 < t_k < t} (1 + \theta_k), \\
 B_i(t) &= b_i(t) \prod_{0 < t_k < t - \sigma_i(t)} (1 + \theta_k), \\
 C_j(t) &= c_j(t) \prod_{0 < t_k < t} (1 + \theta_k), \\
 D_l(t) &= d_l(t) \prod_{0 < t_k < t - \tau_l(t)} (1 + \theta_k).
 \end{aligned} \tag{12}$$

By a solution $y(t)$ of (10) and (11), it means an absolutely continuous function $y(t)$ defined on $[-\tau, 0]$ that satisfies (10) a.e., for $t \geq 0$, and $y(\xi) = \varphi(\xi)$, $y'(\xi) = \varphi'(\xi)$ on $[-\tau, 0]$.

The following lemmas will be used in the proofs of our results, and the proof of the lemma is similar to that of Theorem 1 in [13]. $\square$

Lemma 4. Suppose that $(H_1)-(H_4)$ hold. Then,

- (1) if $y(t)$ is a solution of (10) and (11) on $[-\tau, +\infty)$, then $N(t) = \prod_{0 < t_k < t} e^{(1+\theta_k)} y(t)$ is a solution of (1) and (2) on $[-\tau, +\infty)$;
- (2) if $N(t)$ is a solution of (1) and (2) on $[-\tau, +\infty)$, then $y(t) = \prod_{0 < t_k < t} (e^{(1+\theta_k)})^{-1} N(t)$ is a solution of (10) and (11) on $[-\tau, +\infty)$.

Proof. (1) It is easy to see that $N(t) = \prod_{0 < t_k < t} e^{(1+\theta_k)} y(t)$ is absolutely continuous on every interval $(t_k, t_{k+1}]$, $t \neq t_k$, $k = 1, 2, \dots$,

$$\begin{aligned}
 N'(t) - N(t) &\left[r(t) - a(t) \ln N(t) \right. \\
 &\quad - \sum_{i=1}^n b_i(t) \ln N(t - \sigma_i(t)) \\
 &\quad - \sum_{j=1}^m c_j(t) \int_{-\infty}^t K_j(t - s) \ln N(s) ds \\
 &\quad \left. - \sum_{l=1}^p d_l(t) \frac{d \ln N(t - \tau_l(t))}{dt} \right] \\
 &= \prod_{0 < t_k < t} e^{(1+\theta_k)} y'(t) \\
 &\quad - \prod_{0 < t_k < t} e^{(1+\theta_k)} y(t)
 \end{aligned}$$

$$\begin{aligned}
& \times \left[r(t) - a(t) \prod_{0 < t_k < t} (1 + \theta_k) \ln y(t) \right. \\
& - \sum_{i=1}^n b_i(t) \prod_{0 < t_k < t - \sigma_i(t)} (1 + \theta_k) \\
& \times \ln y(t - \sigma_i(t)) \\
& - \sum_{j=1}^m c_j(t) \int_{-\infty}^t K_j(t-s) \\
& \times \prod_{0 < t_k < t} (1 + \theta_k) \ln y(s) ds \\
& - \sum_{l=1}^p d_l(t) \prod_{0 < t_k < t - \tau_l(t)} (1 + \theta_k) \\
& \left. \times \frac{dy(t - \tau_l(t))}{dt} \right] \\
& = \prod_{0 < t_k < t} e^{(1+\theta_k)} \\
& \times \left\{ y'(t) - y(t) \right. \\
& \times \left[r(t) - A(t) \ln y(t) \right. \\
& - \sum_{i=1}^n B_i(t) \ln y(t - \sigma_i(t)) \\
& - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s) \ln y(s) ds \\
& \left. - \sum_{l=1}^p D_l(t) \frac{dy(t - \tau_l(t))}{dt} \right] \left. \right\} = 0.
\end{aligned} \tag{13}$$

On the other hand, for any $t = t_k$, $k = 1, 2, \dots$,

$$\begin{aligned}
N(t_k^+) &= \lim_{t \rightarrow t_k^+} \prod_{0 < t_j < t} e^{(1+\theta_k)} y(t) \\
&= \prod_{0 < t_j \leq t_k} e^{(1+\theta_k)} y(t_k), \\
N(t_k) &= \prod_{0 < t_j < t_k} e^{(1+\theta_k)} y(t_k),
\end{aligned} \tag{14}$$

thus,

$$N(t_k^+) = e^{(1+\theta_k)} N(t_k), \quad k = 1, 2, \dots \tag{15}$$

It follows from (13)–(15) that $N_i(t)$ is a solution of (1) and (2).

(2) Since $N(t) = \prod_{0 < t_k < t} e^{(1+\theta_k)} y(t)$ is absolutely continuous on every interval $(t_k, t_{k+1}]$, $t \neq t_k$, $k = 1, 2, \dots$, and in view of (15), it follows that for any $k = 1, 2, \dots$,

$$\begin{aligned}
y(t_k^+) &= \prod_{0 < t_j \leq t_k} (e^{(1+\theta_k)})^{-1} N(t_k^+) \\
&= \prod_{0 < t_j < t_k} (e^{(1+\theta_k)})^{-1} N(t_k) = y(t_k), \\
y(t_k^-) &= \prod_{0 < t_j < t_k} (e^{(1+\theta_k)})^{-1} N(t_k^-) \\
&= \prod_{0 < t_j \leq t_k^-} (e^{(1+\theta_k)})^{-1} N(t_k^-) = y(t_k),
\end{aligned} \tag{16}$$

which implies that $y(t)$ is continuous on $[-\tau, +\infty)$. It is easy to prove that $y(t)$ is absolutely continuous on $[-\tau, +\infty)$. Similar to the proof of (1), one can check that $y(t) = \prod_{0 < t_k < t} (e^{(1+\theta_k)})^{-1} N(t)$ are solutions of (10) and (11) on $[-\tau, +\infty)$. The proof of Lemma 3 is completed. $\square$

Here, we take the transformation $y(t) = e^{x(t)}$; then, (10) can be rewritten in the following form:

$$\begin{aligned}
\frac{dx(t)}{dt} &= r(t) - A(t)x(t) \\
& - \sum_{i=1}^n B_i(t)x(t - \sigma_i(t)) \\
& - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)x(s) ds \\
& - \sum_{l=1}^p D_{0l}(t)x'(t - \tau_l(t)),
\end{aligned} \tag{17}$$

where $D_{0l}(t) = D_l(t)(1 - \tau_l'(t))$, $l = 1, 2, \dots, p$. Therefore, we only discuss the existence and uniqueness of a periodic solution for (17).

Definition 5 (see [23]). Let U be a bounded subset in X . Define that

$$\begin{aligned}
\alpha_X(U) &= \inf \left\{ \delta > 0 : \text{there is a finite number of subsets} \right. \\
& \left. U_i \subset U \text{ such that } U = \bigcup_i U_i \text{ and } \text{diam}(U_i) \leq \delta \right\},
\end{aligned} \tag{18}$$

where $\text{diam}(U_i)$ denotes the diameter of the set U_i , obviously, $0 \leq \alpha_X(U) < \infty$. So, $\alpha_X(U)$ is called the (Kuratowski) measure of noncompactness of X .

Definition 6 (see [23]). Let X, Y be two Banach spaces and $D \subset X$; a continuous and bounded map $T : D \rightarrow Y$ is called k -set contractive if for any bounded set $U \subset D$ one has

$$\alpha_Y(T(U)) \leq k\alpha_X(U), \quad (19)$$

T is called strict-set-contractive if it is k -set contractive for some $0 \leq k < 1$.

For a Fredholm operator $L : X \rightarrow Y$ with index zero, according to [9, 24], we define

$$l(L) = \sup \{ r\alpha_X(U) \leq \alpha_Y(L(U)) \text{ for any bounded set } U \subset X \}. \quad (20)$$

Lemma 7 (see [9, 24]). Let $L : X \rightarrow Y$ be a Fredholm operator with zero index and $a \in Y$ be a fixed point. Suppose that $N : \Omega \rightarrow Y$ is called a k -set contractive with $k < l(L)$, where $\Omega \subset X$ is bounded, open, and symmetric about $0 \in \Omega$. Further, one also assumes that

- (1) $Lx \neq \lambda Nx + \lambda a$, for $x \in \partial\Omega$, $\lambda \in (0, 1)$;
- (2) $[QN(x) + Qa, x][QN(-x) + Qa, x] < 0$, for $x \in \text{Ker } L \cap \partial\Omega$;

where $[\cdot, \cdot]$ is a bilinear form on $Y \times X$ and Q is the projection of Y onto $\text{Coker}(L)$, where $\text{Coker}(L)$ is the cokernel of the operator L . Then, there is an $x \in \bar{\Omega}$ such that $Lx - Nx = a$.

In order to use Lemma 7 to study (17), we set $X = C_\omega^1, Y = C_\omega$,

$$L : C_\omega^1 \longrightarrow C_\omega, \quad Lx = \frac{dx}{dt}, \quad (21)$$

and $N : C_\omega^1 \rightarrow C_\omega$ defined by

$$\begin{aligned} Nx = & -A(t)(t)x(t) - \sum_{i=1}^n B_i(t)x(t - \sigma_i(t)) \\ & - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)x(s)ds \\ & - \sum_{l=1}^p D_{0l}(t)x'(t - \tau_l(t)). \end{aligned} \quad (22)$$

It is easy to see from [25] that L is a Fredholm operator with index zero. Thus, (17) has a positive ω -periodic solution if and only if $Lx = Nx + r$ has a solution $x \in C_\omega^1$, where $r = r(t)$.

Lemma 8 (see [25]). The differential operator L is a Fredholm operator with index zero and satisfies $l(L) \geq 1$.

Lemma 9. Let r_0, r_1 be two positive constants and $\Omega = \{x_i \in C_\omega^1, |x_i|_0 < r_0, |x_i'|_0 < r_1\}$. If $k = \sum_{l=1}^p |D_{0l}|_0$, then $N : \Omega \rightarrow C_\omega^1$ is a k -set contractive map.

As Lemma 9 can be proved in the same way as in the proof of Lemma 2.4 in [7], we omit it here.

Lemma 10 (see [7, 9]). Suppose that $\tau \in C_\omega^1$ and $\tau'(t) < 1$, $t \in [0, \omega]$, then the function $t - \tau(t)$ has a unique inverse $\mu(t)$ satisfying $\mu \in C(R, R)$ with $\mu(a + \omega) = \mu(a) + \omega$ for all $a \in R$, and if $g \in PC_\omega$, $\tau'(t) < 1$, $t \in [0, \omega]$, then $g(\mu(t)) \in PC_\omega$.

Lemma 11 (see [26, 27]). Suppose that $x(t)$ is a differently continuous ω -periodic function on R with $(\omega > 0)$, then to any $t_* \in R$, $\max_{t_* \leq t \leq t_* + \omega} |x(t)| \leq |x(t_*)| + (1/2) \int_0^\omega |x'(t)| dt$.

3. Main Theorem

Since $\sigma_i'(t) < 1$, $\tau_l'(t) < 1$, $t \in [0, \omega]$, we see that $\sigma_i(t), \tau_l(t)$ all have its inverse function. Throughout the following part, we set $\zeta_i(t), \xi_l(t)$ that represent the inverse function of $t - \sigma_i(t), t - \tau_l(t)$, respectively. We also denote

$$\begin{aligned} \Gamma(t) := & A(t) + \sum_{i=1}^n \frac{B_i(\zeta_i(t))}{1 - \sigma_i'(\zeta_i(t))} \\ & + \sum_{j=1}^m C_j(t) - \sum_{l=1}^p \frac{D_l'(\xi_l(t))}{1 - \tau_l'(\xi_l(t))}. \end{aligned} \quad (23)$$

Remark 12. From Lemma 9, we get that $\zeta_i(\omega) = \zeta_i(0) + \omega$, $\xi_l(\omega) = \xi_l(0) + \omega$, $i = 1, 2, \dots, n$; $l = 1, 2, \dots, p$; then,

$$\begin{aligned} \int_0^\omega \frac{B_i(\zeta_i(s))}{1 - \sigma_i'(\zeta_i(s))} ds &= \int_{\zeta_i(0)}^{\zeta_i(\omega)} \frac{B_i(t)(1 - \sigma_i'(t))}{1 - \sigma_i'(t)} dt \\ &= \int_{\zeta_i(0)}^{\zeta_i(0) + \omega} B_i(t) dt = \bar{B}_i \omega, \quad i = 1, 2, \dots, n. \end{aligned} \quad (24)$$

Similarly,

$$\begin{aligned} \int_0^\omega \frac{D_l'(\xi_l(s))}{1 - \tau_l'(\xi_l(s))} ds &= \int_{\xi_l(0)}^{\xi_l(\omega)} \frac{D_l'(t)(1 - \tau_l'(t))}{1 - \tau_l'(t)} dt \\ &= \int_{\xi_l(0)}^{\xi_l(0) + \omega} D_l'(t) dt = 0, \quad l = 1, 2, \dots, p. \end{aligned} \quad (25)$$

Thus,

$$\bar{\Gamma} \omega = \int_0^\omega \Gamma(t) dt = \left(\bar{A} + \sum_{i=1}^n \bar{B}_i + \sum_{j=1}^m \bar{C}_j \right) \omega. \quad (26)$$

Theorem 13. In addition to $(H_1)-(H_3)$, suppose that the following conditions hold:

(H_4) there exists a constant $\eta > 0$ such that $|\Gamma(t)| \geq \eta$, for all $t \in [0, \omega]$, where $\Gamma(t)$ is defined by (23);

(H_5) if $(1/2)(|A|_0 \omega + \sum_{i=1}^n |B_i|_0 \omega + \sum_{j=1}^m |C_j|_0 \omega^{1/2}) + \sum_{l=1}^p |1 - \tau_l'|_0^{1/2} |D_l|_0 < 1$, and $\sum_{l=1}^p |1 - \tau_l'|_0 |D_l|_0 < 1$.

Then, (1) has at least one positive ω -periodic solution.

Proof. Let $u(t)$ be an arbitrary ω -periodic solution of the operator equation as follows:

$$Lu = \lambda Nu + \lambda r, \quad \lambda \in (0, 1), \quad (27)$$

where L, N defined by (21) and (22), respectively. Then, $u(t)$ satisfies the following operator equation:

$$\begin{aligned} \frac{du(t)}{dt} = \lambda \left[r(t) - A(t)u(t) \right. \\ \left. - \sum_{i=1}^n B_i(t)u(t - \sigma_i(t)) \right. \\ \left. - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)u(s)ds \right. \\ \left. - \sum_{l=1}^p D_{0l}(t)u'(t - \tau_l(t)) \right]. \end{aligned} \quad (28)$$

Integrating both sides of (28) over $[0, \omega]$, we have

$$\begin{aligned} \bar{r}\omega = \int_0^\omega \left[A(t)u(t) \right. \\ \left. + \sum_{i=1}^n B_i(t)u(t - \sigma_i(t)) \right. \\ \left. + \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)u(s)ds \right. \\ \left. + \sum_{l=1}^p D_{0l}(t)u'(t - \tau_l(t)) \right] dt \\ = \int_0^\omega A(t)u(t)dt \\ + \int_0^\omega \sum_{i=1}^n B_i(t)u(t - \sigma_i(t))dt \\ + \int_0^\omega \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)u(s)dsdt \\ - \int_0^\omega \sum_{l=1}^p D_{0l}'(t)u(t - \tau_l(t))dt. \end{aligned} \quad (29)$$

Let $t - \sigma_i(t) = s$, then $t = \zeta_i(s)$ and

$$\begin{aligned} \int_0^\omega B_i(t)u(t - \sigma_i(t))dt \\ = \int_{-\sigma_i(0)}^{\omega - \sigma_i(\omega)} \frac{B_i(\zeta_i(s))}{1 - \sigma_i'(\zeta_i(s))} u(s)ds. \end{aligned} \quad (30)$$

By Lemma 10, we have

$$\begin{aligned} \int_0^\omega B_i(t)u(t - \sigma_i(t))dt \\ = \int_0^\omega \frac{B_i(\zeta_i(s))}{1 - \sigma_i'(\zeta_i(s))} u(s)ds. \end{aligned} \quad (31)$$

Similarly, we have

$$\begin{aligned} \int_0^\omega D_{0l}'(t)u(t - \tau_l(t))dt \\ = \int_0^\omega \frac{D_{0l}'(\xi_l(s))}{1 - \tau_l'(\xi_l(s))} u(s)ds. \end{aligned} \quad (32)$$

Substituting (31) and (32) into (29), we have

$$\int_0^\omega \Gamma(s)u(s)ds = \bar{r}\omega. \quad (33)$$

Considering assumption (H_4) , we know that $|\Gamma(t)| \geq \eta > 0$, and then, it follows from the integro mean value theorem that there exists a $\alpha \in [0, \omega]$ satisfying

$$|u(\alpha)| = \frac{\bar{r}_i}{|\Gamma(\alpha)|} \leq \frac{\bar{r}_i}{\eta}. \quad (34)$$

By Lemma 11, we can get that

$$|u|_0 \leq \frac{\bar{r}_i}{\eta} + \frac{1}{2} \int_0^\omega |u'(t)|dt. \quad (35)$$

Multiplying both sides of (28) by $u'(t)$ and integrating them over $[0, \omega]$, we have

$$\begin{aligned} \int_0^\omega |u'(t)|^2 dt \\ = \lambda \left| \int_0^\omega r(t)u'(t)dt - \int_0^\omega A(t)u(t)u'(t)dt \right. \\ \left. - \int_0^\omega \sum_{i=1}^n B_i(t)u(t - \sigma_i(t))u'(t)dt \right. \\ \left. - \int_0^\omega \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)u(s)u'(t)dsdt \right. \\ \left. - \int_0^\omega \sum_{l=1}^p D_{0l}(t)u'(t - \tau_l(t))u'(t)dt \right| \\ \leq \left[|r|_0 + \left(|A|_0 + \sum_{i=1}^n |B_i|_0 \right) |u|_0 \right] \int_0^\omega |u'(t)|dt \\ + \int_0^\omega \sum_{j=1}^m \left| C_j(t) \int_{-\infty}^t K_j(t-s)u(s)ds u'(t) \right| dt \\ + \int_0^\omega \sum_{l=1}^p |D_{0l}(t)u'(t - \tau_l(t))u'(t)|dt. \end{aligned} \quad (36)$$

By using the Cauchy-Schwarz inequality, we get that

$$\begin{aligned}
 & \int_0^\omega |u'(t)|^2 dt \\
 & \leq \left[|r|_0 + \left(|A|_0 + \sum_{i=1}^n |B_i|_0 \right) |u|_0 \right] \\
 & \quad \times \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2} \omega^{1/2} \\
 & \quad + \left(\sum_{j=1}^m |C_j|_0 \int_{-\infty}^t K_j(t-s) u(s) ds \right)^2 dt \Big)^{1/2} \\
 & \quad \times \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2} \\
 & \quad + \sum_{l=1}^p \left(\int_0^\omega |D_{0l}(t) u'(t - \tau_l(t))|^2 dt \right)^{1/2} \\
 & \quad \times \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2} \\
 & = \left[|r|_0 \omega^{1/2} + \left(|A|_0 \omega^{1/2} + \sum_{i=1}^n |B_i|_0 \omega^{1/2} + \sum_{j=1}^m |C_j|_0 \right) |u|_0 \right] \\
 & \quad \times \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2} \\
 & \quad + \sum_{l=1}^p \left(\int_0^\omega |D_{0l}(t) u'(t - \tau_l(t))|^2 dt \right)^{1/2} \\
 & \quad \times \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2}.
 \end{aligned} \tag{37}$$

Meanwhile, we see that

$$\begin{aligned}
 & \left(\int_0^\omega |D_{0l}(t) u'(t - \tau_l(t))|^2 dt \right)^{1/2} \\
 & = \left(\int_0^\omega \frac{1}{1 - \tau_l'(\xi_l(t))} |D_{0l}(\xi_l(t)) u'(t)|^2 dt \right)^{1/2} \\
 & = \left(\int_0^\omega (1 - \tau_l'(\xi_l(t))) |D_l(\xi_l(t)) u'(t)|^2 dt \right)^{1/2} \\
 & \leq |1 - \tau_l'|_0^{1/2} |D_l|_0 \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2}.
 \end{aligned} \tag{38}$$

Substituting (38) and (35) into (37), we can find that

$$\begin{aligned}
 & \int_0^\omega |u'(t)|^2 dt \\
 & \leq \left[|r|_0 \omega^{1/2} \right. \\
 & \quad + \left(|A|_0 \omega^{1/2} + \sum_{i=1}^n |B_i|_0 \omega^{1/2} + \sum_{j=1}^m |C_j|_0 \right) \\
 & \quad \times \left(\frac{\bar{r}_i}{\eta} + \frac{1}{2} \int_0^\omega |u'(t)| dt \right) \Big] \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2} \\
 & \quad + \sum_{l=1}^p |1 - \tau_l'|_0^{1/2} |D_l|_0 \int_0^\omega |u'(t)|^2 dt \\
 & \leq \left[|r|_0 \omega^{1/2} + \left(|A|_0 \omega^{1/2} + \sum_{i=1}^n |B_i|_0 \omega^{1/2} + \sum_{j=1}^m |C_j|_0 \right) \frac{\bar{r}_i}{\eta} \right] \\
 & \quad \times \left(\int_0^\omega |u'(t)| dt \right)^{1/2} \\
 & \quad + \left[\frac{1}{2} \left(|A|_0 \omega^{1/2} + \sum_{i=1}^n |B_i|_0 \omega^{1/2} + \sum_{j=1}^m |C_j|_0 \right) \omega^{1/2} \right. \\
 & \quad \left. + \sum_{l=1}^p |1 - \tau_l'|_0^{1/2} |D_l|_0 \right] \\
 & \quad \times \int_0^\omega |u'(t)|^2 dt,
 \end{aligned} \tag{39}$$

which gives that

$$\begin{aligned}
 & \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2} \\
 & \leq \left[|r|_0 \omega^{1/2} + \left(|A|_0 \omega^{1/2} + \sum_{i=1}^n |B_i|_0 \omega^{1/2} + \sum_{j=1}^m |C_j|_0 \right) \frac{\bar{r}_i}{\eta} \right] \\
 & \quad + \left[\frac{1}{2} \left(|A|_0 \omega + \sum_{i=1}^n |B_i|_0 \omega + \sum_{j=1}^m |C_j|_0 \omega^{1/2} \right) \right. \\
 & \quad \left. + \sum_{l=1}^p |1 - \tau_l'|_0^{1/2} |D_l|_0 \right] \left(\int_0^\omega |u'(t)| dt \right)^{1/2}.
 \end{aligned} \tag{40}$$

From (H_5) , we get that $(1/2)(|A|_0 \omega + \sum_{i=1}^n |B_i|_0 \omega + \sum_{j=1}^m |C_j|_0 \omega^{1/2}) + \sum_{l=1}^p |1 - \tau_l'|_0^{1/2} |D_l|_0 < 1$. Then, there exists a constant $N > 0$ such that

$$\left(\int_0^\omega |u_i'(t)|^2 dt \right)^{1/2} \leq N. \tag{41}$$

From (35) and the Hölder inequality, we obtain that

$$\begin{aligned} |u|_0 &\leq \frac{\bar{r}}{\eta} + \frac{1}{2} \int_0^\omega |u'(t)| dt \\ &\leq \frac{\bar{r}}{\eta} + \frac{\omega^{1/2}}{2} \left(\int_0^\omega |u'(t)|^2 dt \right)^{1/2} \leq \frac{\bar{r}}{\eta} + \frac{\omega^{1/2}}{2} N := N_1. \end{aligned} \quad (42)$$

Again from (28), we get that

$$\begin{aligned} |u'|_0 &\leq |r|_0 + \left(|A|_0 + \sum_{i=1}^n |B_i|_0 + \sum_{j=1}^m |C_j|_0 \right) |u|_0 \\ &\quad + \sum_{l=1}^p |D_{0l}|_0 |u'|_0. \end{aligned} \quad (43)$$

From condition $\sum_{l=1}^p |D_{0l}|_0 \leq \sum_{l=1}^p |1 - \tau'_l|_0 |D_l|_0 < 1$, it is easy to see that

$$|u'|_0 \leq \frac{|r|_0 + \left(|A|_0 + \sum_{i=1}^n |B_i|_0 + \sum_{j=1}^m |C_j|_0 \right) N_1}{1 - \sum_{l=1}^p |D_{0l}|_0} := N_2. \quad (44)$$

Now, we take $N_3 > \max\{N_1, N_2, |\bar{r}/(\bar{A} + \sum_{i=1}^n \bar{B}_i + \sum_{j=1}^m \bar{C}_j)|\}$ and $\Omega = \{u \mid u \in C_\omega^1, \|u\| < N_3\}$. Then, $k = \sum_{l=1}^p |D_{0l}|_0 < 1 \leq l(L)$. So by (42) and (44), we can find that all the conditions of Lemma 4 except (2) hold. In what follows, we will prove that condition (2) of Lemma 4 is also satisfied. In order to do this, we defined a bounded bilinear form on C_ω, C_ω^1 by $[\cdot, \cdot]$ as the following $[y, x] = \int_0^\omega y(t)x(t)dt$. Also, we defined $Q : y \rightarrow \text{Coker}(L)$ as $y \rightarrow (1/\omega) \int_0^\omega y(t)dt$. It is obvious that

$$\{u \mid u \in \text{Ker } L \cap \partial\Omega\} = \{u \mid u \equiv N_3, \text{ or } -N_3\}. \quad (45)$$

Without loss of generality, suppose that $u \equiv N_3$; then,

$$\begin{aligned} &[QN(u) + Qr, u] [QN(-u) + Qr, u] \\ &= N_3^2 \left[\int_0^\omega r(t) dt \right. \\ &\quad \left. - N_3 \int_0^\omega \left(A(t) + \sum_{i=1}^n B_i(t) + \sum_{j=1}^m C_j(t) \right) dt \right] \end{aligned}$$

$$\begin{aligned} &\times \left[\int_0^\omega r(t) dt \right. \\ &\quad \left. + N_3 \int_0^\omega \left(A(t) + \sum_{i=1}^n B_i(t) + \sum_{j=1}^m C_j(t) \right) dt \right] \\ &= \omega^2 N_3^2 \left[\bar{r} - N_3 \left(\bar{A} + \sum_{i=1}^n \bar{B}_i + \sum_{j=1}^m \bar{C}_j \right) \right] \\ &\quad \times \left[\bar{r} + N_3 \left(\bar{A} + \sum_{i=1}^n \bar{B}_i + \sum_{j=1}^m \bar{C}_j \right) \right], \end{aligned} \quad (46)$$

since $N_3 > |\bar{r}/(\bar{A} + \sum_{i=1}^n \bar{B}_i + \sum_{j=1}^m \bar{C}_j)|$, then

$$\begin{aligned} N_3 &> \frac{\bar{r}}{\bar{A} + \sum_{i=1}^n \bar{B}_i + \sum_{j=1}^m \bar{C}_j}, \\ -N_3 &< -\frac{\bar{r}}{\bar{A} + \sum_{i=1}^n \bar{B}_i + \sum_{j=1}^m \bar{C}_j}. \end{aligned} \quad (47)$$

By (46), we get that

$$[QN(u) + Qr, u] [QN(-u) + Qr, u] < 0. \quad (48)$$

Therefore, by using Lemma 4, we obtain that (1) has at least one positive ω -periodic solution; the proof of Theorem 13 is completed. $\square$

Since $|1 - \tau'_l|_0 < 1$, for all $l = 1, 2, \dots, p$, it follows that $\sum_{l=1}^p |1 - \tau'_l|_0 |D_l|_0 \leq \sum_{l=1}^p |1 - \tau'_l|_0^{1/2} |D_l|_0 < 1$. So from Theorem 13, we have the following result.

Corollary 14. Suppose that $(H_1)-(H_3)$ and the following conditions hold:

(H'_4) there exists a constant $\eta > 0$ such that $|\Gamma(t)| \geq \eta$, for all $t \in [0, \omega]$, where $\Gamma(t)$ is defined by (23);

(H'_5) if $(1/2)(|A|_0\omega + \sum_{i=1}^n |B_i|_0\omega + \sum_{j=1}^m |C_j|_0\omega^{1/2}) + \sum_{l=1}^p |1 - \tau'_l|_0^{1/2} |D_l|_0 < 1$.

Then (1) has at least one positive ω -periodic solution.

On the other hand, if $|1 - \tau'_l|_0 > 1$, for all $l = 1, 2, \dots, p$, it follows that $\sum_{l=1}^p |1 - \tau'_l|_0 |D_l|_0 \geq \sum_{l=1}^p |1 - \tau'_l|_0^{1/2} |D_l|_0$. So from Theorem 13, we also have the following result.

Corollary 15. Suppose that $(H_1)-(H_3)$ and the following conditions hold:

(H_4^*) there exists a constant $\eta > 0$ such that $|\Gamma(t)| \geq \eta$, for all $t \in [0, \omega]$, where $\Gamma(t)$ is defined by (23);

(H_5^*) if $(1/2)(|A|_0\omega + \sum_{i=1}^n |B_i|_0\omega + \sum_{j=1}^m |C_j|_0\omega^{1/2}) + \sum_{l=1}^p |1 - \tau'_l|_0 |D_l|_0 < 1$.

Then, (1) has at least one positive ω -periodic solution.

Our next theorem is concerned with the global attractivity of periodic solution of the system (1).

Theorem 16. Suppose that (H_1) – (H_5) and the following conditions hold:

(H_6) there is a positive constant μ such that

$$\sum_{l=1}^p |D_l|_0 + \int_0^t \left[\sum_{l=1}^p |H_l|_0 + \sum_{i=1}^n |B_i|_0 + \sum_{j=1}^m |C_j|_0 \right] \times \exp \left(- \int_u^t A(\xi) d\xi \right) du \leq \mu < 1; \quad (49)$$

(H_7) $\exp(-\int_0^t A(\xi) d\xi) \rightarrow 0$, as $t \rightarrow +\infty$.

Then, the positive ω -periodic solution of (1) is globally attractive, where $H_l(t) = [D_l'(t) + D_l(t)A(t)][1 - \tau_l'(t)] + D_l(t)\tau_l''(t)/[1 - \tau_l'(t)]^2$.

Proof. Suppose that $y(t) = e^{v(t)}$ is a positive ω -periodic solution of (10), $y^*(t) = e^{v^*(t)}$ is another positive solution of (10). Similar to (17), we have

$$\begin{aligned} \frac{dv}{dt} &= r(t) - A(t)v(t) - \sum_{i=1}^n B_i(t)v(t - \sigma_i(t)) \\ &\quad - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)v(s) ds \\ &\quad - \sum_{l=1}^p D_{0l}(t)v'(t - \tau_l(t)), \end{aligned} \quad (50)$$

$$\begin{aligned} \frac{dv^*}{dt} &= r(t) - A(t)v^*(t) \\ &\quad - \sum_{i=1}^n B_i(t)v^*(t - \sigma_i(t)) \\ &\quad - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)v^*(s) ds \\ &\quad - \sum_{l=1}^p D_{0l}(t)v'^*(t - \tau_l(t)). \end{aligned} \quad (51)$$

Let $v(t) - v^*(t) = w(t)$; then,

$$\begin{aligned} \frac{dw}{dt} &= -A(t)w(t) - \sum_{i=1}^n B_i(t)w(t - \sigma_i(t)) \\ &\quad - \sum_{j=1}^m C_j(t) \int_{-\infty}^t K_j(t-s)w(s) ds \\ &\quad - \sum_{l=1}^p D_{0l}(t)w'(t - \tau_l(t)). \end{aligned} \quad (52)$$

Multiply both sides of (51) with $\exp(\int_0^t A(\xi) d\xi)$ and then integrate from 0 to t to obtain that

$$\begin{aligned} &\int_0^t \left[w(u) \exp \left(\int_0^u A(\xi) d\xi \right) \right]' du \\ &= - \int_0^t \left[\sum_{i=1}^n B_i(u) w(t - \sigma_i(u)) \right. \\ &\quad \left. + \sum_{j=1}^m C_j(u) \int_{-\infty}^t K_j(u-s)w(s) ds \right. \\ &\quad \left. + \sum_{l=1}^p D_{0l}(u) w'(u - \tau_l(u)) \right] \\ &\quad \times \exp \left(\int_0^u A(\xi) d\xi \right) du, \end{aligned} \quad (53)$$

Then,

$$\begin{aligned} w(t) &= w(0) \exp \left(- \int_0^t A(\xi) d\xi \right) \\ &\quad - \int_0^t \left[\sum_{i=1}^n B_i(u) w(t - \sigma_i(u)) \right. \\ &\quad \left. + \sum_{j=1}^m C_j(u) \int_{-\infty}^t K_j(u-s)w(s) ds \right. \\ &\quad \left. + \sum_{l=1}^p D_{0l}(u) w'(u - \tau_l(u)) \right] \\ &\quad \times \exp \left(- \int_u^t A(\xi) d\xi \right) du. \end{aligned} \quad (54)$$

Meanwhile, we see that

$$\begin{aligned} &\int_0^t \sum_{l=1}^p D_{0l}(u) w'(u - \tau_l(u)) \exp \left(- \int_u^t A(\xi) d\xi \right) du \\ &= \int_0^t \sum_{l=1}^p \frac{D_{0l}(u) w'(u - \tau_l(u)) (1 - \tau_l'(u))}{1 - \tau_l'(u)} \\ &\quad \times \exp \left(- \int_u^t A(\xi) d\xi \right) du \\ &= \int_0^t \sum_{l=1}^p \left[\frac{D_{0l}(u) \exp \left(- \int_u^t A(\xi) d\xi \right)}{1 - \tau_l'(u)} \right] \\ &\quad \times [w'(u - \tau_l(u)) (1 - \tau_l'(u))] du \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=1}^p \left[\frac{D_{0l}(t)}{1 - \tau_l'(t)} w(t - \tau_l(t)) \right. \\
&\quad \left. - \frac{D_{0l}(0)}{1 - \tau_l'(0)} w(-\tau_l(0)) \exp\left(-\int_0^t A(\xi) d\xi\right) \right] \\
&\quad - \sum_{l=1}^p \int_0^t H_l(u) \exp\left(-\int_u^t A(\xi) d\xi\right) w(u - \tau_l(u)) du \\
&= \sum_{l=1}^p \left[D_l(t) w(t - \tau_l(t)) \right. \\
&\quad \left. - D_l(0) w(-\tau_l(0)) \exp\left(-\int_0^t A(\xi) d\xi\right) \right] \\
&\quad - \sum_{l=1}^p \int_0^t H_l(u) \exp\left(-\int_u^t A(\xi) d\xi\right) w(u - \tau_l(u)) du,
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
&H_l(u) \\
&= \frac{[D_l'(u) + D_l(t) A(t)] [1 - \tau_l'(t)] + D_l(t) \tau_l''(t)}{[1 - \tau_l'(t)]^2}.
\end{aligned} \tag{56}$$

Substituting (55) into (54), we get that

$$\begin{aligned}
w(t) &= \left[w(0) + \sum_{l=1}^p D_l(0) w(-\tau_l(0)) \right] \\
&\quad \times \exp\left(-\int_0^t A(\xi) d\xi\right) \\
&\quad + \int_0^t \left[\sum_{l=1}^p H_l(u) w(u - \tau_l(u)) \right. \\
&\quad \left. - \sum_{i=1}^n B_i(u) w(u - \sigma_i(u)) \right. \\
&\quad \left. - \sum_{j=1}^m C_j(u) \int_{-\infty}^u K_j(u-s) w(s) ds \right] \\
&\quad \times \exp\left(-\int_u^t A(\xi) d\xi\right) du - \sum_{l=1}^p D_l(t) w(t - \tau_l(t)).
\end{aligned} \tag{57}$$

Therefore, we have

$$\begin{aligned}
|w|_0 &\leq \left| w(0) + \sum_{l=1}^p D_l(0) w(-\tau_l(0)) \right| \\
&\quad \times \exp\left(-\int_0^t A(\xi) d\xi\right)
\end{aligned}$$

$$\begin{aligned}
&+ \int_0^t \left[\sum_{l=1}^p |H_l|_0 + \sum_{i=1}^n |B_i|_0 + \sum_{j=1}^m |C_j|_0 \right] \\
&\quad \times |w|_0 \exp\left(-\int_u^t A(\xi) d\xi\right) du + \sum_{l=1}^p |D_l|_0 |w|_0.
\end{aligned} \tag{58}$$

From (H_6) , we have

$$\begin{aligned}
|w|_0 &\leq \left| w(0) + \sum_{l=1}^p D_l(0) w(-\tau_l(0)) \right| \exp\left(-\int_0^t A(\xi) d\xi\right) \\
&\quad \times \left(1 - \sum_{l=1}^p |D_l|_0 \right. \\
&\quad \left. - \int_0^t \left[\sum_{l=1}^p |H_l|_0 + \sum_{i=1}^n |B_i|_0 + \sum_{j=1}^m |C_j|_0 \right] \right. \\
&\quad \left. \times \exp\left(-\int_u^t A(\xi) d\xi\right) du \right)^{-1} \\
&\leq \frac{\left| w(0) + \sum_{l=1}^p D_l(0) w(-\tau_l(0)) \right| \exp\left(-\int_0^t A(\xi) d\xi\right)}{1 - \mu}.
\end{aligned} \tag{59}$$

From (H_7) , we have

$$|w|_0 = |v - v^*|_0 = 0, \quad \text{as } t \rightarrow +\infty, \tag{60}$$

thus, $v(t) \rightarrow v^*(t)$, as $t \rightarrow +\infty$; that is, the positive ω -periodic solution of (10) is globally attractive; by Definition 2, the positive ω -periodic solution of (1) is globally attractive. The proof is completed. $\square$

Consider the following equation:

$$\begin{aligned}
\frac{dN(t)}{dt} &= N(t) \left[r(t) - a(t) \ln N(t) \right. \\
&\quad \left. - \sum_{i=1}^n b_i(t) \ln N(t - \sigma_i(t)) \right. \\
&\quad \left. - \sum_{j=1}^m c_j(t) \int_{-\infty}^t K_j(t-s) \ln N(s) ds \right. \\
&\quad \left. - \sum_{l=1}^p d_l(t) \frac{d \ln N(t - \tau_l(t))}{dt} \right],
\end{aligned} \tag{61}$$

which is a special case of the system (1) without impulse. Similarly, we can get the following conclusions.

Corollary 17. Suppose that the following conditions hold:

(H_1^*) $r(t), a(t), b_i(t), c_j(t), d_l(t), \sigma_i(t), \tau_l(t)$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m; l = 1, 2, \dots, p$) are all positive ω -periodic continuous functions with $\sigma_i(t) \geq 0, \tau_l(t) \geq 0, t \in [0, \omega], \sigma'_i(t) < 1, \tau'_l(t) < 1$, for all $i = \{1, 2, \dots, n\}$, for all $l = \{1, 2, \dots, p\}$; furthermore, $d_l(t) \in C^1(R, R), \tau_l(t) \in C^2(R, R), \int_0^\omega K_j(s)ds = 1, \int_0^{+\infty} sK_j(s)ds < +\infty$, for all $j = \{1, 2, \dots, m\}$, for all $l = \{1, 2, \dots, p\}$;

(H_2^*) there exists a constant $\eta > 0$ such that $|\Gamma^*(t)| \geq \eta$, for all $t \in [0, \omega]$, where $\Gamma^*(t)$ is defined by the following:

$$\Gamma^*(t) := a(t) + \sum_{i=1}^n \frac{a_i(\zeta_i(t))}{1 - \sigma'_i(\zeta_i(t))} + \sum_{j=1}^m c_j(t) - \sum_{l=1}^p \frac{d'_l(\xi_l(t))}{1 - \tau'_l(\xi_l(t))}; \quad (62)$$

(H_3^*) if $(1/2)(|a|_0\omega + \sum_{i=1}^n |b_i|_0\omega + \sum_{j=1}^m |c_j|_0\omega^{1/2}) + \sum_{l=1}^p |1 - \tau'_l|_0^{1/2}|d_l|_0 < 1$, and $\sum_{l=1}^p |1 - \tau'_l|_0|d_l|_0 < 1$.

Then, (61) has at least one positive ω -periodic solution.

Corollary 18. Suppose that (H_1^*) and the following conditions hold:

(H_4^*) there is a positive constant μ such that

$$\sum_{l=1}^p |d_l|_0 + \int_0^t \left[\sum_{l=1}^p |h_l|_0 + \sum_{i=1}^n |b_i|_0 + \sum_{j=1}^m |c_j|_0 \right] \times \exp \left(- \int_u^t a(\xi) d\xi \right) du \leq \mu < 1; \quad (63)$$

(H_5^*) $\exp(-\int_0^t A(\xi)d\xi) \rightarrow 0$, as $t \rightarrow +\infty$.

Then, the positive ω -periodic solution of (61) is globally attractive, where $h_l(t) := [d'_l(t) + d_l(t)a(t)] [1 - \tau'_l(t)] + d_l(t)\tau''_l(t)/[1 - \tau'_l(t)]^2$.

Remark 19. One could easily see that the systems (5)–(7) are all special cases of the system (61); we can get the similar results, and we omit them here. Hence, our results improve and generalize the corresponding results in [4–7].

4. An Example

Consider the following impulsive model:

$$\frac{dN(t)}{dt} = N(t) \left[r(t) - a(t) \ln N(t) - b(t) \ln N(t - \sigma(t)) \right]$$

$$- c(t) \int_{-\infty}^t K(t-s) \ln N(s) ds - d(t) \frac{d \ln N(t - \tau(t))}{dt} \Bigg], \quad t \neq t_k,$$

$$\Delta N(t_k) = N(t_k^+) - N(t_k) = \theta_k N(t_k), \quad k = 1, 2, \dots, \quad (64)$$

where $r(t), a(t), b(t), c(t), d(t), \sigma(t), \tau(t)$ are all positive ω -periodic continuous functions with $\sigma(t) \geq 0, \tau(t) \geq 0, t \in [0, \omega], \sigma'(t) < 1, \tau'(t) < 1$; furthermore, $d(t) \in C^1(R, R), \tau(t) \in C^2(R, R), \int_0^\omega K(s)ds = 1, \int_0^{+\infty} sK(s)ds < +\infty$.

Corollary 20. Suppose that (H_1) – (H_3) and the following conditions hold:

(H_4) there exists a constant $\eta > 0$ such that $|\Gamma(t)| \geq \eta$, for all $t \in [0, \omega]$, where $\Gamma(t)$ is defined by the following:

$$\Gamma(t) := A(t) + \frac{B(\zeta(t))}{1 - \sigma'(\zeta(t))} + C(t) - \frac{D'(\xi(t))}{1 - \tau'(\xi(t))}; \quad (65)$$

(H_5) if $(1/2)(|A|_0 + |B|_0)\omega + |C|_0\omega^{1/2} + |1 - \tau'|_0^{1/2}|D|_0 < 1$, and $|1 - \tau'|_0|D|_0 < 1$.

Then, (64) has at least one positive ω -periodic solution, where

$$\begin{aligned} A(t) &= a(t) \prod_{0 < t_k < t} (1 + \theta_k), \\ B(t) &= b(t) \prod_{0 < t_k < t - \sigma(t)} (1 + \theta_k), \\ C(t) &= c(t) \prod_{0 < t_k < t} (1 + \theta_k), \\ D(t) &= d(t) \prod_{0 < t_k < t - \tau(t)} (1 + \theta_k). \end{aligned} \quad (66)$$

Corollary 21. Suppose that (H_1) – (H_5) and the following conditions hold:

(H_6) there is a positive constant μ such that

$$|D|_0 + \int_0^t [|H|_0 + |B|_0 + |C|_0] \times \exp \left(- \int_u^t A(\xi) d\xi \right) du \leq \mu < 1; \quad (67)$$

(H_7) $\exp(-\int_0^t A(\xi)d\xi) \rightarrow 0$, as $t \rightarrow +\infty$.

Then, the positive ω -periodic solution of (64) is globally attractive, where $H(t) := [D'(t) + D(t)A(t)] [1 - \tau'(t)] + D(t)\tau''(t)/[1 - \tau'(t)]^2$.

Remark 22. The results in the work show that by means of appropriate impulsive perturbations, we can control the dynamics of these equations.

Acknowledgments

This work was supported by the Construct Program of the Key Discipline in Hunan Province. Research was supported by the National Natural Science Foundation of China (10971229, 11161015), the China Postdoctoral Science Foundation (2012M512162), and Hunan Provincial Natural Science-Hengyang United Foundation of China (11JJ9002).

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